Subgroups of abelian Polish groups

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1 Introduction

Ilijas Farah and Slawek Solecki showed in [2] that every Polish group contains arbitrarily complicated Borel subgroups and go on to ask whether the same can be shown for *Polishable* subgroups. As a partial answer, this notes shows that an uncountable abelian Polish group contains Polishable subgroups of unbounded Borel complexity.

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2 Notation

In what follows, G is an uncountable abelian Polish group and $d(\cdot,\cdot)$ a compatible, complete, two sided-invariant metric. We write the group operations on G with reference to it being abelian – thus $+$ is the group operation and $n \cdot g$ stands for

$$g + g + ...(n \text{ times}).. + g.$$

0 is the group identity in G.

We will find it convenient to use the notation of an infinite sum. Thus we say that

$$g = \sum_{n=0}^{\infty} g_n$$

if

$$d(\sum_{n=0}^{N} g_n, g) \to 0.$$

Since d is two sided-invariant metric we obtain that

$$\sum_{n=0}^{\infty} g_n = \sum_{n=0}^{\infty} h_n$$

if and only if

$$\sum_{n=0}^{\infty}(g_n - h_n) = 0.$$

For g a group element we use $o(g)$ to denote the *order* – that is to say, the least ℓ with $\ell \cdot g = 0$ if such an ℓ exists, and infinity otherwise.

3 The subgroups

We choose a sequence of non-zero elements

$$(g_i)_i$$

in G such that:

(I) for any $k \in \mathbb{N}$ and $(n_\ell)_{\ell \leq k}$ chosen from $\mathbb{Z}$, with each

$$|n_\ell| \leq 4\ell k$$

and

$$h = n_0 \cdot g_0 + n_1 \cdot g_1 ... + n_k \cdot g_k \in G$$
$$h \neq 0$$

we have that for any choice of $(\hat{n}_\ell)_{\ell > k}$ with each $|\hat{n}_\ell| < 4\ell k$ that

$$h \neq \sum_{\ell > k} \hat{n}_\ell \cdot g_\ell;$$

(II) for any ℓ and n with $|n| \leq \ell^2$ we have

$$d(0, n \cdot g_\ell) < 2^{-\ell};$$

(III) either every g_i has infinite order, or the g_i's all have the same *finite* order, or $o(g_i) \to \infty$ as $i \to \infty$.

We obtain (I) and (II) just by taking any sequence which converges to 0 sufficiently fast. (III) then follows by refining the sequence as needed.

Definition If $h \in G$ then a *k-representation* of h is an expression of the form

$$\sum_{\ell \in \mathbb{N}} n_\ell \cdot g_\ell$$

with

$$\lim_{M \to \infty} \sum_{\ell=0}^{M} n_\ell \cdot g_\ell = h$$

and each $|n_\ell| \leq \ell \cdot k$. A *representation* is a k-representation for some k; a group element is *representable* if it has a representation.

Lemma 3.1 *If*

$$\sum_{\ell \in \mathbb{N}} n_\ell \cdot g_\ell,$$

$$\sum_{\ell \in \mathbb{N}} \hat{n}_\ell \cdot g_\ell,$$

are two $2k$*-representations of* h *then for all* $\ell > k$

$$n_\ell \cdot g_\ell = \hat{n}_\ell \cdot g_\ell.$$

Proof By the definition of $2k$-representation we have $|n_\ell|, |\hat{n}_\ell| \leq 2\ell k$ each ℓ, and then by our assumption (I) on the (g_i) sequence we have that at all $M \geq k$

$$\sum_{\ell \leq M} (n_\ell - \hat{n}_\ell) \cdot g_\ell = 0.$$

From this the lemma follows. □

Definition We let H be the subgroup of all h which have some representation

$$\sum_{\ell \in \mathbb{N}} n_\ell \cdot g_\ell$$

with

$$\frac{|n_\ell|}{\ell} \to 0.$$

Definition If

$$\sum_{\ell \in \mathbb{N}} n_\ell \cdot g_\ell$$

is a representation of h, then we let

$$\varphi(\sum_{\ell \in \mathbb{N}} n_\ell \cdot g_\ell) = \sup_\ell \frac{|n_\ell|}{\ell}.$$

If $h \in H$ we let $\psi(h)$ be the infinum of

$$\varphi(\sum_{\ell \in \mathbb{N}} n_\ell \cdot g_\ell)$$

as $\sum n_\ell \cdot g_\ell$ ranges over representations of h.

