Classification and Orbit Equivalence Relations

Greg Hjorth

Author address:

Department of Mathematics, University of California at Los Angeles, Los Angeles, California, 90095-1555
E-mail address: greg@math.ucla.edu

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To Noela and Bob Hjorth

Contents

Preface

What does it mean to *classify* the equivalence classes of some equivalence relation?

We have some tangible space whose points are very definite. Let X be the space. Maybe it is the reals. Or the complex numbers. Or perhaps all groups with underlying set $\mathbb{N}$. In any case, a concrete object.

In addition there is an equivalence relation E and we consider the quotient X/E. When should we judge this quotient object consisting of all equivalence classes to be comprehensible? When should we allow that the set of all equivalence classes, $\{[x]_E : x \in X\}$, is classifiable?

There is no absolute agreement on what constitutes a good classification theorem. It is necessarily a vague concept. But even granting its vagueness, there are probably some attempts one could dismiss out of hand.

For instance, assigning the equivalence class of x as a complete invariant for each point in the space is hardly satisfactory, since it provides us with no progress at all. The complete invariants should be objects we feel to be reasonably well understood – or at least, less mysterious than the equivalence classes with which we began.

Similarly we do have some standards concerning how the invariants can be produced. Simply appealing to the axiom of choice to well order the equivalence classes of E, and then using some corresponding well order of $\mathbb{R}$ to assign real numbers as complete invariants does not constitute a satisfactory system of classification, even though the complete invariants in this case are indeed well understood. The system of classification should assign invariants to the points in the space X based on their intrinsic properties; even though the properties we may use could be highly subtle or extremely complex, we would have much greater respect for a system of classification that is fantastically difficult than one that just pulls down the axiom of choice and then goes home to bed.

In general terms a complete classification of E should consist of a *reasonably intrinsic* or *definable* function

$$\theta : X \to I$$

from X to a *reasonably well understood* collection of invariants I so that for all x and y in X

$$xEy \Leftrightarrow \theta(x) = \theta(y).$$

That much is a platitude. Beyond this there is great disagreement.

EXAMPLE 0.1. **Low dimensional topology** For compact orientable surfaces a complete invariant can be obtained by simply counting the number of handles, and moreover, this invariant can be produced in a *recursive* or *computable* fashion from a finite triangularization of the manifold; for non-orientable surfaces the structure theorem is more complicated, but again the invariant can be taken to be a finite object – indeed it can be represented by a natural number – and may be computed from a triangularization using a purely finite search. ([**12**], [**46**].) In higher dimensions it is known from [**64**] that the isomorphism relation is not computable in the sense of Turing machines, and this is *sometimes* taken by topologists as indicating that there is no satisfactory system of complete invariants for higher dimension manifolds.

This is an extremely restrictive definition. Here a classification is a computable function

$$\theta : X \to \mathbb{N}$$

where X is a collection of finite triangularizations of manifolds so that $x, y \in X$ code homeomorphic manifolds if and only if $\theta(x) = \theta(y)$. This is the most stringent notion of classification: The invariants are finite, the process by which we assign them finitary.

EXAMPLE 0.2. **Linear algebra** A rather different example is suggested by linear algebra. We may reasonably regard two unitary operators over a finite dimensional (complex) Hilbert space as somehow equivalent if there is a third that conjugates them – so in this sense we obtain an orbit equivalence relation on the space of all unitary operators on $(\mathbb{C}^n, \cdot)$. A complete invariant for an operator is given by its finite set of eigenvalues, considered up to multiplicity, and it is always possible to encode finitely many complex numbers by a single real; thus we can assign in a Borel fashion to each element of U_n, the group of unitary operators over $(\mathbb{C}^n, \cdot)$, a real number as a complete invariant.

Equivalence relations which in a Borel fashion allow real numbers as complete invariants are known as *smooth* or *tame*; thus E_G on X is smooth if there is a Borel

$$\theta : X \to \mathbb{R}$$

that reduces E_G to equality on $\mathbb{R}$, in the sense

$$x E_G y \Leftrightarrow \theta(x) = \theta(y).$$

[**30**], like [**14**] that it followed, takes *classifiable* to mean smooth.

EXAMPLE 0.3. **Ergodic theory** Consider the classification problem for measure preserving transformations of the unit interval. It is natural to say that $\pi_1, \pi_2 : [0,1] \to [0,1]$ are *equivalent* or *isomorphic* if there is some measure preserving bijection $\sigma : [0,1] \to [0,1]$ with

$$\sigma \circ \pi_1 \circ \sigma^{-1} = \pi_2 \text{ a.e.}$$

This equivalence relation arises from a group action – the action of this group on itself by conjugation. In two special cases there are classification theorems.

Ornstein's classification of Bernoulli shifts in [**71**] provides a proof that isomorphism on this class of measure preserving transformations is smooth. One can assign to each Bernoulli shift its *entropy* – a real number that completely classifies a Bernoulli shift up to isomorphism. This is one of the most celebrated theorems of ergodic theory, but it is not true that the *only* notion of classification in ergodic theory is that of being smooth or reducible to the equality relation on $\mathbb{R}$. A rather more generous notion of classification is suggested by a classic paper of Halmos and von Neumann.

In [**29**] Halmos and von Neumann show that for *discrete spectrum* measure preserving transformations we can assign a countable collection $\{c_i(\pi) : i \in \mathbb{N}\}$

of complex numbers that completely describe the equivalence class of π. This assignment is indeed natural, since it arises from taking the eigenvalues of the form $\lambda \in \mathbb{C}$ for which there is some non-zero $f \in L^2([0,1])$ with $f \circ \pi = \lambda f$ a.e.

The notion of classification here cannot be reduced to that of 0.2. For instance [**20**] shows that there is no reasonable method for representing *countably infinite* sets of real or complex numbers by single points in $\mathbb{R}$; indeed without appeal to the axiom of choice we may find it impossible to produce *any* injection from $\mathcal{P}_{\aleph_0}(\mathbb{R})$ (the set of all countable collections of reals) to $\mathbb{R}$. The conjugacy relation on discrete spectrum measure preserving transformations is non-smooth, and yet the perspective of say [**82**] would uphold this as a complete classification for discrete spectrum measure preserving transformations.

EXAMPLE 0.4. **Locally compact group actions** In [**49**] Kechris proves that the orbit equivalence relations induced by locally compact Polish groups are all reducible to countable equivalence relations. If we have locally compact G acting continuously on Polish X, with orbit equivalence relation E_G, then we can find a Borel equivalence relation F all of whose equivalence classes are countable such that we may assign to each point $x \in X$ some corresponding $\theta(x)$ so that

$$x_1 E_G x_2 \Leftrightarrow \theta(x_1) F \theta(x_2).$$

This result can be viewed as a classification theorem for orbit equivalence relations induced by locally compact group actions. We may assign to each $x \in X$ the countable set $\{y : yF\theta(x)\}$ to obtain an invariant similar in structure to the Halmos-von Neumann spectral invariants.

EXAMPLE 0.5. **Point set topology** The Cantor-Bendixson derivation as described in [**52**] can be used to provide a classification for countable compact metric spaces. At the first stage we remove the isolated points. At the next we remove the isolated points from the remaining space, and so on, through however many countable ordinals as are needed. This analysis provides a complete invariant of the space consisting of two parts: The ordinal length of this process, along with the number of points left standing before the termination at the final stage. Two countable compact metric spaces will be homeomorphic if and only if their Cantor-Bendixson derivations require the same ordinal number of steps and at the penultimate moment they share the same finite number of points remaining.

EXAMPLE 0.6. **Abelian group theory** In [**21**] ordinals also enter stage in the famed *Ulm invariants* from abelian group theory. In essence the Ulm invariants are bounded subsets of $\aleph_1$, the first uncountable ordinal, that completely describe the isomorphism type of a countable *torsion* abelian group.

EXAMPLE 0.7. **Topological dynamics** The authors of [**24**] classify so called *minimal Cantor systems* up to *strong orbit equivalence* by assigning countable ordered abelian groups. Two continuous

$$\varphi_1 : X_1 \to X_1,$$

$$\varphi_2 : X_2 \to X_2$$

which are *minimal* in the sense of having no non-trivial closed invariant sets and are *Cantor* in the sense of X_1, X_2 being compact, uncountable, and zero-dimensional metric spaces, are said to be *strong orbit equivalent* if there is a homeomorphism $F : X_1 \to X_2$ which respects the orbits structure set wise and with the resulting conjugation suffering at most a single discontinuity – so that if $m : X_1 \to \mathbb{Z}$, $n : X_2 \to \mathbb{Z}$ are defined by

$$F \circ \varphi_1(x) = \varphi_2^{n(x)}(F(x)),$$
$$F \circ \varphi_1^{m(x)}(x) = \varphi_2(F(x)),$$

then n, m are continuous on $X \setminus \{x_0\}$ for some $x_0 \in X$.

The countable group associated to (φ, X) itself arises as a homomorphic image of $C(X, \mathbb{Z})$, the space of all continuous maps from X to $(\mathbb{Z}, +)$ with the product topology. As important as the details of the construction may be it is also remarkable that there is *any* reasonable way to assign a countable structure as a complete invariant.

EXAMPLE 0.8. **Stone spaces** Perhaps this preceding example is reminiscent of the duality theorem of [**75**] for compact separable zero-dimensional Hausdorff spaces. To each such space we can assign a countable Boolean algebra with two spaces homeomorphic if and only if there exists an algebraic isomorphism between the Boolean algebras. Similarly Pontryagin duality, as it is found in [**31**], allows us to completely classify compact abelian metric groups by their countable discrete dual groups.

While it is true that complete invariants are provided this is not to say that these dualities are only or even primarily theorems of classification.

EXAMPLE 0.9. $\mathbf{Hom}^+([0,1])$ Let $\mathrm{Hom}^+([0,1])$ be the group of all orientation preserving ($\pi(0) = 0$, $\pi(1) = 1$) homeomorphisms of the unit interval. The natural equivalence relation is that of conjugation: Two homeomorphisms, π_1, π_2 are equivalent if a homeomorphic "relabeling" of the underlying space transforms one to the other, so that there is some $\sigma \in \mathrm{Hom}^+([0,1])$ with

$$\pi_1 = \sigma^{-1} \circ \pi_2 \circ \sigma.$$

Parallel to 0.3, the equivalence relation arises by the self-action of $\mathrm{Hom}^+([0,1])$ through conjugation.

It is sometimes felt that homeomorphisms of the unit interval are completely understood since we may represent each transformation *symbolically* by indicating the maximal regions on which we have either $\pi(x) > x$, $\pi(x) = x$, or $\pi(x) < x$. This can be made more precise by providing a classification of elements of $\mathrm{Hom}^+([0,1])$ by *countable models*. We naturally assign to each $\pi \in \mathrm{Hom}^+([0,1])$ a countable model $\mathcal{M}(\pi)$ such that for all $\pi_1, \pi_2 \in \mathrm{Hom}^+([0,1])$

$$\exists \sigma \in \mathrm{Hom}^+([0,1])(\sigma \circ \pi_1 \circ \sigma^{-1} = \pi_2) \Leftrightarrow \mathcal{M}(\pi_1) \cong \mathcal{M}(\pi_2).$$

The model $\mathcal{M}(\pi)$ consists of the maximal open intervals on which π displays one of the three possible behaviors indicated above. The language of $\mathcal{M}(\pi)$ encodes the linear ordering between these intervals and indicates which of the three possibilities hold. We will have an ordering $\leq$, and predicates P_-, P_+, and $P_=$. For $I_1 = (a_1, b_1)$, $I_2 = (a_2, b_2)$ maximal open intervals on which the behavior of π is unvarying, we have:

$$I_1 \leq^{\mathcal{M}(\pi)} I_2 \Leftrightarrow a_1 \leq a_2;$$
$$P_-^{\mathcal{M}(\pi)}(I_1) \Leftrightarrow \forall x \in I_1(\pi(x) < x);$$
$$P_+^{\mathcal{M}(\pi)}(I_1) \Leftrightarrow \forall x \in I_1(\pi(x) > x);$$
$$P_=^{\mathcal{M}(\pi)}(I_1) \Leftrightarrow \forall x \in I_1(\pi(x) = x).$$

The outcome is similar for the homeomorphism group of Cantor space – $(\{0,1\})^{\mathbb{N}}$ – in the product topology. Since the homeomorphism group of the Cantor space is isomorphic to a closed subgroup of the infinite symmetric group it follows from [**4**] (see 2.39 below) that we may classify these homeomorphisms by countable models.

This is more than enough examples to be impressed by the diversity. But despite the variation, there are some common themes.

In the above we have natural numbers, real numbers, countable sets of complex numbers, countable ordinals, countable sets of countable ordinals, and various kinds of countable structures considered up to isomorphism being used as complete invariants. The connection between these examples is that in *every one* we may take a countable structure as a complete invariant; as in §2.3 below, we may code complex numbers, countable sets of complex numbers, countable ordinals, and countable sets of countable ordinals by appropriately chosen models. This suggests a notion of classification found at the opposing end of the spectrum to that of 0.1 and which is extreme in its generosity.

