Borel versus measure theoretic notions in the study of equivalence relations

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Some classical descriptive set theory

Definition A topological space is said to be *Polish* if it is separable and it admits a complete compatible metric.

We then say that the *Borel sets* are those appearing in the smallest σ-algebra containing the open sets.

A set X equipped with a σ-algebra is said to be a *standard Borel space* if there is some choice of a Polish topology giving rise to that σ-algebra as its collection of Borel sets.

A function between two Polish spaces,

$$f : X \to Y,$$

is said to be *Borel* if for any Borel $B \subset Y$ the pullback $f^{-1}[B]$ is Borel.

Theorem 0.1 *(Classical) If X is a Polish space and $B \subset X$ is a Borel set, then B (equipped in the σ-algebra of Borel subsets from the point of view of X) is standard Borel.*

Theorem 0.2 *(Classical; the "perfect set theorem") If X is a Polish space and $B \subset X$ is a Borel set, then exactly one of:*

1. B is countable; or

2. B contains a homeomorphic copy of Cantor space, $2^{\mathbb{N}}$ (and hence has size $2^{\aleph_0}$).

Theorem 0.3 *(Classical) If X is a standard Borel space, then the cardinality of X is one of* $\{1, 2, 3, ..., \aleph_0, 2^{\aleph_0}\}$.

Moreover!

Theorem 0.4 *(Classical) Any two standard Borel spaces of the same cardinality are Borel isomorphic.*

Here we say that X and Y are Borel isomorphic if there is a Borel bijection

$$f : X \to Y$$

whose inverse is Borel.[1]

Thus, as sets equipped with their σ-algebras they are isomorphic.

There is a similar theorem for quotients of the form X/E, E a Borel equivalence relation.

[1] In fact it is a classical theorem that any Borel bijection must have a Borel inverse

The analogues for Borel equivalence relations

Definition If X is a standard Borel space, an equivalence relation E on X is *Borel* if it appears in the σ-algebra on $X \times X$ generated by the rectangles $A \times B$ for A and B Borel subsets of X.

Theorem 0.5 *(Silver, 1980) Let X is a standard Borel space and assume E is a Borel equivalence relation on X. Then the cardinality of X/E is one of*

$$\{1, 2, 3, ..., \aleph_0, 2^{\aleph_0}\}.$$

However here there is no *moreover*.

In terms of Borel structure, and the situation when X/E is uncountable, there are vastly many possibilities at the level of Borel structure.

In the last twenty years, one of the major projects of set theorists has to be try to understand the range of possibilities.

This was initiated by two key papers.

A Borel Reducibility Theory for Classes of Countable Structures, H. Friedman and L. Stanley, **The Journal of Symbolic Logic**, Vol. 54, No. 3 (Sep., 1989), pp. 894-914

A Glimm-Effros Dichotomy for Borel Equivalence Relations, L. A. Harrington, A. S. Kechris and A. Louveau, **Journal of the American Mathematical Society**, Vol. 3, No. 4 (Oct., 1990), pp. 903-928

Neither paper referenced the other, and yet they used the exact same terminology and notation to introduce a new concept.

Definition Given equivalence relations E and F on X and Y we say that E is Borel reducible to F, written

$$E \leq_B F,$$

if there is a Borel function

$$f : X \to Y$$

such that

$$x_1 E x_2 \Leftrightarrow f(x_1) F f(x_2).$$

In other words, the Borel function f induces an injection

$$\hat{f} : X/E \to Y/F.$$

The perspective of Friedman and Stanley was to compare various classes of countable structures under the ordering $\leq_B$. The Harrington, Kechris, Louveau paper instead generalized earlier work of Glimm and Effros in foundational issues involving the theory of unitary group representations.

"Borel cardinality"

Definition The *cardinality of* X *is less than or equal to* Y,

$$|X| \leq |Y|,$$

if there is an injection from X to Y.

This might suggest a notion of "cardinality" where we restrict our attention to some restricted class of injections.