Lemma 3.2 *In the metric*

$$d_H(h, h') = \psi(h - h') + d(h, h')$$

we have that H *is a Polish group.*

Proof First we need to check that this is indeed a metric, and that follows from the observation that given any two representations

$$\sum n_\ell \cdot g_\ell,$$

$$\sum \hat{n}_\ell \cdot g_\ell,$$

for $h - h'$ and $h' - h''$ respectively, we have

$$\varphi(\sum (n_\ell + \hat{n}_\ell) \cdot g_\ell) \leq \varphi(\sum n_\ell \cdot g_\ell) + \varphi(\sum \hat{n}_\ell \cdot g_\ell).$$

The metric *is* separable since the representations with finite support ($\forall^\infty \ell(n_\ell = 0)$) give rise to a dense subgroup. Continuity of the group operations follows since

$$\psi(h) + \psi(h') \geq \psi(h + h').$$

This only leaves us with a campaign to show that the metric is complete.

For this purpose, consider a Cauchy sequence $(h_i)_i$ in H, with each h_i having

$$\sum_{\ell \in \mathbb{N}} n^i_\ell \cdot g_\ell$$

as a representation. Since

$$|\psi(h_i) - \psi(h_j)| \leq \psi(h_i - h_j)$$

we get that $(\psi(h_i))_{i \in \mathbb{N}}$ is Cauchy and so there is a fixed k such that each is a k-representation. Since the space of k-representations is precompact, we may assume there exist $(n^\infty_\ell)_{\ell \in \mathbb{N}}$ such that

$$\forall^\infty j(n^j_\ell = n^\infty_\ell).$$

Claim(1): As $j \to \infty$ and then $M \to \infty$

$$d_H(h_j, \sum_{\ell=0}^{M} n^\infty_\ell \cdot g_\ell) \to 0.$$

Proof of Claim: Fix $\epsilon > 0$. We need to show that for all sufficiently large j we have

$$\forall^\infty M(d_H(h_j, \sum_{\ell=0}^{M} n^\infty_\ell \cdot g_\ell) < \epsilon).$$

Choose j large enough that $d_H(h_j, h_{j'}) < \epsilon$ all $j' \geq j$ and $n^j_\ell = n^\infty_\ell$ all $\ell \leq k$. We may assume without loss of generality that $\epsilon < \frac{1}{k}$. Thus at each $j' \geq j$ we have that if

$$\sum_{\ell \in \mathbb{N}} m_\ell \cdot g_\ell$$

is a $2k$-representation of $h_j - h'_j$ with φ value less than ϵ then

$$m_\ell = 0$$

at all $\ell \leq k$. Since by 3.1 we have

$$\sum (n^j_\ell - n^{j'}_\ell) \cdot g_\ell$$

as the unique $2k$-representation of $h_j - h_{j'}$ with coefficient 0 at the ℓ^{th} coordinate all $\ell \leq k$, and thus

$$\sum (n^j_\ell - n^{j'}_\ell) \cdot g_\ell$$

is the unique $2k$-representation of $h_j - h_{j'}$ with φ value less than ϵ.