QUESTION 0.10. Let E be an equivalence relation on a space X. When can we assign countable *models* or *structures* considered up to isomorphism as complete invariants?

Recall that HC is the collection of all hereditarily countable sets, and may be defined as the smallest collection of sets containing the natural numbers and closed under the operation of taking a countable subset. These therefore include all countable subsets of $\aleph_1$, all countable sets of subsets of $\mathbb{N}$, and – appropriately understood – all real numbers, all countable sets of real numbers, and so on.

In virtue of the Scott analysis of [**59**] we may equivalently ask:

QUESTION 0.11. For which equivalence relations can we assign elements of HC as complete invariants?

[**4**] allows one more reworking of the question:

QUESTION 0.12. Let E be an equivalence relation on a space X. When can we find a Polish space Y on which the infinite symmetric group S_∞ acts continuously and a reasonable function $\theta : X \to Y$ so that for all $x_1, x_2 \in X$

$$x_1 E x_2 \Leftrightarrow \exists g \in S_\infty(g \cdot \theta(x_1) = \theta(x_2))?$$

Of course we clearly need to have an assumption that the function θ or the assignments of models or HC sets be *reasonable*. On the whole I will take *reasonable* to mean Borel in an appropriate Borel structure, the technicalities of which are addressed in §2.1, §2.2, §2.3, and §3.1.[1] But I should stress that relatively little change occurs if we extend to much broader classes of functions and far more generous methods of reduction. A point made in the course of §6.2 and §9.1-2 is that if there is any remotely definable assignment of countable models or HC sets in the context of Polish group actions, then we may find a reduction that is at worst only slightly more complicated than Borel.

This monograph can be viewed as part of a broad project to understand *effective* cardinalities in the sense raised by Luzin in [**62**], in the sense which reappears

[1] In the context of Borel reduction, 0.10 and 0.12 are known to be equivalent problems from 2.7.3 of [**4**]. Provided we are willing to countenance reductions somewhat more complicated than Borel the Scott analysis of [**59**] shows both equivalent to 0.11. §6.1 and §9.1 return to these points.

briefly in the opening parts of [**11**], but which finds its most forthright statement in modern works of descriptive set theory such as [**60**] and [**50**]. Here the concept is to calculate cardinalities using only functions that lay some claim on being *reasonable* or *definable*. Formally $\mathbb{R}$ and $\mathbb{R}/\mathbb{Q}$ both have cardinality $2^{\aleph_0}$; a well ordering of $\mathbb{R}$ will enable us to find a bijection. *Effectively* $\mathbb{R}$ is smaller than $\mathbb{R}/\mathbb{Q}$, since we may find reasonable injections of $\mathbb{R}$ into $\mathbb{R}/\mathbb{Q}$ (2.59, 2.63 below), but not the converse (3.8).

Papers such as [**14**], [**63**], and [**25**] have previously addressed the question of which naturally occuring objects have effective cardinality no greater than that of $\mathbb{R}$. In a great many specific instances the answer has been determined, and one finds in [**30**] and [**14**] a kind of theory, recounted in §3.1 and §7.1, regarding when a reduction exists and why in certain cases it cannot. In turn this monograph tries to understand which objects have effective cardinality below HC and develop a parallel theory of why some do not.

We will only be concerned with the case that E arises from a Polish group action. Admittedly this may seem very restrictive, and it would certainly be desirable to have an analysis for all Borel or even $\underset{\sim}{\Sigma}^1_1$ equivalence relations. On the other hand most naturally occurring examples can be subsumed under an appropriately chosen Polish group action, and the impression left by chapter 8 is that the exceptions are somehow pathological. This is the point of question 10.9 near the end of the book, and even if that conjecture should fail it seems plausible that a similar outlook is justified.

§3 isolates a dynamical property for analyzing which Polish group actions allow reduction to countable models:

DEFINITION 0.13. Let G be a topological group acting on a space X. The action is said to be *turbulent* if:

(i) every orbit is dense;

(ii) every orbit is meager;

(iii) for all $x, y \in X$, $U \subset X$, $V \subset G$ open with $x \in U$, $1 \in V$, there exists $y_0 \in [y]_G =_{df} G \cdot y$ and $(g_i)_{i\in\mathbb{N}} \subset V$, $(x_i)_{i\in\mathbb{N}} \subset U$ with

$$x_0 = x,$$

$$x_{i+1} = g_i \cdot x_i,$$

and for some subsequence $(x_{n(i)})_{i\in\mathbb{N}} \subset (x_i)_{i\in\mathbb{N}}$

$$x_{n(i)} \to y_0.$$

Turbulence is a sufficient condition for the orbit equivalence relation of a Polish group to refuse classification by countable structures; further: for a turbulent orbit equivalence relation any function assigning countable models up to isomorphism as invariants must be constant on a comeager set. (3.18, 3.19)

A transfinite analysis shows turbulence to be necessary for non-classification:

THEOREM 0.14. *Let G be a Polish group acting continuously on a Polish space X. Exactly one of the following holds:*

1. *the orbit equivalence relation E^X_G is reasonably reducible to isomorphism on countable models;*
2. *there is a turbulent Polish G-space Y and a continuous G-embedding from Y to X.*

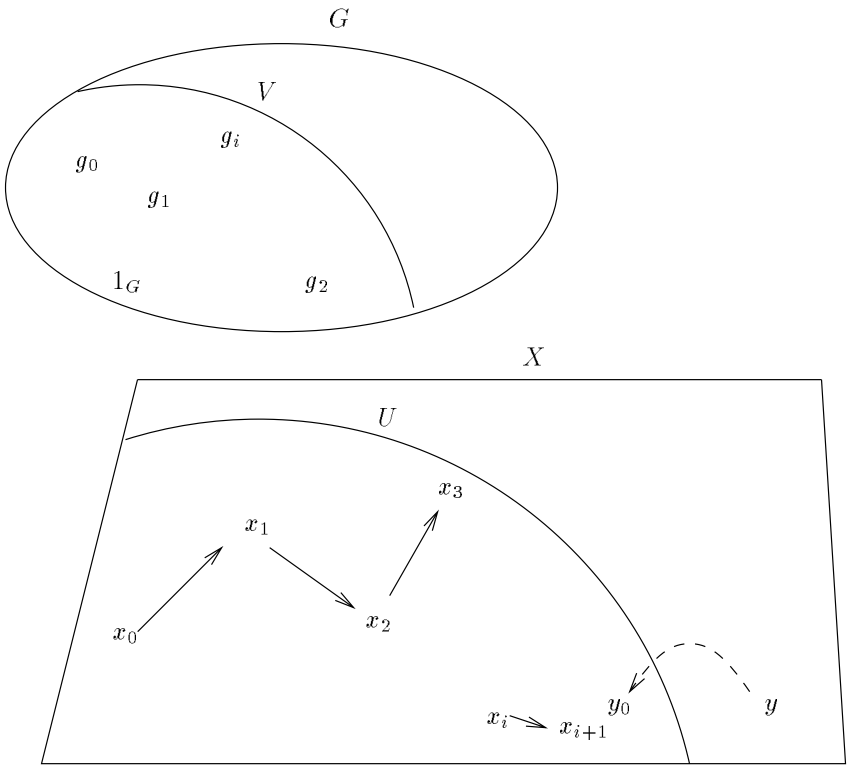

FIGURE 0.1. Turbulence

Here the notion of reduction is somewhat more complicated than Borel. In special cases, such as for G abelian or invariantly metrizable, one can obtain a reduction in 1. that is not only Borel but admits a Borel inverse up to orbit equivalence (6.40 and 6.30). The proof that 2. implies the negation of 1. is given in §3.2; the converse requires a long argument, finally concluding in chapter 9, and uses an elaboration on the Scott analysis presented in §6.2 that may have independent interest.

The form of 0.14 is intended to make it into a tool that can be easily applied in concrete cases.

CHAPTER 1

An outline

This chapter discusses a few specific classification problems, but leaves until later the precise definitions.

1.1. Some specific classification problems

In light of 0.2, we may very well ask regarding the difficulty of classifying unitary operators on *infinite dimensional* Hilbert space. Let

$$l^2 = \{(x_i)_{i\in\mathbb{N}} | x_i \in \mathbb{C}, \sum |x_i|^2 < \infty\}$$

with the usual inner product and let U_∞ be the set

$$\{T : l^2 \to l^2 | T \text{ linear}, \forall \vec{x}, \vec{y} \in l^2(\langle \vec{x}, \vec{y}\rangle = \langle T(\vec{x}), T(\vec{y})\rangle)\}.$$

We can set

$$T_1 \approx T_2 \Leftrightarrow \exists S \in U_\infty (ST_1S^{-1} = T_2)$$

and obtain an infinite dimensional analog of the classification problem for unitary matrices. It was known from [**10**] that $\approx$ is non-smooth, in the sense of not allowing points in $\mathbb{R}$ or any other Polish space as complete invariants.

Building on earlier results of Kechris' the classifiability of $\approx$ by countable structures has been refuted in the following dramatic form:

Theorem 1.1. *(Kechris, Sofronidis) The self action of U_∞ by conjugation is turbulent (on an invariant dense G_δ).*

While a satisfactory system of complete invariants have been established for the class of discrete spectrum measure preserving transformations of the unit interval and Bernoulli shifts, there is no known classification theorem for arbitrary measure preserving transformations. Certainly we might ask for classification by countable models or even by countable sets of reals.

It can be shown that we may reduce a turbulent orbit equivalence and conclude:

Theorem 1.2. *(See [**34**]) The isomorphism relation on measure preserving transformations does not allow classification by countable models.*

The classification theorem for locally compact groups from 0.4 hints at the existence of classifiability for broader classes of Polish groups actions, in particular those with a kinship to locally compact. Even though an infinite product group such as $\mathbb{R}^\mathbb{N}$ would seem to bear little topological or algebraic kinship to S_∞:

Theorem 1.3. *(§7.4) Let $\prod_\mathbb{N} H_n$ be the product of countably many locally compact Polish groups and suppose it acts continuously on a Polish space. Then the resulting orbit equivalence relation allows classification by countable models.*

Given the existence of a classification for orientation preserving homeomorphisms of the unit interval (0.9 and §4.2), we might hope for a similar classification of the homeomorphisms of the unit square. Here there is a distinction between dimensions 1 and 2.

THEOREM 1.4. *(§4.3) Let $H = Hom([0,1]^2)$ act upon itself by conjugation. Then the resulting equivalence relation does not admit classification by countable models.*

After showing that the equivalence relation of conformal equivalence for Riemann surfaces is not reducible to the equality relation on any reasonably concrete space, the authors of [**5**] ask whether a classification may be produced using *countable unordered sets* of reals as complete invariants. [**40**] uses the uniformization theorem to reduce this classification problem to the orbit equivalence relation of a locally compact group, and then appeals to Kechris' theorem as mentioned at 0.4 to obtain an affirmative answer.

No such result can be obtained for general complex manifolds:

THEOREM 1.5. *(Hjorth, Kechris; see [**40**]) Biholomorphism of 2-dimensional complex manifolds reduces a turbulent orbit equivalence relation; thus it does not admit classification by countable models, and in particular there is no reasonable method of assigning countable sets of reals as complete invariants.*

Finally some words about the study of infinite dimensional group representations, since this is the area in which the smooth/non-smooth division received perhaps its first clear formulation. Given a countable group G we can consider the space of all representations on Hilbert space – that is to say, all homomorphisms of G into the infinite dimensional unitary group U_∞. The natural notion of classification is up to pointwise conjugation. Given $\rho_1, \rho_2 : G \longrightarrow U_\infty$ let us write $\rho_1 \sim_r \rho_2$ if there is some $T \in U_\infty$ such that at each $g \in G$

$$T \circ \rho_1(g) \circ T^{-1} = \rho_2(g).$$

Already the Kechris-Sofronidis result shows that the classification problem for $G = \mathbb{Z}$ does not allow reduction to countable models, but one might argue that here we should restrict attention to the irreducible transformations. Building on [**79**]:

THEOREM 1.6. *([**35**]) The irreducible representations considered up to $\sim_r$ admit classification by countable models if and only if G is abelian-by-finite.*

In §5 we prove this just for the specific case of G being the free group on infinitely many generators.

1.2. The form of this book (and one time paper)

This began as some seminars given at Caltech. The paper grew longer and longer, and at some point there entered the optimistic idea that by adding just a couple more pages it could be made totally self contained.

The desire to avoid any black boxes and reach a completely general audience has determined the choice of material. I have not tried to cover everything, but concentrated only on developing what seemed like the main ideas. In the exercises of §6.1 a set theorist may recognize that some calculations can be given swifter proofs using the relatively sophisticated techniques of [**52**] and [**67**]. §3.2 could be completed more quickly using "forcing", but I have instead taken every step by hand.