This in turn could relate to the idea that an equivalence relation E on a set X is in some sense *classifiable* if there is a "reasonably nice" or "natural" or "explicit" function

$$f : X \rightarrow I$$

which induces (via $x_1 E x_2 \Leftrightarrow f(x_1) = f(x_2)$) an injection

$$\hat{f} : X/E \rightarrow I.$$

In the context of unitary group representation a definition exactly along these lines was proposed by G. W. Mackey.

Definition (Mackey) An equivalence relation E on a Polish space X is *smooth* if there is a Polish space Y and a Borel function $f : X \to Y$ such that

$$x_1 E x_2 \Leftrightarrow f(x_1) = f(x_2).$$

Many well known classification theorems can be viewed, in part, as giving a Borel reduction of one Borel equivalence relation to another. For instance Baer's classification of rank one torsion free groups can be viewed as demonstrating a Borel reduction from the isomorphism relation on rank one torsion free abelian groups to elements of $2^{\mathbb{N}\times\mathbb{P}}$ considered up to finite agreement. The spectral theorem for infinite dimensional operators implies isomorphism of unitary representations of $\mathbb{Z}$ is Borel reducible to measures considered up to absolute continuity.

Definition Let E_0 be the equivalence relation of eventual agreement on $2^{\mathbb{N}}$. For X a Polish space let $\mathrm{id}(X)$ be the equivalence relation of equality on X.

Thus in the above notation we can recast Mackey's definition of *smooth*: An equivalence relation E is smooth if for some Polish X we have

$$E \leq_B \mathrm{id}(X).$$

It turns out that for any uncountable Polish space X we have

$$\mathrm{id}(\mathbb{R}) \leq_B \mathrm{id}(X)$$

and

$$\mathrm{id}(X) \leq_B \mathrm{id}(\mathbb{R}).$$

Theorem 0.6 *(Silver, rephrased) Let E be a Borel equivalence relation. Then exactly one of the following two conditions holds:*

1. $E \leq_B \mathrm{id}(\mathbb{N})$*;*

2. $\mathrm{id}(\mathbb{R}) \leq_B E$*.*

Theorem 0.7 *(Harrington, Kechris, Louveau) Let E be a Borel equivalence relation. Then exactly one of the following two conditions holds:*

1. $E \leq_B \mathrm{id}(\mathbb{R})$*;*

2. $E_0 \leq_B E$*.*

This breakthrough result, this paradigm *dichotomy theorem*, suggested the possibility of understanding the structure of the Borel equivalence relations up to Borel reducibility.

It turns out that these are the only *global* dichotomy theorems of this nature. The research in the area has generally proceeded by dividing the Borel equivalence relations in to certain natural classes, and working inside these.

For instance it is natural to divide Borel equivalence relations on the basis of whether they are "countable" (or at least Borel reducible to a countable equivalence relation).

Definition A Borel equivalence relation is *countable* if every equivalence class is countable.

Another natural class, at least for a logician, are those equivalence relations classifiable by countable structures.

Definition An equivalence relation E on a Polish space X is *classifiable by countable structures* if there is a countable language $\mathcal{L}$ and a Borel function

$$f : X \to \mathrm{Mod}(\mathcal{L})$$

such that for all $x_1, x_2 \in X$

$$x_1 E x_2 \Leftrightarrow f(x_1) \cong f(x_2).$$

This is something like asking for an equivalence relation to have complete *algebraic* invariants, and it has many equivalent forms. For instance, it is equivalent to asking that there be a Borel way of assigning a countable linear order or a countable group whose isomorphism type is a complete invariant. It is also equivalent to E being Borel reducible to the orbit equivalence relation induced by a Borel action of S_∞ (the Polish group of *all* permutations of $\mathbb{N}$).

It follows quickly from the definitions and some minor massaging that if a Borel equivalence relation is countable then it is classifiable by countable structures.

We might then divide the situation up in to three classes:

1. Borel equivalence relations which are not classifiable by countable structures.
2. Borel equivalence relations which are classifiable by countable structures, but not Borel reducible to countable Borel equivalence relations.
3. Countable Borel equivalence relations.