Therefore if at arbitrarily large M we have

$$d_H(h_j, \sum_{\ell=0}^{M} n^\infty_\ell \cdot g_\ell) > \epsilon$$

then for arbitrarily large M we have

$$d_H(\sum_{\ell=0}^{M} n_\ell^j \cdot g_\ell, \sum_{\ell=0}^{M} n_\ell^\infty \cdot g_\ell) > \epsilon,$$

but then going to some j' with $n_\ell^{j'} = n_\ell^\infty$ for all $\ell \leq M$ we have

$$\varphi(\sum (n_\ell^j - n_\ell^{j'}) \cdot g_\ell) > \epsilon,$$

with a contradiction. (□Claim)

Claim(2): As $N \to \infty$

$$\varphi(\sum_{\ell=N}^{\infty} n_\ell^\infty \cdot g_\ell) \to 0.$$

Proof of Claim: Consider $\epsilon > 0$. Choose M_0 such that at all $j \geq M_0$ and $\ell \leq k$ we have $n_\ell^j = n_\ell^\infty$. Choose $M_1 > M_0$ such that at all $j, j' \geq M_1$ we have $\psi(h_j - h_{j'}) < \epsilon$. Choose N_0 such that at all $\ell \geq N_0$

$$\frac{|n_\ell^{M_1}|}{\ell} \leq \epsilon,$$

$$\therefore \varphi(\sum_{\ell=N_0}^{\infty} n_\ell^{M_1} \cdot g_\ell) \leq \epsilon.$$

As in the proof of claim(1) we may assume that $\epsilon < \frac{1}{k}$, and again as in that proof we have that for all sufficiently large j'

$$\sum_0^\infty (n_\ell^{j'} - n_\ell^{M_1}) \cdot g_\ell$$

is the only possible representation of $h_{j'} - h_{M_1}$ with

$$\varphi(\sum_0^\infty (n_\ell^{j'} - n_\ell^{M_1}) \cdot g_\ell) < \epsilon,$$

and hence by 3.1 for j' sufficiently large we have $\sum_0^\infty (n_\ell^{j'} - n_\ell^{M_1}) \cdot g_\ell$ is *the* representation of $h_{j'} - h_j$ with $\varphi(\sum_0^\infty (n_\ell^{j'} - n_\ell^{M_1}) \cdot g_\ell) < \epsilon$. Hence at each $N > N_0$

$$\varphi(\sum_{\ell=N_0}^{N} n_\ell^\infty \cdot g_\ell) \leq \varphi(\sum_{\ell=N_0}^{N} (n_\ell^{j'} - n_\ell^{M_1}) \cdot g_\ell) + \varphi(\sum_{\ell=N_0}^{N} n_\ell^{M_1} \cdot g_\ell)$$

$$\leq \varphi(\sum_0^\infty (n_\ell^{j'} - n_\ell^{M_1}) \cdot g_\ell) + \epsilon = \psi(h_{j'} - h_{M_1}) + \epsilon < 2\epsilon.$$

Letting $\epsilon \to 0$ this proves the claim. (□Claim)

These two claims between them demonstrate that

$$\sum n_\ell^\infty \cdot g_\ell$$

represents some $h_\infty \in H$ and moreover

$$\psi(h_j - h_\infty) \to 0$$

as $j \to \infty$.

□

Definition We define the notion of α-calibration by induction on the countable ordinal α.

A *0-calibration* is an infinite $A \subset \mathbb{N}$. An $\alpha+1$-calibration is an infinite $A \subset \mathbb{N}$ equipped with a partition

$$A = \dot{\bigcup} B_n,$$

where each B_n is an α-calibration. For λ a limit, with a specified[1]

$$\alpha_n \nearrow \lambda,$$

we say that $A \subset \mathbb{N}$ is λ-calibration if it is partitioned

$$A = \dot{\bigcup} B_n,$$

with each B_n an α_n-calibration.

Definition For A a 0-calibration, a representation

$$\sum n_\ell \cdot g_\ell$$

is A-0-*good* if there is an n_∞ such that

$$\forall^\infty \ell \in A(n_\ell = n_\infty);$$

that is to say, the n_ℓ's, for $\ell \in A$, converge to n_∞. We then let $\pi_{A,0}(\sum n_\ell \cdot g_\ell)$ be this indicated n_∞ and

$$\varphi_{A,0}(\sum n_\ell \cdot g_\ell) = \varphi(\sum n_\ell \cdot g_\ell) + |\pi_{A,0}(\sum n_\ell \cdot g_\ell)| + |\{\ell \in A : n_\ell \neq \pi_{A,0}(\sum n_\ell \cdot g_\ell)\}|;$$

in other words, we add into the norm on representations for H information about the eventual value n_∞ along with a counter for the number of times this value is missed.