Chapter 2 gives some background in classical descriptive set theory and equivalence relations. 3 provides the sufficient condition for non-classification:

THEOREM 1.7. *(See 3.19) A turbulent orbit equivalence relation does not allow classification by countable models.*

and 4 and 5 applies this to homeomorphism groups and the unitary group of Hilbert space. Returning to the theory at 6 it is shown that turbulence is essential to non-classification for a broad class of groups:

THEOREM 1.8. *Let G be a GE group (for instance abelian) and X a Polish G-space. Then the following are equivalent:*

1. *the orbit equivalence relation admits classification by countable models;*
2. *there is no turbulent Polish G-space Y and continuous G-embedding from Y to X.*

In 7 a complete and self contained account is given of the GE groups and the Glimm-Effros dichotomies, and more generally the smooth\non-smooth dichotomy. There is a new result to the effect that countable products of locally compact groups admit a Glimm-Effros dichotomy. This in turn is used in the proof 1.3 at 7.43.

8 passingly covers Borel groups.

9 presents the general classifiability versus turbulence analysis, and is unlike the other chapters in the sense of being written purely with the expert in mind. I regret that it seemed impossible to give a self contained exposition of these proofs.

10 lists some open problems.

There is an appendix discussing ordinals, another clarifying notation, especially conventions that are idiosyncratic among set theorists, and there is an index.

Although the direction of this book is towards explicating a notion of classification there are also comments along the way regarding the broader area of Polish group actions. General results are not presented axiomatically in layered fashion, but are mostly developed at the point of use. Even as late as §7.1 basic facts appear for the first time.

The logical order of the chapters is that 2 precedes 3, and from there one can read any of chapters 4 through 8 just assuming those two. Chapter 9 however requires the reader to be familiar with most of the book.

For many purposes, shorter routes may be possible. Simply reading chapter §2.1, §2.2, and then up through the first half of §2.4, and §3.1, and §7.1-3 and §7.5 covers most of what is known about the Glimm-Effros dichotomy for Polish group actions. In a 10 week course at UCLA we went up through to 2.51, then §3.1, then lemma 3.14, which constitutes the essence of why turbulent equivalence relations are complicated, a few examples, and then finally we discussed §6.1.

There are many exercises, some of which are difficult and some of which are even theorems I wanted to mention briefly without taking a great deal of time to prove.

1.3. Acknowledgments

The work here was partially supported by NSF grant DMS 96-22977 and a generous grant from the Alfred P. Sloan Foundation.

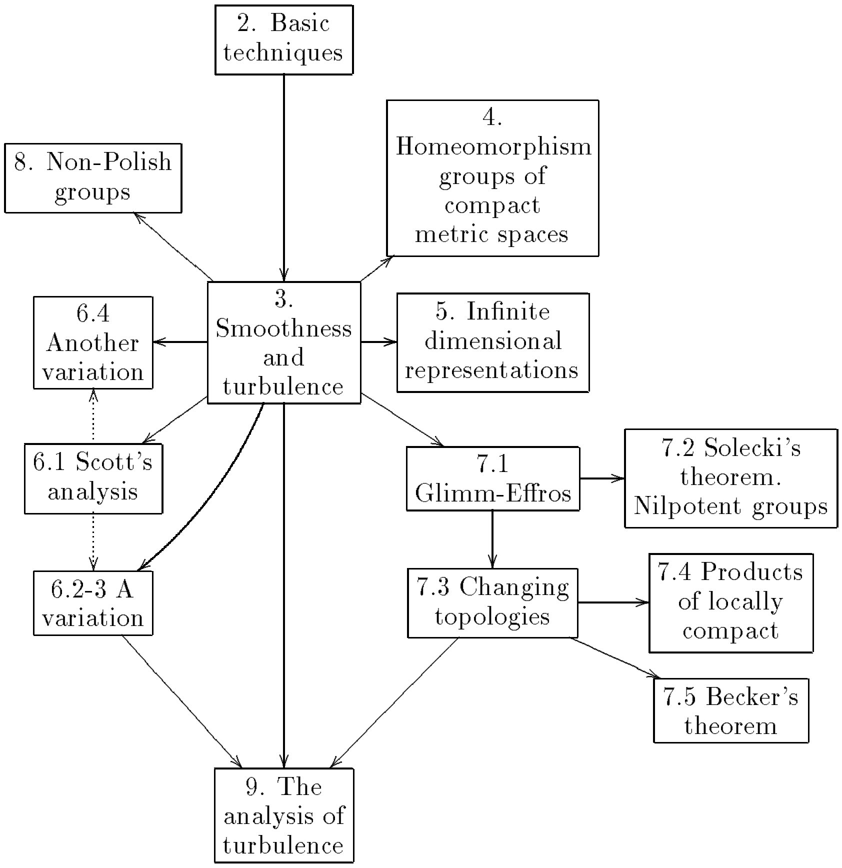

FIGURE 1.1. The order of dependence

I am indebted to Harvey Friedman for showing me the unpublished proof alluded to in [**20**].

I must twice thank Alexander Kechris.

Once for his general works on Polish groups, and many conversations, freely given, that served to introduce me to this area.

And once more for some very specific discussions on the notion of turbulence. During the Caltech seminars the definitions were the focus of his unrelenting criticism; this in turn molded the order of presentation at §3.2, and the decisions of where to make a theorem from a definition and where to do the reverse.

I am further grateful to Alexander Andretta, Lyman Chaffee, David Fremlin, Su Gao, and Nathan Schroeder for reading through various parts of earlier drafts and making invaluable comments. Undoubtedly obscurities and oversights remain, but they would number far greater without the suggestions I received. The research underpinning chapter 4 was especially influenced by a number of conversations with Su.

CHAPTER 2

Definitions and technicalities

2.1. Polish groups and Polish spaces

DEFINITION 2.1. A topological space is said to be *Polish* if it is separable and admits a complete metric. When it becomes necessary we can write (X, τ) for a Polish space with underlying set X and topology τ. Mostly there is no danger of confusion and no harm in identifying a space with its underlying set. We will say that a metric d on a Polish space (X, τ) is *compatible* if it generates the topology; there is in particular no assumption of completeness.

For X a Polish space we form the *Borel sets* by taking the smallest collection containing the open sets and closed under complements and countable unions and intersections. In particular a countable intersection of open sets in X is G_δ, a countable union of closed is F_σ, countable intersections of F_σ are $F_{\sigma\delta}$, countable unions of G_δ are $G_{\delta\sigma}$, and so on. (This moves through the hierarchy of countable ordinals, with $\undertilde{\Sigma}^0_1$ = open, $\undertilde{\Pi}^0_1$ = closed, $\undertilde{\Sigma}^0_\alpha$ = countable unions of sets in $\undertilde{\Pi}^0_\beta$ for various $\beta < \alpha$, and $\undertilde{\Pi}^0_\alpha$ indicating the complements of $\undertilde{\Sigma}^0_\alpha$.) A function $\theta : X \to Y$ between Polish spaces is said to be *Borel* if the preimage of any open set is Borel.

For the purposes of verifying that certain maps and sets are Borel we will frequently be more interested in the fact that a space *can* be made Polish, and less interested in the details of how this is actually performed. If X is a set and $\mathcal{B}$ is a σ-algebra of subsets of X then one says that $(X, \mathcal{B})$ is a *standard Borel space* if there is some Polish topology τ on X so that $\mathcal{B}$ is the collection of Borel sets for (X, τ).

EXAMPLES 2.2. Examples of Polish spaces include: $\mathbb{R}$ with the usual topology; $\mathbb{N} =_{df} \{0, 1, 2, ...\}$ with the discrete topology; the unit interval; the space of all real valued continuous functions from the unit interval with the sup norm –

$$d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$

(compare §4.1 below). Note that the class of Polish spaces is closed under countable products, since for $X_0, X_1, ...X_i, ...$ Polish, as witnessed by metrics $d_0, d_1, ...d_i, ...$, we can find equivalent bounded metrics

$$d'_i(\cdot, \cdot) = d_i(\cdot, \cdot)/(d_i(\cdot, \cdot) + 1)$$

and take

$$d_\Pi(\vec{x}, \vec{y}) = \sum 2^{-i} d'_i(x_i, y_i)$$

for $\vec{x}, \vec{y}$ in $\prod X_i$. In particular any countable product of countable discrete spaces is Polish. For instance let $\mathbb{N}^\mathbb{N}$ denote

$$\{f \mid f : \mathbb{N} \to \mathbb{N}\}$$

with the product topology. (Similarly for $n \in \mathbb{N}$ let $\mathbb{N}^n$ be the set of functions from $\{0, 1, ..., n-1\}$ to $\mathbb{N}$, and let $\mathbb{N}^{<\mathbb{N}} = \bigcup \mathbb{N}^n$; one ruthlessly considers $\emptyset$, the empty function, as the unique element of $\mathbb{N}^0$.) Then we obtain that $\mathbb{N}^{\mathbb{N}}$, sometimes known as *Baire space*, is Polish.

A compact Polish space is presented by *Cantor space*:

$$2^{\mathbb{N}} =_{df} \{f | \ f : \{0, 1, 2, ...\} \rightarrow \{0, 1\}\},$$

in the product topology, where we understand that $2 =_{df} \{0, 1\}$ is given the discrete topology and $2^{\mathbb{N}}$ is the collection of functions from $\mathbb{N}$ to $\{0, 1\}$. (And for $n \in \mathbb{N}$ we will use 2^n to denote the set of functions from $\{0, 1, ...n-1\}$ to $\{0, 1\}$, and $2^{<\mathbb{N}}$ for the union over $n \in \mathbb{N}$ of 2^n). It is natural in turn to identify $2^{\mathbb{N}}$ with the collection of all subsets of $\mathbb{N}$ – viewing each element $f \in 2^{\mathbb{N}}$ as being married to the subset X_f of $\mathbb{N}$ that has

$$n \in X_f \Leftrightarrow f(n) = 1.$$

A more exotic example of a Polish space is given by $M(\lambda)$, the measurable subsets of $[0, 1]$ considered up to null sets, so sets agreeing almost everywhere are identified. The metric here is

$$d(A, B) = \lambda(A \Delta B),$$

where $A\Delta B$ is the set of points lying in one of the sets but not the other and where λ is Lebesgue measure. This is a complete metric, since if $(A_i)_{i\in\mathbb{N}}$ is Cauchy then by accelerating the sequence we can assume

$$\lambda(A_i \Delta A_{i+1}) < 2^{-i},$$

from which it follows that taking

$$A_\infty = \{x \in [0, 1] : \forall i \in \mathbb{N}\, \exists j > i(x \in A_j)\}$$

will be a limit.

For our purposes there is an especially important example of a Polish space, namely the space of all $\mathcal{L}$-structures on $\mathbb{N}$, for a countable language $\mathcal{L}$. This example is the subject of §2.3.

LEMMA 2.3. *(classical) Let (X, τ) be a Polish space.*

(i) If $B \subset X$ is G_δ with respect to τ, then B is a Polish space in the relative topology.

(ii) If $C \subset X$ is τ-closed and $\mathcal{O} \subset X$ is τ-open, then there is a Polish topology $\tau_{C,\mathcal{O}}$ on X with basis

$$\{C \cap \mathcal{O} \cap U : U \in \tau\} \cup \{U : U \in \tau\}.$$

(iii) If τ is a Polish topology on X and for each $i \in \mathbb{N}$ the topology $\tau_i \supset \tau$ is a richer Polish topology, then the topology generated by taking finite intersections of sets from $\bigcup_{i\in\mathbb{N}} \tau_i$ is Polish.

(iv) If $B \subset X$ is Borel in τ then there is a richer Polish topology $\hat{\tau} \supset \tau$ all of whose open sets are Borel with respect to τ and in which B is clopen, that is to say both $\hat{\tau}$-open and $\hat{\tau}$-closed.

PROOF. (i): First consider the case of a single open set $\mathcal{O} \subset X$. Choose closed $C_0 \subset ...C_n \subset C_{n+1} \subset ...$ with each C_n included in the interior of C_{n+1} and $\mathcal{O} = \bigcup_{n\in\mathbb{N}} C_n$. Let d be a complete metric on X bounded by 1. For a closed set C and $x \in X$ let $d(x, C) = \inf\{d(x, y) : y \in C\}$. Let $X \setminus \mathcal{O}$ indicate the points in X *not* appearing in $\mathcal{O}$.

For each n, define $f_n(x)$ to be

$$\frac{d(x, C_n)/d(x, X \setminus \mathcal{O})}{[d(x, C_n)/d(x, X \setminus \mathcal{O})] + 1},$$

where in the case $d(x, X \setminus \mathcal{O}) = 0$ (and thus $x \in X \setminus \mathcal{O}$) we let $f_n(x) = 1$.

Thus we have a continuous function

$$f_n : X \to [0, 1]$$

with f_n constantly 0 on C_n and f_n constantly 1 on $X \setminus \mathcal{O}$.

Then

$$d_{\mathcal{O}}(x_0, x_1) = d(x_0, x_1) + \sum_{n \in \mathbb{N}} |f_n(x_0) - f_n(x_1)|$$

is a complete metric on $\mathcal{O}$ compatible with its topology as a subspace.