I will go ahead in order and try to quickly summarize the state of knowledge about these three classes, but it is really the last class which is relevant to this talk.

Borel equivalence relations which are not classifiable by countable structures

Here we know nothing and there is no known body of techniques which seem to be helpful.[2]

[2] For the purposes of this talk that is probably an on target summary, but it is not quite fully accurate. Ilijas Farah has a sequence of technically demanding papers which borrow ideas from Banach space theory to provide spectacular counterexamples to various nature conjectures. It is also true that logicians have developed something we call *the theory of turbulence*, which has proven very succesful in determining *when* an equivalence relation is classifiable by countable structures. For instance:

1. (Hjorth) The homeomorphism group of the unit square,

$$\mathrm{Hom}([0,1]^2),$$

considered up to homeomorphism does not admit classification by countable structures.

2. (Gao) Countable metric spaces up to homeomorphism does not admit classification by countable structures.
3. (Törnquist) Measure preserving actions of $\mathbb{F}_2$ up to orbit equivalence do not admit classification by countable structures.

Borel equivalence relations which *are* classifiable by countable structures but are not Borel reducible to a countable equivalence relation

Here we know a *lot*, and all the techniques arise from logic, or set theory, in some general sense.[3]

[3] These were the kinds of problems considered in the Friedman-Stanley paper.

Countable Borel equivalence relations

Every major *negative* result (showing some countable E not Borel reducible to some F) relies on measure theory, and uses techniques arising from outside logic.

Definition Let E and F be equivalence relations on standard Borel probability spaces (X, μ), (Y, ν). Then we say that E is *orbit equivalent* to F if there is a measure preserving bijection

$$\theta : X \to Y$$

with

$$\theta[[x]_E] = [\theta(x)]_F$$

for almost all $x \in X$.

Definition Given countable groups Γ, Δ and a measurable action of Γ on a standard Borel probability space (X, μ), a measurable map

$$\alpha : \Gamma \times X \to \Delta$$

is a *cocycle* if for all $\gamma_1, \gamma_2 \in \Gamma$ and a.e. $x \in X$ we have

$$\alpha(\gamma_1\gamma_2, x) = \alpha(\gamma_1, \gamma_2 \cdot x)\alpha(\gamma_2, x).$$

Here it might be worth introducing a third, rather ad hoc, definition simply for comparison.

Definition Let E and F be equivalence relations on standard Borel probability spaces (X, μ), (Y, ν). Write

$$E \leq_{B,m} F$$

if there is a conull $X_0 \subset X$ with

$$E|_{X_0} \leq_B F.$$

In the case that Δ acts freely (and in a Borel manner) on Y, Γ acts (in a Borel manner) on X,

$$E \leq_{B,m} F$$

implies that there is a resulting cocycle from Γ to Δ.

The typical proofs that one countable Borel equivalence relation is not Borel reducible to actually end up showing that there is a failure of this $\leq_{B,m}$. It is pretty much true (though one can argue on the edges) that every proof of non-reduction with respect to $\leq_B$ among countable Borel equivalence relation actually derives from a proof of non-reduction with respect to $\leq_{B,m}$, and that these in turn often use ideas from orbit equivalence.[4]

In many cases, however, theorems for orbit equivalence turn out to be insufficient, as far as we can tell, to establish the parallel theorem for Borel reducibility.

[4]It should be underscored though that the applications of measure theoretic ideas can be quite indirect. For instance, there isn't any non-trivial invariant Borel probability measure on the space of rank n torsion free abelian groups, but Thomas' proof that rank $n+1$ is not Borel reducible to rank n ultimately imports measure theoretic ideas in an indirect way.

A comparison between orbit equivalence theorems and Borel reducibility theorems

Theorem 0.8 *(Gaboriau-Popa, c, 2000[?]) There are continuum many free, measure preserving action of* $\mathbb{F}_2$ *(on standard Borel probability spaces) up to orbit equivalence.*

None of us were able to adapt their proof to say anything about the Borel reducibility version of this problem. In fact, the final solution to this borrowed as much from a paper of Ioana.