For A an $\alpha+1$-calibration, $A = \dot{\bigcup} B_n$, we say that

$$\sum n_\ell \cdot g_\ell$$

is A-$\alpha+1$-*good* if it is B_n-α-good at every n and there is an m_∞ such that

$$\forall^\infty n(\pi_{B_n,\alpha}(\sum n_\ell \cdot g_\ell) = m_\infty).$$

We then let $\pi_{A,\alpha+1}(\sum n_\ell \cdot g_\ell)$ be this eventual m_∞ value and let

$$\varphi_{A,\alpha+1}(\sum n_\ell \cdot g_\ell) = \varphi(\sum n_\ell \cdot g_\ell) + \sum_n 2^{-n} \frac{\varphi_{B_n,\alpha}(\sum n_\ell \cdot g_\ell)}{1 + \varphi_{B_n,\alpha}(\sum n_\ell \cdot g_\ell)}$$

$$+ |\pi_{A,\alpha+1}(\sum n_\ell \cdot g_\ell)| + |\{n : \pi_{B_n,\alpha}(\sum n_\ell \cdot g_\ell) \neq \pi_{A,\alpha+1}(\sum n_\ell \cdot g_\ell)\}|;$$

that is to say, we merge into a single norm information about the eventual value m_∞, the set of exceptional n's at which this value is not yet reached, along with a bounded version of norms for the various B_n sets.

For λ a limit ordinal, $\alpha_n \nearrow \lambda$, A a λ-calibration, we can define A-λ-goodness in the exactly parallel manner, replacing each mention of B_n-α-goodness by B_n-α_n-goodness.

[1] For the purpose of this definition, imagine that all our limit ordinals come with a previously decided cofinal sequence.

We want to now define $\pi_{A,\alpha}(h)$ in a way that is independent of the representation. Here there is a split in cases, depending on the situation dictated at (III).

Observe first of all by 3.1 we have that if $\sum n_\ell \cdot g_\ell$ and $\sum n'_\ell \cdot g_\ell$ are two A-α-representations of $h \in H$ then the terms

$$n_\ell \cdot g_\ell, \qquad n'_\ell \cdot g_\ell$$

differ at only finitely many places.

Therefore in the case of every g_ℓ having infinite order we get

$$\pi_{A,\alpha}(\sum n_\ell \cdot g_\ell) = \pi_{A,\alpha}(\sum n'_\ell \cdot g_\ell)$$

for any two distinct representations. Therefore we just do the natural thing and let $\pi_{A,\alpha}(h)$ be the common value realized on all the A-α-good representations.

The next case is when the g_ℓ's *all* have the *same* finite order. Use M to denote this common value. We then let $\pi_{A,\alpha}(h)$ be the unique $m \in \{0, 1, ..., M-1\}$ for which there is some A-α-good representation having

$$\pi_{A,\alpha}(\sum n_\ell \cdot g_\ell) = m.$$

In the case that the g_ℓ's have strictly increasing but finite orders, we let $\pi_{A,\alpha}(h)$ be the unique $m \in \mathbb{Z}$ for which there is some A-α-good representation with

$$\pi_{A,\alpha}(\sum n_\ell \cdot g_\ell) = m.$$

Note also that the collection of A-α-good elements is closed under the group operations – a critical fact underpinning the next definition.