For the general case of a countable intersection of open sets and $A = \bigcap_i \mathcal{O}_i$ we may take

$$d_A(\cdot, \cdot) = \sum_i 2^{-i} d_{\mathcal{O}_i}(\cdot, \cdot)/(1 + d_{\mathcal{O}_i}(\cdot, \cdot)).$$

(ii): Write

$$\mathcal{O} = \bigcup_{n \in \mathbb{N}} C_n,$$

with each C_n closed and included in the interior of C_{n+1}. We let $f_n : X \to [0, 1]$ be a continuous function with

$$f_n^{-1}[\{1\}] \supset C_n,$$
$$f_n^{-1}[\{0\}] \supset X \setminus \mathcal{O}.$$

Then define $f : X \to [0, 1]$ by

$$f(x) = \sum_{n \in \mathbb{N}} 2^{-n-1} f_n(x).$$

f is continuous and constantly 0 off of $\mathcal{O}$.

Let $X_{C,\mathcal{O}}$ be the set of $(x, r) \in X \times [0, 1]$ such that either

$$x \notin C,\ r = 0,$$

or

$$x \in C,\ r = f(x).$$

Here we have twisted X and immersed it into $X \times [0, 1]$. $(X \setminus C) \cup (X \setminus \mathcal{O})$ is taped down to the bottom of the unit interval and $C \cap \mathcal{O}$ pulled up though $(0, 1]$, but with the end points, obtained as $x \to X \setminus \mathcal{O}$, trailing back to $X \times \{0\}$.

Claim: $X_{C,\mathcal{O}}$ is a G_δ subset of $X \times [0, 1]$.

Proof of Claim: We have here that

$$X_{C,\mathcal{O}} = (A_{00} \cap A_{01}) \cup (A_{10} \cap A_{11}),$$

with each A_{ij} either closed or open, and therefore G_δ. Since the intersection of two G_δ sets are immediately G_δ, it suffices to see that the union of two G_δ sets is G_δ.

But given intersections of open sets

$$\bigcap_{n \in \mathbb{N}} O_n$$

and

$$\bigcap_{n \in \mathbb{N}} U_n$$

we can represent their unions as

$$\bigcap_{n\in\mathbb{N}}((\bigcap_{k<n} O_n)\cup(\bigcap_{k<n} U_n)).$$

(Claim□)

Thus by part (i), $X_{C,\mathcal{O}}$ is Polish. We moreover have a canonical bijection

$$\pi : X \twoheadrightarrow X_{C,\mathcal{O}},$$

$$x \mapsto (x, f(x))$$

if $x \in C$,

$$x \mapsto (x, 0)$$

if $x \notin C$. We can therefore let $\tau_{C,\mathcal{O}}$ be the Polish topology on X obtained by sets of the form

$$\pi^{-1}[O],$$

where $O \subset X_{C,\mathcal{O}}$ is open.

This clearly includes the original topology, and without difficulty can be seen to have the basis indicated by the statement of (ii).

(iii): For τ_∞ the topology thus generated in the statement of (iii) the space (X, τ_∞) is homeomorphic to

$$\{\vec{x} \in \prod_{i\in\mathbb{N}}(X, \tau_i) : \forall n, m \in \mathbb{N}(x_n = x_m)\},$$

which in turn is a closed subspace of a Polish space.

(iv): The class of $B \subset X$ satisfying the conclusion of (iv) includes the open sets by (ii), is closed under complementation by the structure of the definition, and is closed under countable unions by (iii) and (ii) again. So it forms a σ-algebra including the open sets, and thus the Borel. □

In particular from (iv) we have that the class of standard Borel spaces is closed under taking Borel subsets.

Example 2.4. For X a Polish space and $\mathcal{F}(X)$ the space of closed subsets of X, we may find it natural to give $\mathcal{F}(X)$ the Borel structure generated by sets of the form $\{F \in \mathcal{F}(X); F \cap O \neq \emptyset\}$ for $O \subset X$ open. This is known as the *Effros* Borel structure. $\mathcal{F}(X)$ equipped with the Effros Borel structure is a standard Borel space. One way to see this is to note that if we fix some countable basis $\mathcal{B}$, and if we choose some complete metric d, and if we endow

$$2^{\mathcal{B}} =_{df} \{f|\ f : \mathcal{B} \to \{0, 1\}\}$$

with the product topology – under which it is not just Polish but homeomorphic to Cantor space – then we may identify $\mathcal{F}(X)$ with the G_δ subset X_0 of $2^{\mathcal{B}}$ consisting of all $x : \mathcal{B} \to \{0, 1\}$ such that

(i) $U \subset V$ and $x(U) = 1$ implies $x(V) = 1$;

(ii) for all $n \in \mathbb{N}, U \in \mathcal{B}$, if $x(U) = 1$ then there is some $V \in \mathcal{B}$ with

$$x(V) = 1,$$

$$d(V)(=_{df}\ \sup\{d(y, z) : y, z \in V\}) < 2^{-n},$$

$$V \neq \emptyset,$$

and the closure of V is included in U.

So for $C \in \mathcal{F}(X)$ we define $x_C : \mathcal{B} \to \{0,1\}$ by the requirement $x_C(V) = 1$ if and only if $C \cap V \neq \emptyset$.

Claim: $C \mapsto x_C$ defines a bijection between $\mathcal{F}(X)$ and X_0.

Proof of claim: The map is clearly one to one, so there is only the task of showing it onto. Take $x \in X_0$ and define $C \subset X$ by

$$C = X \setminus (\bigcup\{V : x(V) = 0\}).$$

We need to show that $x = x_C$.

Clearly $x(V) = 0$ implies $V \cap C = \emptyset$, and hence $x_C(V) = 0$. On the other hand, if $x(V) = 1$ we may use clause (ii) in the definition of X_0 to find $V_0 = V \supset V_1 \supset V_2 ... \supset V_i ...$ such that at each i we have $x(V_i) = 1$, $d(V_i) < 2^{-i}$, and the closure of V_{i+1} included in V_i. Letting z be the unique point in

$$\bigcap_{n \in \mathbb{N}} V_n$$

it suffices to see that $z \in C$. But otherwise we would obtain some open $U \in \mathcal{B}$ with $x(U) = 0$ and $z \in U$. From the construction we obtain $V_n \subset U$ for some n, and a contradiction to clause (i) in the definition of X_0. (Claim$\square$)

The claim implies that X_0 equipped with its Borel σ-algebra is isomorphic to $\mathcal{F}(X)$ equipped with the Effros σ-algebra.

DEFINITION 2.5. A *Polish group* is a topological group that is Polish as a space. In the case that G is a Polish group acting continuously on a Polish space X we say that X is a *Polish G-space*; if X is a standard Borel space and G acts in a Borel fashion, then X is said to be a *Borel G-space*. In either case we write E_G^X – or E_G when context indicates X – for the orbit equivalence relation given by

$$x_1 E_G^X x_2 \Leftrightarrow \exists g \in G(g \cdot x_1 = x_2).$$

Here $g \cdot x$ denotes the result of $g \in G$ acting on $x \in X$. The *orbit* of x is the set $\{g \cdot x : g \in G\}$ and denoted by $[x]_G$; thus $x_1 E_G^X x_2$ if and only if $[x_1]_G = [x_2]_G$. We let

$$G_x =_{df} = \{g \in G : g \cdot x = x\}$$

be the *stabilizer* of x, consisting of group elements that leave x undisturbed.

For $A \subset X$

$$g \cdot A = \{g \cdot x : x \in A\};$$

A is said to be *invariant* if for all $g \in G$

$$g \cdot A = A.$$

Given $F \subset G$ and $A \subset X$ we let $F \cdot A$ equal the set $\{g \cdot x : g \in F, x \in A\}$. $G \cdot A =_{df} \{g \cdot x : g \in G, x \in A\}$ is said to be the *saturation* of A.

EXAMPLES 2.6. Any countable discrete group is Polish. $(\mathbb{R}, +)$ is a Polish group. $\mathbb{R}^{\mathbb{N}}$ in the product group structure. Any separable Banach space; for instance if we let

$$c_0 = \{\vec{x} \in \mathbb{R}^{\mathbb{N}} : x_i \to 0\},$$

then

$$d(\vec{x}, \vec{y}) = \sup|x_i - y_i|$$

is a metric witnessing it Polish. $GL_2(\mathbb{R})$, along with the other Lie groups, is Polish. $2^{\mathbb{N}}$ may be viewed as a Polish group under the identification with $(\mathbb{Z}_2)^{\mathbb{N}}$ given by thinking of $\mathbb{Z}_2$, the abelian group of order 2, as having elements 0 and 1. $M(\lambda)$, the measurable subsets of $[0,1]$, with the group operation of symmetric difference is a Polish group.

DEFINITION 2.7. A group G along with an algebra $\mathcal{B}$ is said to be a *Borel group* if $(G,\mathcal{B})$ is a standard Borel space and the group operation of inversion is measurable with respect to $\mathcal{B}$ and the group operation of multiplication for G is measurable with respect to the σ-algebra on $G\times G$ generated by $\mathcal{B}\times\mathcal{B}$. Equivalently, if τ is any Polish topology on G generating $\mathcal{B}$ as its Borel sets,

$$((\,)^{-1})^G : (G,\tau) \to (G,\tau),$$
$$g \mapsto g^{-1},$$

and

$$((\,)\cdot(\,))^G : (G,\tau)\times(G,\tau) \to (G,\tau),$$
$$(g,h) \mapsto g\cdot h,$$

are Borel functions.

A Borel group $(G,\mathcal{B})$ is said to be *Polishable* if there is some Polish topology τ giving rise to $\mathcal{B}$ as its collection of Borel sets and for which (G,τ) is a Polish group.

EXAMPLES 2.8. $l^1 = \{\vec{x}\in\mathbb{R}^{\mathbb{N}} : \sum_{i\in\mathbb{N}}|x_i| < \infty\}$ is a standard Borel group under the operation of pointwise multiplication and in the Borel structure it obtains from being a Borel subset of $\mathbb{R}^{\mathbb{N}}$. The metric derived from the l^1-norm,

$$d(\vec{x},\vec{y}) = \sum_{i\in\mathbb{N}}|x_i - y_i|,$$

shows it to be Polishable. $c_{00} = \{\vec{x}\in\mathbb{R}^{\mathbb{N}} : \exists m\forall i > m(x_i = 0)\}$ is an example of a non-Polishable group to which chapter 8 will return.

DEFINITION 2.9. Let λ denote the usual Lebesgue measure on $[0,1]$. We then define $L^2([0,1],\lambda)$, or just L^2 for short, to be the separable Hilbert space consisting of all square integrable complex valued functions considered up to λ-a.e. equality. Then we have the inner product

$$\langle f,g\rangle = \int_{[0,1]} f\cdot\overline{g}\,\lambda,$$

where here $\overline{g}$ denotes the complex conjugate of g. Let $||\cdot||_2$ denote the Hilbert space norm. Let U_∞ be the unitary group of L^2, consisting of all continuous linear operators that respect the inner product; we give this group the operator topology, so that a subbasic open set will have the form

$$\{T\in U_\infty : ||T(e) - T_0(e)||_2 < \epsilon\}$$

for some $T_0\in U_\infty$, $\epsilon > 0$, $e\in L^2$.

Let M_∞ be the group of invertible measure preserving transformations of the unit interval where we identify two transformations that agree a.e. M_∞ is given the topology arising from viewing it as a subgroup of the unitary group on $L^2([0,1],\lambda)$. To each measure preserving bijection $\pi:[0,1]\to[0,1]$ we can associate

$$T_\pi : L^2([0,1],\lambda) \to L^2([0,1],\lambda)$$

by

$$T_\pi(f) = f\circ\pi^{-1}.$$

Finally, let S_∞, or Sym($\mathbb{N}$), be the infinite symmetric group on $\mathbb{N}$ consisting of all permutations of $\mathbb{N}$ with the topology of pointwise convergence. Thus a basic open set will have the form

$$\{f \in S_\infty : f(n_1) = k_1, ...f(n_l) = k_l\}$$

for some sequence $n_1, n_2, ..., n_l, k_1, ..., k_l$.

S_∞ in the above topology is a G_δ subset of $\mathbb{N}^\mathbb{N}$ and hence Polish by 2.3. The proofs for the other two groups are not quite so quick.

LEMMA 2.10. *U_∞ is a Polish group.*

PROOF. Let $d(\cdot,\cdot)$ be the bounded complete metric on L^2 given by

$$d(e, e') = \frac{||e - e'||_2}{(||e - e'||_2 + 1)}.$$

Let $(e_i)_{i\in\mathbb{N}}$ be a dense subset of the Hilbert space.

Then a metric on U_∞ is formed by

$$d^*(T, S) = \sum 2^{-i}(d(T(e_i), S(e_i)) + d(T^{-1}(e_i), S^{-1}(e_i))).$$

This is easily seen to give a metric compatible with the topology, since if

$$U = \{T \in U_\infty : ||T(e) - T_0(e)||_2 < \epsilon\}$$

and

$$T \in U,$$

then we may find some $\delta > 0$ with

$$||T(e) - T_0(e)||_2 < \epsilon - 3\delta$$

and some $i \in \mathbb{N}$ with

$$||e - e_i||_2 < \delta.$$

Then for any other $S \in U_\infty$ with

$$d^*(S, T) < 2^{-i}(\delta/(\delta + 1))$$

we obtain

$$||S(e_i) - T(e_i)||_2 < \delta$$

and hence

$$||S(e) - T(e)||_2 < 3\delta$$

since both transformations are isometries of L^2. Hence

$$||S(e) - T_0(e)||_2 < \epsilon$$

by the triangle inequality.