Theorem 0.9 [5] *(Hjorth, 2008) There are continuum many free, Borel actions of* $\mathbb{F}_2$ *(on standard Borel spaces) up to Borel reducibility,* $\leq_B$.

[5] Here we were really concerned with the problem of *treeable* equivalence relations, but it turns out to be equivalent to the theorem as stated.

Theorem 0.10 *(Epstein, 2007) Every countable non-amenable group has continuum many actions (on standard Borel probability spaces) up to orbit equivalence.*

Her proof does *not* appear to adapt show the analogous statement for Borel reducibility.

Question If Γ is a countable non-amenable group, must it have continuum many Borel actions (on standard Borel spaces) up to Borel reducibility?

It is not at all clear how to tackle this problem, but there might be some glimmer for optimism given recent history. On the other hand, there is a whole sequence of open questions relating to the notion of hyperfiniteness where the situation appears much more obscure.

Definition An equivalence relation E is *hyperfinite* if there are Borel equivalence relations

$$F_0 \subset F_1 \subset F_2 \subset ...$$

with each F_n having all its classes finite and

$$E = \bigcup_{n \in \mathbb{N}} \uparrow F_n.$$

Perhaps the canonical example of this is E_0.

Definition E_0 is the equivalence relation of eventual agreement on infinite binary sequences. So for $\vec{x}, \vec{y} \in 2^{\mathbb{N}}$

$$\vec{x} E_0 \vec{y}$$

if there is an $N \in \mathbb{N}$ such that for all $n > N$ we have $x_n = y_n$.

Proposition 0.11 *(various authors) For E a countable Borel equivalence relation the following are equivalent:*

1. *$E \leq_B E_0$;*
2. *E is hyperfinite;*
3. *there exists a Borel action of $\mathbb{Z}$ with $E = E_{\mathbb{Z}}$.*

Theorem 0.12 *(Ornstein, Weiss) If Γ is a countable amenable group acting in a measure preserving manner on a standard Borel probability space (X, μ), then there is a conull $X_0 \subset X$ such that $E_\Gamma|_{X_0}$ is hyperfinite.*

One doesn't even need the action to be measure preserving.[6] With respect to *any* Borel probability measure, one can always go down to a measure one set on which E_Γ is hyperfinite.

[6] The first place I have seen it observed is in some work by Kechris, but I don't know when it was first observed

I think it is fair to say that in the measure theoretic context hyperfiniteness is well understood.

In the Borel context, we have two theorems, the first of which builds on a proof by Benji Weiss for finitely generated abelian groups.

Theorem 0.13 *(Jackson, Kechris, Louveau) If Γ is a finitely generated nilpotent by finite group acting in a Borel manner on a standard Borel space X, then E_Γ is hyperfinite.*

The second theorem has a proof which largely occupies the entirety of a fifty page paper, and is perhaps the most technical difficult argument in the whole field.

Theorem 0.14 *(Gao, Jackson) If* Γ *is a countable abelian group acting in a Borel manner on a standard Borel space* X*, then* E_Γ *is hyperfinite.*

So much for the good news, here is the unsettling part: Their proof gives no hint whatsoever that one may be able to prove this for all amenable groups. In fact, even extending to nilpotent groups seems technically infeasible.

On the one hand, we do not have any real evidence to think Ornstein-Weiss goes through in the Borel context. On the other hand, we know that measure theoretic counterexamples do not exist. This *may* suggest the need for new techniques.

Question If Γ is a countable amenable group acting in a Borel manner on a standard Borel space, must the resulting orbit equivalence relation E_Γ be hyperfinite?

Although this is perhaps *the* question right now, which would receive the most enthusiastic response among set theorists working in the field, there is another question for which a positive answer would have far reaching structural consequences for Borel equivalence relations under $\leq_B$.

Question Let G be an abelian Polish group acting in a continuous manner on a Polish space X. Suppose E_G is Borel reducible to a countable Borel equivalence relation. Must E_G be hyperfinite?