Definition For A an α-calibration we let

$$H_{A,\alpha}$$

be the subgroup of H consisting of all elements which admit an A-α-good representation. For h such an element, we let

$$\psi_{A,\alpha}(h)$$

be the infinum of

$$\varphi_{A,\alpha}(\sum_{\ell \in \mathbb{N}} n_\ell \cdot g_\ell)$$

as $\sum n_\ell \cdot g_\ell$ ranges over all possible representations. We then let $d_{A,\alpha}(\cdot,\cdot)$ be defined by

$$d_{A,\alpha}(h, h') = \psi_{A,\alpha}(h - h') + d_H(h, h').$$

Lemma 3.3 *$d_{A,\alpha}$ defines a complete, separable metric on $H_{A,\alpha}$ for any α-calibration A.*

Proof That this is a metric follows as in the proof for H. Separability follows by induction on α. For $\alpha = 0$ we observe that the set of elements with a representation

$$\sum n_\ell \cdot g_\ell$$

for which there is some m_∞ with

$$\forall^\infty \ell (n_\ell = m_\infty)$$

forms a countable dense subset. For $A = \dot{\bigcup} B_n$ with each B_n a β_n-calibration, we assume that there are countable dense subsets D_n of each H_{B_n,β_n} and we take as our countable dense subset those representations which on finitely many B_n's are an element of a corresponding D_n, and elsewhere are eventually constant.

Again, these arguments are quickly dismissed, and the main battle is for completeness. So we suppose that we have some $d_{A,\alpha}$-Cauchy sequence $(h_j)_{j\in\mathbb{N}}$, each h_j having

$$\sum_{\ell\in\mathbb{N}} n_\ell^j \cdot g_\ell$$

as a representation. Again by the compactness of the space of k-representations, as in 3.2, we may assume there are $(n_\ell^\infty)_{\ell>k}$ such that

$$\forall^\infty j(n_\ell^j = n_\ell^\infty).$$

Note that for all sufficiently large j, j' we have

$$\psi_{A,\alpha}(h_j - h_{j'}) < 1,$$

and hence

$$\pi_{A,\alpha}(h_j - h_{j'}) = 0$$

by definition of $\psi_{A,\alpha}(\cdot)$.

Thus we may find a single m_∞ such that

$$\forall^\infty j(\pi_{A,\alpha}(h_j) = m_\infty).$$

Let h_∞ be the limit in the $d_H(\cdot,\cdot)$ metric of the $(h_j)_j$ sequence, with representation

$$\sum_\ell n_\ell^\infty \cdot g_\ell.$$

Note that each $|n_\ell^\infty| \leq \ell \cdot k$ since $\forall^\infty j(n_\ell^j = n_\ell^\infty)$ and each $|n_\ell^j| \leq \ell \cdot k$. Thus $\sum n_\ell^\infty \cdot g_\ell$ is a k-representation.

In the case that $\alpha = 0$ we have the following claim:

Claim: There are only finitely many $a \in A$ with

$$n_a^\infty \neq m_\infty.$$

Proof of Claim: We again use the fact that $2k$-representations are unique on coordinates bigger than k. Thus we obtain that if for some $c \geq 2k$ we have $2c$-many a's with $n_a^\infty \neq m_\infty$, then for all sufficiently large j we have *all* $2k$ representations of h_j have at least c many coordinates which present a value other than

$$m_\infty = \pi_{A,0}(h_j),$$

and hence we are left with $\psi_{A,0}(h_j) > c$.

Thus to recap the argument, we have seen that if at *every* c there exist more than c many $a's$ with $n_a^\infty \neq m_\infty$ then $\psi_{A,0}(h_j) \to \infty$ as $j \to \infty$, contradicting the assumption that $(\psi_j)_{j\in\mathbb{N}}$ is Cauchy. ($\Box$Claim)