As for completeness of the metric, note that if $(T_j)_{j\in\mathbb{N}}$ is d^* Cauchy, then we obtain two linear isometries S and T from the sets $\{e_i : i \in \mathbb{N}\}$ to L^2 such that at each i

$$T_j(e_i) \to T(e_i)$$

and

$$T_j^{-1}(e_i) \to S(e_i)$$

as $j \to \infty$. Since T and S are isometries defined on dense subsets of L^2 they extend to linear isometries $\hat{T}, \hat{S} : L^2 \to L^2$, which we then obtain to be isomorphisms of the Hilbert space by observing that $\hat{T}^{-1} = \hat{S}$. □

LEMMA 2.11. *M_∞ is a Polish group.*

PROOF. Fix $\{B_n : n \in \mathbb{N}\}$ a countable algebra of subsets of [0,1] including the intervals with rational endpoints. Then at each n we can let $\chi_{B_n} : [0,1] \to \{0,1\}$ be the characteristic function of B_n,

$$\chi_{B_n}^{-1}[\{1\}] = B_n,$$

$$\chi_{B_n}^{-1}[\{0\}] = [0,1] \setminus B_n,$$

where we choose to consider this function as an element of $L^2([0,1],\lambda)$. For $\pi_0, \pi_1 \in M_\infty$ let

$$d(\pi_0,\pi_1) = \sum_{i\in\mathbb{N}} 2^{-i}[\lambda(\pi_0(B_i)\Delta\pi_1(B_i)) + \lambda(\pi_0^{-1}(B_i)\Delta\pi_1^{-1}(B_i))].$$

Since the finite linear combinations of the functions $\{\chi_{B_n} : n \in \mathbb{N}\}$ are dense in $L^2([0,1],\lambda)$ we obtain that d is compatible with the topology on M_∞.

If $(\pi_i)_{i\in\mathbb{N}}$ is Cauchy then we can assume that for all $j, j' > l > i$

$$\lambda(\pi_j(B_i)\Delta\pi_{j'}(B_i)) + \lambda(\pi_j^{-1}(B_i)\Delta\pi_{j'}^{-1}(B_i)) < 2^{-l}$$

and thus, as in the proof of the completeness of $M(\lambda)$ from 2.2, obtain the needed A_i and C_i measurable subsets of $[0,1]$ such that

$$\pi_j(B_i) \to A_i$$

$$\pi_j^{-1}(B_i) \to C_i$$

as

$$j \to \infty.$$

Note that the map $B_i \mapsto A_i$ defines an isometry on a dense subset of $M(\lambda)$, the space of all measurable subset of [0,1]. Hence it extends to an isometry

$$\hat{\pi} : M(\lambda) \to M(\lambda).$$

Since $B_i \mapsto C_i$ extends to an inverse for $\hat{\pi}$ we obtain that it is onto. Since the Boolean operations are continuous, $\hat{\pi}$ is actually an automorphism of the Boolean algebra of the measurable subsets of [0,1]. We will be finished if we can somehow produce $\pi : [0,1] \to [0,1]$ such that for all $A \subset [0,1]$ in $M(\lambda)$ we have $\pi[A] = \hat{\pi}(A)$.

Let $\{I_s : s \in 2^{<\mathbb{N}}\}$ be a family of closed intervals such that at each $n \in \mathbb{N}$

(i) $d(I_s) = 2^{-n}$ for each $s : \{0,1,...,n-1\} \to \{0,1\}$;

(ii) for $s,t : \{0,1,...,n-1\} \to \{0,1\}$ with $s \neq t$, $I_s \cap I_t$ contains at most a single point;

(iii) [0,1] equals

$$\bigcup_{s:\{0,1,\ldots,n-1\}\to\{0,1\}} I_s.$$

It follows from the assumption that $\hat{\pi}$ defines an automorphism of the algebra of measurable sets that for λ a.e. $x \in [0,1]$ there will be a unique point in the intersection

$$\bigcap_{\{s:x\in\hat{\pi}^{-1}(I_s)\}} I_s,$$

which we can then define to be $\pi(x)$, the value of the transformation π at x. Letting $\mathcal{B}_0$ be the dense subset of $M(\lambda)$ generated by $\{I_s : s \in 2^{<\mathbb{N}}\}$ we have that for all $A \in \mathcal{B}_0$

$$\pi^{-1}[A] = \hat{\pi}^{-1}(A).$$

But since for any sequence $(D_i)_{i\in\mathbb{N}}$ of measurable sets

$$\pi^{-1}(\bigcup_{i\in\mathbb{N}} D_i) = \bigcup_{i\in\mathbb{N}} \pi^{-1}(D_i)$$
$$\pi^{-1}(\bigcap_{i\in\mathbb{N}} D_i) = \bigcap_{i\in\mathbb{N}} \pi^{-1}(D_i)$$

and since $\hat{\pi}^{-1}$ is an automorphism of the algebra of measurable sets, we obtain that π^{-1} and $\hat{\pi}^{-1}$ agree on the σ algebra generated by $\mathcal{B}_0$, as required. □

One can also establish 2.11 by showing M_∞ to be a closed subgroup of U_∞.

EXAMPLES 2.12. A very simple kind of example of a Polish G-space is when G may be continuously injected into some other Polish group and we take the induced action by translation. c_0 with the sup norm continuously injects into $\mathbb{R}^\mathbb{N}$ with the usual product Polish structure; the orbit equivalence relation resulting from this action is the coset equivalence relation.

More interesting are the self actions of reasonably rich groups such as U_∞, M_∞, and S_∞ by conjugation. So for σ and π in M_∞ we have the action $\pi\cdot\sigma = \pi\circ\sigma\circ\pi^{-1}$. The orbit equivalence relation resulting from this action of M_∞ is the subject of [**34**]. The action of U_∞ by self conjugation is analyzed in the course of the *spectral theorem* (see [**83**]). The action of S_∞ on itself by conjugation is on the other hand trivial, and completely described by 3.11 below.

EXERCISE 2.13. Let G be a topological group.

(i) If is V an open neighborhood of the identity then there is an open neighborhood of the identity W with $W^2(=_{df} \{gh : g, h \in W\}) \subset V$ and $1_G \in W$ and $W^{-1}(=_{df} \{g^{-1} : g \in W\}) = W$.

(ii) If $H < G$ is an open subgroup then H is clopen.

(iii) If G acts continuously on a metric space X then for all $x \in X$ the set

$$G_x =_{df} \{g \in G : g \cdot x = x\}$$

forms a closed subgroup.

(iv) If $Q \subset G$ is dense and $U \subset G$ open then $G = QU(=_{df} \{g\cdot h : g \in Q, h \in U\})$.

EXERCISE 2.14. Let G be a Polish group and X be Polish G-space.

(i) If $\mathcal{O} \subset X$ is open so is

$$\{x \in X : \exists g \in G(g \cdot x \in \mathcal{O})\},$$

the set of all points whose orbits meet $\mathcal{O}$.

(ii) If $C \subset X$ is a dense G_δ and $g \in G$, then $g \cdot C$ is again a dense G_δ.

EXERCISE 2.15. (i) Let $(g_i)_{i\in\mathbb{N}}$ be a sequence of group elements in S_∞ such that for all $i, j \geq l$

$$g_i(l) = g_j(l)$$
$$g_i^{-1}(l) = g_j^{-1}(l).$$

Show there is some limit $g \in S_\infty$

$$g_i \to g.$$

(ii) Show that simply assuming for all $i, j \geq l$

$$g_i(l) = g_j(l)$$

does not suffice to establish a limit.

EXERCISE 2.16. Let X be Polish and suppose for each n we have $B_n \subset X$ Borel.

(i) The set

$$\{x \in X : \exists n \in \mathbb{N}(x \in B_n)\}$$

equals

$$\bigcup_{n \in \mathbb{N}} B_n$$

and is thus Borel.

(ii) The set

$$\{x \in X : \forall n \in \mathbb{N}(x \in B_n)\}$$

equals

$$\bigcap_{n \in \mathbb{N}} B_n$$

and is thus Borel.

The catch phrase is that "Borel sets are closed under number quantifiers." A similar result holds for the $\exists^\infty$ quantifier, where for $x \in X$ write $\exists^\infty n(x \in B_n)$ if there are infinitely many n with $x \in B_n$.

(iii) Then the set of $\{x \in X : \exists^\infty n(x \in B_n)\}$ is Borel.

(iv) If each B_n is open then $\{x \in X : \exists^\infty n(x \in B_n)\}$ is G_δ.

(v) Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of Borel functions from X to $[0, 1]$. Then the function

$$x \mapsto \lim \inf_{n \in \mathbb{N}} f_n(x)$$

is again a Borel function.

EXERCISE 2.17. (i) If $f : X \to Y$ is a continuous function between Polish spaces, and $B \subset Y$ is G_δ (or F_σ, $G_{\delta\sigma}$, $F_{\sigma\delta}$, and so on), then $f^{-1}[B]$ is G_δ (or respectively F_σ, $G_{\delta\sigma}$, $F_{\sigma\delta}$).

(ii) If X, Y are Polish and $B \subset X \times Y$ is G_δ (or F_σ, $G_{\delta\sigma}$, $F_{\sigma\delta}$, and so on) in the product topology, then for any $x \in X$ the set $B_x = \{y \in Y : (x, y) \in B\}$ is G_δ (or respectively F_σ, $G_{\delta\sigma}$, $F_{\sigma\delta}$).

(iii) If $f : X \to Y$ is a Borel function between Polish spaces then there is a richer Polish topology in which f is continuous.

(iv) If $f : X \to Y$ is a Borel function between Polish spaces then for any Borel $B \subset Y$ we have $f^{-1}[B]$ to be a Borel subset of X.

EXERCISE 2.18. If $f : X \to Y$ is a Borel function between Polish spaces, then the graph of f,

$$\{(x, y) \in X \times Y : f(x) = y\},$$

is Borel as a subset of $X \times Y$.

EXERCISE 2.19. Let (X, d) be a complete metric space. Suppose we have non-empty open sets $(U_n)_{n \in \mathbb{N}}$ such that at each n

$$\overline{U_{n+1}} \subset U_n,$$

$$d(U_n) =_{df} \sup\{d(x_0, x_1) : x_0, x_1 \in U_n\} < 2^{-n}.$$

Then there is a unique point

$$x \in \bigcap_{n \in \mathbb{N}} U_n.$$

EXERCISE 2.20. Let X be a Polish space and suppose $X_0 \subset X$ is a subspace which is Polish in the induced subspace topology. Then X_0 is G_δ in X.

2.2. Equivalence relations

DEFINITION 2.21. For X Polish, an equivalence relation $E \subset X \times X$ is *Borel* if it is Borel with respect to the product topology on $X \times X$. For E an equivalence relation on a space X and $x \in X$,

$$[x]_E =_{df} \{z \in X : zEx\}$$

indicates the equivalence class of x. We say that $A \subset X$ is *E-invariant* if every E-equivalence class is either included in A or included in $X \setminus A$ (where here $X \setminus A$ is the complement of A). For E and F equivalence relations on X and Y we say that E is *Borel reducible* to F, written $E \leq_B F$, if there is a Borel function $\theta : X \to Y$ so that for all $x_1, x_2 \in X$

$$x_1 E x_2 \Leftrightarrow \theta(x_1) F \theta(x_2).$$

We write $E \leq_c F$ if there is a continuous reduction, and $E \sqsubseteq_B F$ and $E \sqsubseteq_c F$ if the reduction may respectively chosen Borel and one to one or continuous and one to one. We can write $E <_B F$ if $E \leq_B F$ holds and $F \leq_B E$ fails. $E \not\leq_B F$ and $E \not\leq_c F$ indicate that E is not Borel reducible, in the first case, or not to continuously reducible, in the second, to F. In the case that G acts on the Polish space space X we may as well write $X/G \leq_B F$ and $F \leq_B X/G$ for $E_G^X \leq_B F$ and $F \leq_B E_G^X$ respectively; similarly for group actions $[x]_G$ stands in place of $[x]_{E_G^X}$.

DEFINITION 2.22. E_0 is the equivalence relation of eventual agreement on elements of $2^{\mathbb{N}}$, so that

$$xE_0y \Leftrightarrow \exists m \forall n > m(x(n) = y(n)).$$

E_0 is a Borel equivalence relation, and in fact it is F_σ in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$. For a Polish space X, we let $\mathrm{id}(X)$ denote the equality relation on X. For $x, y \in 2^{\mathbb{N}\times\mathbb{N}}$ we write xF_2y if

$$\{x(n, \cdot) : n \in \mathbb{N}\} = \{y(n, \cdot) : n \in \mathbb{N}\},$$

where $x(n, \cdot)$ is the element of x_n of $2^{\mathbb{N}}$ defined by $x_n(m) = x(n, m)$; so in other words, x and y are F_2 equivalent if they code the same countable set of elements of Cantor space. Similarly for $x, y \in 2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ we write xF_3y if

$$\{\{x(n, l, \cdot) : n \in \mathbb{N}\} : l \in \mathbb{N}\} = \{\{y(n, l, \cdot) : n \in \mathbb{N}\} : l \in \mathbb{N}\}.$$

More generally we can iterate this definition, defining for $x, y \in 2^{(\mathbb{N}^k)}$ the equivalence relation xF_ky to hold if $\{[x(n, \cdot, \cdot, ...)]_{F_{k-1}} : n \in \mathbb{N}\} = \{[y(n, \cdot, \cdot, ...)]_{F_{k-1}} : n \in \mathbb{N}\}$. (And here the authors of **[20]** continue these definitions through all the countable ordinals, providing F_α for $\alpha < \omega_1$.) By induction on k each F_k can be shown to be Borel – for instance F_2 is $F_{\sigma\delta}$. With a minor blurring of vision we may view these F_k's as orbit equivalence relations induced by S_∞. For instance let B_2 be the set of $x \in 2^{\mathbb{N}\times\mathbb{N}}$ such that for all $m \neq n$

$$x(n, \cdot) \neq x(m, \cdot)$$

and let S_∞ act on B_2 by

$$g \cdot x(n, m) = x(g^{-1}(n), m).$$

Here we have that $E_{S_\infty}^{B_2}$ (the orbit equivalence relation from this action of S_∞ on B_2) equals $F_2|_{B_2}$ (the restriction of F_2 to B_2) and

$$F_2 \leq_B F_2|_{B_2} \leq_B F_2.$$

One direction of this is trivial; $F_2 \leq_B F_2|_{B_2}$ is left as an exercise below.

LEMMA 2.23. *(folklore) Let G be a countable discrete group and X a Borel G-space. Then $E_G^X \leq_B F_2$.*

PROOF. Rephrasing our assumptions, there is a countable group $(f_n)_{n\in\mathbb{N}}$ of Borel bijections such that for each $x \in X$

$$[x]_G =_{df} \{y \in X : xE_G^X y\} = \{f_n(x) : n \in \mathbb{N}\}.$$

Let $\{U_l : l \in \mathbb{N}\}$ be a countable basis for X, the space on which E is defined. For $x \in X$ let $\hat{f}(x) \in 2^{\mathbb{N}}$ be given by

$$(\hat{f}(x))(l) = \chi_{U_l}(x),$$

where χ_{U_l} is the characteristic function of U_l, taking the value 1 if $x \in U_l$ and 0 otherwise. $\hat{f} : X \to 2^{\mathbb{N}}$ is Borel.

$$\theta : X \to 2^{\mathbb{N}\times\mathbb{N}}$$

defined by

$$(\theta(x))(n, m) = (\hat{f} \circ f_n(x))(m)$$

witnesses $E_G^X \leq_B F_2$. □

Another class of group actions are those arising from shifting subsets of G or some collection of functions on a G-set.

DEFINITION 2.24. If G is a Polish group then we can let it act by *shift* on the standard Borel space $\mathcal{F}(G)$ of all closed subsets of G by

$$h \in g \cdot F \Leftrightarrow g^{-1}h \in F.$$

For G a countable group we have the *shift* action on $2^G (=_{df} \{f \mid f : G \to \{0, 1\}\})$ given by

$$(g \cdot \vec{x})(h) = \vec{x}(g^{-1}h)$$

where $\vec{x} \in 2^G, g, h \in G$; note that in the product topological structure 2^G is a compact metrizable space and the action is continuous.

LEMMA 2.25. *(classical)* $\mathrm{id}(\mathbb{R}) <_B E_0 \leq_B \mathbb{R}/\mathbb{Q}, 2^{\mathbb{Z}}/\mathbb{Z}$.

PROOF. The first strict inequality is in §3.1; $E_0 \leq_B 2^{\mathbb{Z}}/\mathbb{Z}$ and $E_0 \leq_B \mathbb{R}/\mathbb{Q}$ follow from 7.9. □

DEFINITION 2.26. An equivalence relation E is said to be *smooth* if $E \leq_B \mathrm{id}(\mathbb{R})$.

EXERCISE 2.27. (i) $\mathrm{id}(\mathbb{R}) \leq_B \mathrm{id}(2^{\mathbb{N}}) \leq_B \mathrm{id}(\mathbb{R})$. (For instance: For $r \in \mathbb{R}$ and $(q_n)_{n\in\mathbb{N}}$ an enumeration of $\mathbb{Q}$ define $\theta(r) \in 2^{\mathbb{N}}$ by $(\theta(r))(n) = 1$ if and only if $r < q_n$. Conversely, to each $x \in 2^{\mathbb{N}}$ we can associate the real $r_x \in [0, 1]$ that has the value $x(i)$ in the ith position of its decimal expansion.)

(ii) Let E be an equivalence relation on a Polish space X. E is smooth if and only if there is a countable sequence of $(A_n)_{n\in\mathbb{N}}$ of Borel invariant sets such that for all $x_1, x_2 \in X$

$$x_1 E x_2 \Leftrightarrow \forall n (x_1 \in A_n \Leftrightarrow x_2 \in A_n).$$

(iii) For E and X as in (ii), let $\mathcal{B}_E$ be the subsets of X/E generated by the Borel invariant sets, so that for each E-invariant Borel set $B \subset X$ we will have the corresponding $B/E = \{[x]_E : x \in B\}$ in $\mathcal{B}_E$. If $(X/E, \mathcal{B}_E)$ is a standard Borel space then E is smooth.[1]

EXERCISE 2.28. Show $\mathbb{R}/\mathbb{Z} \leq_B \mathrm{id}(\mathbb{R})$.

EXERCISE 2.29. (i) Define $E_0(\mathbb{N}^\mathbb{N})$ on $\mathbb{N}^\mathbb{N}$ by analogy with E_0, so that for $x, y \in \mathbb{N}^\mathbb{N}$ we set $xE_0(\mathbb{N}^\mathbb{N})y$ if

$$\exists m \in \mathbb{N} \forall n \geq m(x(n) = y(n)).$$

Show $E_0(\mathbb{N}^\mathbb{N}) \leq_c E_0 \leq_c E_0(\mathbb{N}^\mathbb{N})$.

(ii) More generally, if A is a countably infinite set and B is a countable or finite set of size at least 2, we may define $E_0(B^A)$ by the specification that for $x, y \in B^A$ we have $xE_0(B^A)y$ if there is some finite $F \subset A$ such that

$$\forall a \in A \setminus F(x(a) = y(a)).$$

Here again $E_0(B^A) \leq_c E_0 \leq_c E_0(B^A)$.

EXERCISE 2.30. Show that $F_2 \leq_B F_2|_{B_2}$. (Hint: For $m \in \mathbb{N}$ and $x \in \mathbb{N}^\mathbb{N}$ we will define $(0^m 1)^\frown x$ by

1. $((0^m 1)^\frown x)(k) = 0$ for $k < m$;
2. $((0^m 1)^\frown x)(k) = 1$ for $k = m$;
3. $((0^m 1)^\frown x)(k) = x(k - m - 1)$ for $k > m$.

It is then possible to find a Borel function $x \mapsto y_x$ such that for all $x \in 2^{\mathbb{N}\times\mathbb{N}}$ we have $\{y_x(n, \cdot) : n \in \mathbb{N}\} = \{((0^m 1)^\frown x)(n, \cdot) : n \in \mathbb{N}\}$ and $y_x \in B_2$.)

EXERCISE 2.31. Let F be an equivalence relation on a Polish space all of whose equivalence classes are countable. Show $F_2 \not\leq_c F$ as follows:
(i) Observe that B_2 from 2.22 is a Polish S_∞-space and it suffices to show

$$E^{B_2}_{S_\infty} \not\leq_c F.$$

(ii) For $N \in \mathbb{N}$ let E_N be the set of $g \in S_\infty$ such that for all $i \leq N$

$$g(i) = i.$$

If $x \in B_2$ is such that for all $s \in 2^{<\mathbb{N}}$ there exists $n \in \mathbb{N}$

$$x(n, \cdot) \supset s$$

then

$$\overline{E_N \cdot x} = \{y \in B_2 : \forall i \leq N(x(i, \cdot) = y(i, \cdot))\}.$$

(Here $\overline{A}$ indicates the topological closure of a set A.)
(iii) If we have a continuous

$$\theta : B_2 \to X$$

such that for all $x \in B_2$ and $y \in [x]_{S_\infty}$

$$\theta(x) F \theta(y)$$

then for each $x \in B_2$ there exists some $x_0 \in [x]_{S_\infty}$ and $N \in \mathbb{N}$ so that $\theta(E_N \cdot x_0)$ consists in a single point. (The Baire category theorem implies that S_∞ is not a countable union of closed nowhere dense sets, and so we can find $z_0 \in [\theta(x)]_F$ for which $\{g \in S_\infty : \theta(g \cdot x) = z_0\}$ contains an open set.)

[1] Although this implication does not in general reverse for arbitrary Borel equivalence relations, there is a converse for the cases of most interest to us here. Burgess in **[8]** shows that if a Polish G acts continuously on Polish X then $E^X_G \leq_B \mathrm{id}(\mathbb{R})$ if and only if $(X/E^X_G, \mathcal{B}_{E^X_G})$ is standard Borel.

(iv) Conclude that for x as in (ii) and x_0 and N as in (iii) there exists some $y \notin [x]_{S_\infty}$ with

$$y \in \overline{E_N \cdot x_0}$$
$$\therefore \theta(x_0) = \theta(y).$$

We will see at 2.64 that F_2 is not even *Borel* reducible to a countable equivalence relation.

EXERCISE 2.32. For E a Borel equivalence relation on a Polish space X, the following three things are equivalent:

(i) $E \leq_B F_2$;

(ii) There is a countable sequence $(f_n)_{n\in\mathbb{N}}$ of Borel functions from X to $\mathbb{R}$ such that for all $x_0, x_1 \in X$ we have $x_0 E x_1$ if and only if

$$\{f_n(x_0) : n \in \mathbb{N}\} = \{f_n(x_1) : n \in \mathbb{N}\}.$$

(Here we are only asking that $\{f_n(x_0) : n \in \mathbb{N}\}$ and $\{f_n(x_1) : n \in \mathbb{N}\}$ be equal as *unordered* sets.)

(iii) There is a countable sequence $(f_n)_{n\in\mathbb{N}}$ of Borel functions from X to $2^{\mathbb{N}}$ such that for all $x_0, x_1 \in X$ we have $x_0 E x_1$ if and only if

$$\{f_n(x_0) : n \in \mathbb{N}\} = \{f_n(x_1) : n \in \mathbb{N}\}.$$

In general for a set A we let $\mathcal{P}_{\aleph_0}(A)$ indicate the countable subsets of A. In light of 2.32 it seems natural to say that $2^{\mathbb{N}\times\mathbb{N}}/F_2$ has the same *effective cardinality* as $\mathcal{P}_{\aleph_0}(\mathbb{R})$.

2.3. Spaces of countable structures

Intuitively the idea of a class of discrete structures should be clear. For instance, any abstract group (freed from any topology) is an example of a discrete structure. Similarly we can consider the class of all rings or fields, or any other notion of algebraic object. Of course we do not need to restrict ourselves to discrete structures appearing in abstract algebra. Other classes of structures are given by graphs and linear orderings.

The purpose of this particular section is to fix a precise definition of the space, $\mathrm{Mod}(\mathcal{L})$, of countable structures or countable *models* for a given language, $\mathcal{L}$. This will provide clear meaning to the idea of a Borel map into this space and the notion of a Borel subset of this space. Most of the natural classes of countable structures, such as all those alluded to above, can be identified with a Borel subset of some appropriately chosen $\mathrm{Mod}(\mathcal{L})$.

DEFINITION 2.33. Let $\mathcal{L} = \{R_i : i \leq k\}$ be a finite relational language. So each R_i will be a relation with some fixed arity, $a(i)$ – which is to say its interpretation in any $\mathcal{L}$-structure will be as some relation on sequences of length $a(i)$.

Then $\mathrm{Mod}(\mathcal{L})$ is the space of all $\mathcal{L}$-structures with underlying set $\mathbb{N}$. Thus each $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ is a sequence of assignments: to each R_i we have a set

$$R_i^{\mathcal{M}} \subset \mathbb{N}^{a(i)}.$$

In this respect $\mathcal{M}$ can be formally viewed as an element of

$$\prod_{i\leq k} \mathcal{P}(\mathbb{N}^{a(i)}),$$

where each $\mathcal{P}(\mathbb{N}^{a(i)})$ is the set of all subsets of $\mathbb{N}^{a(i)}$.

This space is equipped with a topology whose subbasic open sets have form

$$\{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}) : (k_1, k_2, ..., k_{a(i)}) \in R_i^{\mathcal{M}}\}$$

or

$$\{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}) : (k_1, k_2, ..., k_{a(i)}) \notin R_i^{\mathcal{M}}\},$$

for some $i \leq k$, $k_1, k_2, ..., k_{a(i)} \in \mathbb{N}$.