Thus we have $h_\infty \in H_{A,0}$, $\pi_{A,0}(h_\infty) = m_\infty$; the function

$$j \mapsto \{a \in A : n_a^j \neq m_\infty\}$$

is eventually constant by the definition of $\psi_{A,0}(\cdot)$ and uniqueness of the k-representation on coordinates bigger than k. Therefore with respect to $d_{A,0}(\cdot,\cdot)$ we have

$$h_j \to h_\infty.$$

If $\alpha > 0$, $A = \dot{\bigcup} B_n$, each B_n a β_n-calibration, we may assume inductively that there are (p_n) such that at every n

$$\forall^\infty j(\pi_{B_n,\beta_n}(h_j) = p_n),$$
$$h_\infty \in H_{B_n,\beta_n},$$

and

$$h_j \to h_\infty$$

in $d_{B_n,\beta_n}(\cdot,\cdot)$. By Cauchyness of the (h_j) sequence we must eventually have that for all sufficiently large j, j'

$$\forall n(\pi_{B_n,\beta_n}(h_j) = \pi_{B_n,\beta_n}(h_{j'})),$$

and hence the (p_n)-sequence is eventually constant, with value m_∞.

□

Lemma 3.4 *Given any $\underset{\sim}{\Delta}^0_\alpha$ set $D \subset \omega^\omega$ and α-calibration $A = \dot{\bigcup} B_n$, each B_n a β_n-calibration, there is continuous*

$$f^{A,D}_\alpha : \omega^\omega \to G$$

such that

(0) for every y there is a representation

$$\sum n_\ell \cdot g_\ell$$

of $f^{A,D}_\alpha(y)$ with each $n_\ell \in \{0,1\}$.
(i) if $y \in D$ then $\forall^\infty n(\pi_{B_n,\beta_n}(f^{A,D}_\alpha(y)) = 1)$;
(ii) if $y \notin D$ then $\forall^\infty n(\pi_{B_n,\beta_n}(f^{A,D}_\alpha(y)) = 0)$;
(iii) the representation of each $f^{A,D}_\alpha(y)$ has support in A and each $f^{A,D}_\alpha(y) \in H_{A,\alpha}$.

Proof This is proved by induction on α, in a manner entirely parallel to the similar claims from [3]. First of all one sees that for any clopen $D \subset \omega^\omega$ and A a 0-calibration we can certainly find continuous $f^{A,D} : \omega^\omega \to G$ with

$$f^{A,D}(x) = 0$$

if $x \notin D$, but

$$f^{A,D}(x) = \lim_{N\to\infty} \sum_{\ell \in A, \ell \leq N} g_\ell,$$

if $x \in D$. This forms the base of our induction.

Now we do the inductive step, and suppose that D is $\underset{\sim}{\Delta}^0_\alpha$, $A = \dot{\bigcup} B_n$ is an α-calibration, each B_n a β_n-calibration. We recall the classical fact (compare [3]) that there must be $\underset{\sim}{\Delta}^0_{\beta_n}$ sets $(E_n)_n$ such that

(a) if $y \in D$ then $\forall^\infty n(y \in E_n)$;
(b) if $y \notin D$ then $\forall^\infty n(y \notin E_n)$.

Appealing to our inductive hypothesis we may find

$$f_n : \omega^\omega \to H_{B_n,\beta_n},$$

continuous with respect to the Polish G-topology, each $f_n(x)$ having support inside B_n, with $\pi_{B_n,\beta_n}(f_n(x)) = 1$ if $x \in E_n$, $\pi_{B_n,\beta_n}(f_n(x)) = 0$ if $x \notin E_n$, and every $f_n(x)$ having a representation with coordinates taken solely from $\{0,1\}$. We can then simply let

$$f^{A,D}(x) = \lim_{N\to\infty} f_0(x) + f_1(x) + f_2(x) + \ldots + f_N(x).$$

Our inductive assumptions on the f_n's suffice to give that this is in H, and then we obtain it will be in $H_{A,\alpha}$ with either (a) or (b) above by the choice of the E_n sets. It is continuous by condition (II) on the g_ℓ-sequence and the fact that each $f_n(x)$ has support inside B_n. □

Corollary 3.5 *If $A = \dot{\bigcup} B_n$ is an $\alpha+1$-calibration, then $H_{A,\alpha+1}$ is not $\underset{\sim}{\Pi}^0_\alpha$.*

Proof Let $C \in \underset{\sim}{\Sigma}^0_\alpha$ be of the form

$$C = \bigcup D_n,$$

where each $D_n \in \underset{\sim}{\Delta}^0_\alpha$. Following the last lemma we may find continuous

$$f_n : \omega^\omega \to H_{B_n,\beta_n}$$

such that

(0) for every y there is a representation

$$\sum_{\ell \in B_n} n_\ell \cdot g_\ell$$

of $f_n(y)$ with each $n_\ell \in \{0,1\}$.