We can similarly define $\mathrm{Mod}(\mathcal{L})$ in the case that $\mathcal{L} = \{R_i : i \in \mathbb{N}\}$ is a countably infinite relational language, and obtain a natural identification with

$$\prod_{i \in \mathbb{N}} \mathcal{P}(\mathbb{N}^{a(i)}).$$

EXAMPLES 2.34. Thus if $\mathcal{L} = \{R\}$, where R has arity 2, then every $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ is a directed graph on $\mathbb{N}$, and can be naturally thought of as an element of $2^{\mathbb{N}\times\mathbb{N}}$, by associating to each $\mathcal{M}$ the function

$$\chi_{R^{\mathcal{M}}} : \mathbb{N} \times \mathbb{N} \to \{0, 1\},$$

$$\chi_{R^{\mathcal{M}}}^{-1}[\{1\}] = R^{\mathcal{M}}.$$

If $\mathcal{L} = \{P_0, P_1\}$, where P_0, P_1 have arity 1 – that is to say they are unary predicates – then each element of $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ consists of the assignment of subsets $P_0^{\mathcal{M}}$ and $P_1^{\mathcal{M}}$ of $\mathbb{N}$.

DEFINITION 2.35. It is customary to write

$$\mathcal{M} \models R_i(k_1, k_2, ..., k_{a(i)})$$

in place of $(k_1, k_2, ..., k_{a(i)}) \in R_i^{\mathcal{M}}$, and then to define $\mathcal{M} \models \varphi(k_1, ...)$ for all formulas of $\mathcal{L}$ by induction on the complexity of φ:-

$$\mathcal{M} \models \exists x \varphi(k_1, ..., x)$$

if and only if for some $l \in \mathbb{N}$

$$\mathcal{M} \models \varphi(k_1, ..., l);$$

and then

$$\mathcal{M} \models \varphi \wedge \psi(k_1, ...)$$

if and only if

$$\mathcal{M} \models \varphi(k_1, ...)$$

and

$$\mathcal{M} \models \psi(k_1, ...);$$

finally

$$\mathcal{M} \models \neg\varphi(k_1, ...)$$

if and only if it is *not* the case that

$$\mathcal{M} \models \varphi(k_1, ...).$$

As well as these primitive logical connectives, there are also the usual composites defined from them:

$$\mathcal{M} \models \varphi \Rightarrow \psi(k_1, ...)$$

stands for

$$\mathcal{M} \models \neg(\varphi \wedge (\neg\psi(k_1, ...));$$

$$\varphi \Leftrightarrow \psi$$

is short hand for $\varphi \Rightarrow \psi \wedge \psi \Rightarrow \varphi$; $\forall$ for $\neg\exists\neg$; $\varphi \vee \psi$ for $\neg(\neg\varphi \wedge \neg\psi)$. Thus for example if $\mathcal{M}$ is a structure on $\mathbb{N}$ for the language generated by a single binary relation R, then

$$\mathcal{M} \models \forall a \exists b R(a, b)$$

if and only if it really is the case that for all $a \in \mathbb{N}$ there exists $b \in \mathbb{N}$ with

$$\mathcal{M} \models R(a, b).$$

The equality symbol is usually given a special status. One writes

$$\mathcal{M} \models a = b$$

exactly when it is indeed the case that $a = b$, and therefore without reference to any individual properties of $\mathcal{M}$; consequently one would not normally allow the symbol $=$ as part of a language $\mathcal{L}$.

A formula is said to be *quantifier free* if it is built up from the atomic formulas of the form $x_0 = x_1$ and $R_i(x_1, ..., x_{a(i)})$ using just the boolean connectives $\neg$ and $\wedge$.

In the special case of R having arity 2 – that is to say, R is a binary relation – one uses the expressions

$$\mathcal{M} \models R(a, b)$$

and

$$\mathcal{M} \models aRb$$

interchangeably.

$\cong|_{\mathrm{Mod}(\mathcal{L})}$ denotes the isomorphism relation on elements in $\mathrm{Mod}(\mathcal{L})$. Here we understand isomorphism in the natural sense for classes of $\mathcal{L}$-structures: two $\mathcal{L}$-structures $\mathcal{M}, \mathcal{N}$ are isomorphic if there is some bijection π between these structures such for all relations R in $\mathcal{L}$ and $a_1, ..., a_k \in \mathcal{M}$,

$$\mathcal{M} \models R(a_1, ..., a_k)$$

if and only if

$$\mathcal{N} \models R(\pi(a_1), ..., \pi(a_k)).$$

Note that $\mathrm{Mod}(\mathcal{L})$ is homeomorphic to $\prod_{i \in \mathbb{N}} 2^{\mathbb{N}^{a(i)}}$. Since each $\mathbb{N}^{a(i)}$ $(=_{df} \{f | f : \{1, 2, ..., a(i)\} \to \mathbb{N}\})$ is countable, we obtain that $\mathrm{Mod}(\mathcal{L})$ is homeomorphic to $2^{\mathbb{N}}$ and therefore Polish.

The definition is restricted to relational languages, but this entails no real loss of generality. If f is a function symbol appearing in the language of some model $\mathcal{M}$ we may pass to a new model $\mathcal{M}'$ having a fresh relation R_f with the understanding

$$\mathcal{M} \models f(k_1, k_2, ...) = l \Leftrightarrow \mathcal{M}' \models R_f(k_1, k_2, ..., l).$$

So for instance the axiom in group theory stating the existence of a right identity

$$\exists e \forall g (g \cdot e = g)$$

would seamlessly reappear as

$$\exists e \forall g (R_\bullet(g, e, g)).$$

EXAMPLE 2.36. If $\mathcal{L}$ is generated by a single binary relation E, then $\mathrm{Mod}(\mathcal{L})$ can be thought of as the space of directed graphs having the natural numbers as the underlying set. Then $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ is the complete graph if

$$\mathcal{M} \models \forall a \forall b (aEb).$$

DEFINITION 2.37. The *logic action* of S_∞ on $\mathrm{Mod}(\mathcal{L})$ for $\mathcal{L}$ a countable relational language is defined as follows. Given $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ and $f \in S_\infty$ we specify $f \cdot \mathcal{M}$ by

$$f \cdot \mathcal{M} \models R_i(k_1, k_2, ...) \Leftrightarrow \mathcal{M} \models R_i(f^{-1}(k_1), f^{-1}(k_2), ...).$$

In other words, we let $(R_i)^{f \cdot \mathcal{M}}$ be

$$\{(f(k_1), f(k_2), ...) : (k_1, k_2, ...) \in (R_i)^{\mathcal{M}}\}.$$

Note then that this action is continuous and $\cong|_{\mathrm{Mod}(\mathcal{L})} = E_{S_\infty}^{\mathrm{Mod}(\mathcal{L})}$.

The manuscript below is a study of which orbit equivalence relations may in principle be drawn under the umbrella of the orbit equivalence relation induced by the logic action for some sufficiently rich language. This is not to say that it is a study of the logic action, or even of any sort of Polish group action. The object of investigation is one level of abstraction further removed, since in terms of developing a theory of invariants and classification we should be interested in the quotient of a space by an equivalence relation, not the details of how it arises.

DEFINITION 2.38. An equivalence relation E is said *to admit classification by countable models* or *admit classification by countable structures* if for some countable language $\mathcal{L}$

$$E \leq_B \cong|_{\mathrm{Mod}(\mathcal{L})};$$

in other words, we may in a Borel fashion assign to each element of the space X on which E is defined some $\mathcal{L}$-structure considered up to isomorphism as a complete invariant.

In this context the terms *model* and *structure* are used more or less interchangeably.

THEOREM 2.39. *(Becker, Kechris) Let G be a closed subgroup of S_∞ and X a Polish G-space. Then E_G^X admits classification by countable models.*

PROOF. [**4**] derives this from their result that for $G = S_\infty$ we can find a Borel S_∞-embedding from X to $\mathrm{Mod}(\mathcal{L})$. This proof is summarized at 6.19 and 7.18 below and another argument is implicit in §6.2. □

LEMMA 2.40. *There is a countable language $\mathcal{L}$ with* $\mathrm{id}(\mathbb{R}) \leq_B \cong|_{\mathrm{Mod}(\mathcal{L})}$.

PROOF. Let $\mathcal{L}$ be the language generated by $\{P_q : q \in \mathbb{Q}\}$, each P_q is a unary predicate. Assign to $r \in \mathbb{R}$ the $\mathcal{L}$-structure $\mathcal{M}_r$ on $\mathbb{N}$ such that for all $n \in \mathbb{N}$

$$(\mathcal{M}_r \models P_q(n)) \Leftrightarrow q < r.$$

□

LEMMA 2.41. *There is a countable language $\mathcal{L}$ with $F_2 \leq_B \cong|_{\mathrm{Mod}(\mathcal{L})}$.*

PROOF. Let $(P_n)_{n\in\mathbb{N}}$ be a countable sequence of predicates. To each $x \in 2^{\mathbb{N}\times\mathbb{N}}$ assign the structure $\mathcal{M}_x$ on $\mathbb{N}$ that has at each $k, m, n \in \mathbb{N}$

$$(\mathcal{M}_x \models P_m(3^n 5^k)) \Leftrightarrow x(n, m) = 1,$$

and

$$\mathcal{M}_x \models \neg P_m(l)$$

for l not of the form $l = 3^n 5^k$. (Thus we associate to each strip $x(n, \cdot)$ of x infinitely many $a \in \mathcal{M}_x$ such that at each m we have $\mathcal{M}_x \models P_m(a)$ if and only if $x(n, m) = 1$.) □

Any hereditarily countable set may be represented by a model for the language generated by a single binary relation R. So if A is hereditarily countable, and thus we can find an enumeration $\vec{a} = (a_i)_{i\in\mathbb{N}}$ of

$$\{A\} \cup \mathrm{TC}(A) =_{df} \{A\} \cup A \cup \bigcup A \cup \bigcup\bigcup A ...,$$

where $\bigcup A = \{y_1 : \exists y_0 \in A(y_1 \in y_0)\}$, $\bigcup\bigcup A = \{y_2 : \exists y_1 \in \bigcup A(y_2 \in y_1)\}$, and so on, then we can let $\mathcal{M}(\vec{a})$ be a model with

$$\mathcal{M}(\vec{a}) \models nRm$$

if and only if $a_n \in a_m$. The model $\mathcal{M}(\vec{a})$ depends on the enumeration $\vec{a} = (a_i)_{i\in\mathbb{N}}$, since in general there is no reasonable method of choosing $\vec{a}$ purely from A; the isomorphism type of $\mathcal{M}(\vec{a})$ however *is* a function of A alone.

We likewise obtain a parameterization of elements of HC (the hereditarily countable sets) by points in $2^{\mathbb{N}\times\mathbb{N}}$, since the structures for a single binary relation and with underlying set $\mathbb{N}$ may be naturally identified with $2^{\mathbb{N}\times\mathbb{N}}$.

EXERCISE 2.42. Let $\mathcal{L}$ be generated by a single binary relation $<$. Let Mod(LO) be the set of $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ which are linear orderings – that is to say

$$\forall a \in \mathbb{N}(\mathcal{M} \models \neg(a < a)),$$

$$\forall a \neq b \in \mathbb{N}(\mathcal{M} \models a < b \Leftrightarrow \neg(b < a)),$$

$$\forall a, b, c \in \mathbb{N}(\mathcal{M} \models (a < b \wedge b < c) \Rightarrow a < c)).$$

Then Mod(LO) is a closed subset of $\mathrm{Mod}(\mathcal{L})$.

EXERCISE 2.43. Let $\mathcal{L}$ be a relational language. Let τ_{FO} be the topology on $\mathrm{Mod}(\mathcal{L})$ generated by taking as basic open all sets of the form

$$\{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}) : \mathcal{M} \models \varphi(k_1, ..., k_l)\},$$

where φ is a formula from the language $\mathcal{L}$ and $\models$ is the satisfaction relation from 2.35. Show that $(\mathrm{Mod}(\mathcal{L}), \tau_{FO})$ is a Polish space. (One may realize this as a G_δ subspace of Polish space. The main point is that the inductive definition of $\models$ for $\mathrm{Mod}(\mathcal{L})$ amounts to a G_δ set in a suitable topology. The hardest condition – namely $\mathcal{M} \models \exists x\varphi(x)$ if and only if there is some $k \in \mathbb{N}$ with $\mathcal{M} \models \varphi(k)$ – is truly G_δ, while the others are only closed. See [**26**].)

EXERCISE 2.44. (i) Let $\mathcal{L}$ be a relational language and T a (countable set) of sentences in the language. Let B_T be the set of $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ such that for all $\varphi \in T$

$$\mathcal{M} \models \varphi.$$

Then B_T is a Borel set, and therefore a standard Borel space in the relative Borel structure. (A structure $\mathcal{M} \in B_T$ is said to be a *model* for T.)

(ii) Now let $(\varphi_n(\cdot))_{n\in\mathbb{N}}$ be a countable sequence of formulas in $\mathcal{L}$. Then the set of $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ such that for all $a \in \mathbb{N}$ there exists n with

$$\mathcal{M} \models \varphi_n(a)$$

is a Borel set.