(i) if $y \in \bigcup_{i \leq n} D_i$ then $\pi_{B_{2n},\beta_{2n}}(f_{2n}(y)) = 1$;

(ii) if $y \notin \bigcup_{i \leq n} D_i$ then $\pi_{B_{2n},\beta_{2n}}(f_{2n}(y)) = 0$;

(iii) the representation of each $f_n(y)$ has its support in B_n;

(iv) for m odd at every y we have $\pi_{B_m,\beta_m}(f_m(y)) = 1$.

After this we can simply take

$$f^C(y) = \lim_{N \to \infty} f_0(y) + f_1(y) + ... f_N(y).$$

Then in order for the values of the $\pi_{B_n,\beta_n}(f^C(y))$ to settle down into a limit we must have that $y \in C$, and thus membership C is a necessary condition for $f(y)$ to be in $H_{A,\alpha+1}$. But conversely membership in C is also a sufficient condition for $f(y)$ to be in this subgroup, and thus we have continuously reduced membership in an arbitrary $\underset{\sim}{\Sigma}^0_\alpha$ set to membership in $H_{A,\alpha+1}$.

□

4 After thoughts

As observed by Slawek Solecki, this result for uncountable abelian Polish groups implies the same for uncountable locally compact Polish groups.

The main point is that we can use approximation by Lie groups – in particular the theorem that every connected locally compact group has arbitrarily small normal subgroups for which the resulting quotient is Lie. (See [5].)

One case is that the group is totally disconnected, and hence, by local compactness (see [5]) it will have a neighborhood basis of clopen compact subgroups. Appealing to [6] we obtain an uncountable abelian compact subgroup, and finish by the results of the last section.

The other case is that there exists a closed connected subgroup, which in turn will have an uncountable Lie group as a quotient. But any Lie group contains one-parametrized subgroups, arrived at via the exponential map from the tangent bundle at the identity, and hence a continuous image of $\mathbb{R}$ or $\mathbb{T}$, and again we fall back into the cases covered by the last section.

The most optimistic conjecture, then, would be that every uncountable Polish group contains an uncountable abelian subgroup. From this and the results above we could obtain arbitrarily complicated Polishable subgroups of any uncountable Polish group.

However the spirit of fearless scientific research demands that we should never take a position of optimism when pessimism is unrefuted. Thus:

Conjecture 4.1 *There is an uncountable Polish group all of whose abelian subgroups are discrete.*

It may even be that some minor variation of the the surjectively universal two-sided invariant group (see [4], [1]) provides a counterexample. Accordingly it seems unlikely that the construction above will answer the question raised in [2].

References

[1] H. Becker and A. S. Kechris, *The descriptive set theory of Polish group actions*, London Mathematical Society Lecture Note Series, 232, Cambridge University Press, Cambridge, 1996.

[2] I. Farah and S. Solecki, Borel subgroups of Polish groups, preprint, `http://www.math.yorku.ca/~ifarah/preprints.html`.

[3] G. Hjorth, A. S. Kechris, and A. Louveau, The Borel equivalence relations induced actions of the infinite symmetric group, *Annals of Pure and Applied Logic*, vol. 92(1998), pp. 63–112.

[4] A. S. Kechris, *Definable equivalence relations and Polish group actions,* manuscript, Caltech, 1993.

[5] D. Montgomery and L. Zippin, *Topological transformation groups,* reprint of the 1955 original, Robert E. Krieger Publishing Co., Huntington, N.Y., 1974. xi+289 pp.

[6] E. I. Zelmanov, On periodic compact groups, *Israel Journal of Mathematics,* vol. 77(1992), pp. 83–95.

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