(iii) Let $\mathcal{L}$ be generated by a single binary relation R. Let Mod(Gph) be the $\mathcal{L}$-structures on $\mathbb{N}$ which are (undirected) graphs; in other words, for $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ we have $\mathcal{M} \in \mathrm{Mod(Gph)}$ if and only if

$$\mathcal{M} \models \forall x\forall y(xRy \Rightarrow yRx).$$

Show that Mod(Gph) is closed in Mod($\mathcal{L}$), the connected graphs in Mod(Gph) are a G_δ subset of Mod($\mathcal{L}$), and the graphs with only finitely many components form a Borel subset.

2.4. Baire category methods

DEFINITION 2.45. For X a Polish space we say that $A \subset X$ is *meager* if it is disjoint from a dense G_δ and *comeager* if it includes one. Since every countable intersection of dense open sets is dense by the Baire category theorem we may choose to think of comeager sets as being *large*. In particular the class of comeager sets is closed under supersets and countable intersections. It is customary to write $\forall^* x \in X$("such and such about x") if there is a comeager $A \subset X$ on which we have that "such and such about x"; we write $\exists^* x \in X$("such and such about x") if such and such on a non-meager set. A set $B \subset X$ is said to have *the Baire property* if there is some $\mathcal{O} \subset X$ that is open and $A \subset X$ comeager with $A \cap \mathcal{O} = A \cap B$.

The sets with the Baire property form a σ-algebra, and so include the Borel.

THEOREM 2.46. *(Kuratowski, Ulam) Let X and Y be Polish. If $B \subset X \times Y$ is comeager (in the product topology) then for a comeager set of $x \in X$*

$$B_x =_{df} \{y : (x, y) \in B\}$$

is comeager in Y; if B has the Baire property then B is non-meager if and only if there is a non-meager set of $x \in X$ for which B_x is non-meager.

PROOF. For the first proposition it suffices to consider the case that B is open dense. But if $\{U_n : n \in \mathbb{N}\}$ is a basis for Y then for each n

$$\mathcal{O}_n = \{x \in X : U_n \cap B_x \neq \emptyset\}$$

is open dense by assumption on B. Then B_x is comeager for all

$$x \in \bigcap_{n \in \mathbb{N}} \mathcal{O}_n.$$

The second part of the theorem follows from the first and 2.3: B non-meager implies there are open, non-empty $U \subset X$ and $V \subset Y$ with $B \cap U \times V$ comeager in $U \times V$. Since U and V are Polish B_x will be comeager in V for a comeager set of $x \in U$. Alternatively if B is meager in the product then $C = (X \times Y) \setminus B$ will be comeager, and we can conclude from the first part that C_x will be a comeager for a comeager collection of $x \in X$. $\square$

Thus if $B \subset X \times Y$ has the Baire property and

$$\forall^* x \in X \forall^* y \in Y((x, y) \in B)$$

then

$$\forall^* x \in X(Y \setminus B_x \text{ must be meager}),$$

implying B comeager in $X \times Y$, and so

$$\forall^* y \in Y \forall^* x \in X((x, y) \in B).$$

This is the right way to think about 2.46: Under reasonable assumptions the quantifiers $\forall^* x \in X$ and $\forall^* y \in Y$ commute.

COROLLARY 2.47. *Let G be a Polish group, X a Polish G-space. If $A \subset X$ is comeager, then $\forall^* x \in X \forall^* g \in G(g \cdot x \in A)$.*

PROOF. Note that for all $g \in G$ the set $g^{-1} \cdot A$ is comeager. We may assume A is Borel, in fact G_δ. Thus for

$$A = \bigcap_{i \in \mathbb{N}} U_i,$$

each U_i open and $\mathcal{O}_i$ the open set in $G \times X$ consisting of all (g, x) with $g \cdot x \in U_i$, we have

$$\{(g, x) \in G \times X : g \cdot x \in A\} = \bigcap_{i \in \mathbb{N}} \mathcal{O}_i$$

to be Borel. From these observations and 2.46 we have that $B = \{(g, x) \in G \times X : g \cdot x \in A\}$ is comeager in the plane. By a further application of 2.46 we see that $\forall^* x \in X (B_x$ is comeager). □

DEFINITION 2.48. Let X and Y be Polish spaces. A function $f : X \to Y$ is said to be *Baire measurable* if the preimage of every open set has the Baire property . The function f is said to be *universally Baire measurable* if given any Borel function g into X we have that $f \circ g$ is Baire measurable.

So in particular every Borel function is Baire measurable.

LEMMA 2.49. *Let $f : X \to Y$ be a Baire measurable function. Then there is a comeager set $A \subset X$ on which f is continuous.*

PROOF. Since we choose A so $f^{-1}(O) \cap A$ is relatively open for all basic open O. □

LEMMA 2.50. *Let G be a Polish group, X a Polish G-space, $V \subset G$ open.*
(i) If $U \subset X$ is open then $\{x \in X : \exists^* g \in V(g \cdot x \in U)\}$ is open.
(ii) If $A \subset X$ is G_δ then $\{x \in X : \forall^* g \in V(g \cdot x \in A)\}$ is G_δ.
(iii) If $B \subset X$ is Borel then $\{x \in X : \forall^* g \in V(g \cdot x \in B)\}$ is Borel.

PROOF. For (i) we obtain from the continuity of the action that

$$\{x \in X \exists^* g \in V(g \cdot x \in U)\}$$

equals

$$\{x \in X \exists g \in V(g \cdot x \in U)\}.$$

(ii) follows (i), since if we fix a countable basis $\mathcal{B}$ of G

$$\forall^* g \in V(g \cdot x \in \bigcap_{n \in \mathbb{N}} U_n)$$

if and only if for all $n \in \mathbb{N}$ and non-empty basic open $W \subset V$

$$\exists^* g \in W(g \cdot x \in U_n).$$

For (iii): let $\mathcal{C}(X)$ be the collection of $B \subset X$ for which

$$\{x : \forall^* g \in V(g \cdot x \in B)\}$$
$$\{x : \exists^* g \in V(g \cdot x \in B)\}$$

are both Borel whenever $V \subset G$ open. We need to show $\mathcal{C}(X)$ forms a σ-algebra including the open sets.

First note that

Claim: If $B \subset X$ is Borel and $x \in X$ then

$$\{g \in G : g \cdot x \in B\}$$

is Borel.

Proof of claim. Let $\mathcal{D}$ be the collection of subsets of X satisfying the conclusion of the claim. It includes the open sets and at once is closed under complementation. Since

$$\{g \in G : g \cdot x \in \bigcap_{n \in \mathbb{N}} B_n\}$$

$$= \bigcap_{n \in \mathbb{N}} \{g \in G : g \cdot x \in B_n\}$$

$\mathcal{D}$ forms a σ-algebra and thus includes all Borel sets. (Claim$\square$)

Thus in particular for each x and Borel $B \subset X$ and open $V \subset G$, the set of $g \in V$ with

$$g \cdot x \in B$$

has the Baire property , and thus

$$X \setminus \{x : \exists^* g \in V(g \cdot x \in B)\} = \{x : \forall^* g \in V(g \cdot x \in (X \setminus B))\}.$$

Thus $\mathcal{C}(X)$ is closed under complements. By (i) it includes the open sets and from the argument of (ii) it is closed under countable intersections. $\square$

The sets arising 2.50 appear so frequently that there is a further notation for describing them. This was introduced in [**80**].

NOTATION 2.51. Let X and G be as in 2.50 and $V \subset G$ open. Then for any $B \subset X$ we let $B^{\Delta V} = \{x \in X : \exists^* g \in V(g \cdot x \in B)\}$ and $B^{*V} = \{x \in X : \forall^* g \in V(g \cdot x \in B)\}$. The sets $B^{\Delta V}$ and B^{*V} are *Vaught transforms* of B.

Some authors also use B^* for B^{*G} and B^{Δ} for $B^{\Delta G}$.

There are two essential facts about the Vaught transforms, both of which were implicit in 2.50: For $B \subset X$ Borel,

$$X \setminus (B^{\Delta V}) = (X \setminus B)^{*V}$$

and if $(B_i)_{i \in \mathbb{N}}$ are Borel subsets of X and $\{V_n : n \in \mathbb{N}\}$ an enumeration of the non-empty basic open sets in G, then

$$(\bigcup_{i \in \mathbb{N}} B_i)^{\Delta V} = \bigcup_{i,j \in \mathbb{N}, V_j \subset V} B_i^{*V_j}.$$

DEFINITION 2.52. A subset A of a Polish space X is said to be $\underset{\sim}{\Sigma}{}^1_1$, or *analytic*, if there is a Polish space Y and a Borel set $B \subset Y$ and a Borel function $f : Y \to X$ with $A = \{f(y) : y \in B\}$. In other words, the $\underset{\sim}{\Sigma}{}^1_1$ sets are the Borel images of the Borel sets.

THEOREM 2.53. *(Luzin) Every $\underset{\sim}{\Sigma}{}^1_1$ set has the property of Baire. Every $\underset{\sim}{\Sigma}{}^1_1$ set is measurable with respect to any Borel measure.*

PROOF. We concentrate on the proof for the Baire property , with only a passing aside on measurability. For $(A_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ a collection of Borel sets in a Polish space X, indexed by finite functions of the form $s\colon \{0, 1, ..., n\} \to \mathbb{N}$, let $\mathcal{A}((A_s)_{s \in \mathbb{N}^{<\mathbb{N}}})$ be the set of $x \in X$ such that for some $f \in \mathbb{N}^{\mathbb{N}}$

$$x \in \bigcap_{n \in \mathbb{N}} A_{f|_{\{0,1,\ldots,n\}}}.$$

In other words, $x \in \mathcal{A}((A_s)_{s \in \mathbb{N}^{<\mathbb{N}}})$ if there is some sequence of choices $s_1 \in \mathbb{N}^1$, $s_2 \in \mathbb{N}^2$,...$s_k \in \mathbb{N}^k$,..., which are *consistent* in the sense of

$$s_k \subset s_{k+1}$$

all k and for which

$$x \in \bigcap_{k \in \mathbb{N}} A_{s_k}.$$

Claim(I): For all $B \subset X$ Borel there exists a family $(A_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ of closed sets with $B = \mathcal{A}((A_s)_{s \in \mathbb{N}^{<\mathbb{N}}})$.

Proof of Claim. Clearly closed sets can be represented in this form and the Borel sets may be obtained from the closed sets by iterating the operations of taking countable intersections and unions. Therefore it suffices to show that the class of sets that allow such representation are closed under countable unions and intersections.

So suppose $(B_i)_{i \in \mathbb{N}}$ are Borel sets and and at each $n \in \mathbb{N}$ we have

$$\mathcal{A}((A^n_s)_{s \in \mathbb{N}^{<\mathbb{N}}}) = B_n,$$

with each A^n_s closed, providing a representation of B_n as stated in the claim. If we let $A_{n^\frown s} = A^n_s$ for each $n \in \mathbb{N}$ and finite sequence s (where here $n^\frown s$ is the finite sequence defined by $n^\frown s(0) = n$, $n^\frown s(i+1) = s(i)$), then

$$\bigcup_{n \in \mathbb{N}} B_n = \mathcal{A}((A_s)_{s \in \mathbb{N}^{<\mathbb{N}}}).$$

For intersections we can partition $\mathbb{N}$ into infinite subsets $\{C_i : i \in \mathbb{N}\}$, with each C_i enumerated in increasing order by $(N(i,j))_{j \in \mathbb{N}}$. Then for $s \colon \{0,1,2,...,n\} \to \mathbb{N}$ with $n = N(i,j)$ we may define A'_s to be $A^i_{(s(N(i,0)),s(N(i,1)),...,s(N(i,j)))}$; in other words, if the final element of domain of s is the jth element in C_i and we define

$$t : \{0,1,...,j\} \to \mathbb{N}$$

by $t(k)$ being equal to the the value s assumes on the kth element of C_i, then

$$A'_s = A^i_t.$$

We then have

$$\bigcap_{n \in \mathbb{N}} B_n = \mathcal{A}((A'_s)_{s \in \mathbb{N}^{<\mathbb{N}}}).$$

(Claim(I)□)

Claim(II): For all $D \subset X$ with $D \in \underset{\sim}{\Sigma}^1_1$ there is a collection of closed sets $(A_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ in X with $D = \mathcal{A}((A_s)_{s \in \mathbb{N}^{<\mathbb{N}}})$.

Proof of claim: Let

$$f : Y \to X$$

be a Borel function and

$$B \subset Y$$

a Borel set with $f[B] = D$. Note that

$$\{(x,y) \in X \times Y : y \in B, f(y) = x\}$$

is Borel.

The idea of the proof of the claim is to produce $(A_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ such that an infinite function $h \in \mathbb{N}^{\mathbb{N}}$ will encode three functions $h_0, h_1, h_2 \in \mathbb{N}^{\mathbb{N}}$; these will be such that if $x \in \bigcap_n A_{h|_n}$, then h_2 will describe some $y \in Y$, h_0 will witness $(x', y) \in B$ and $f(y) = x'$, for some $x' \in X$, and h_1 will certify that $x' = x$.

By claim(I) there is a collection $(C_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ of closed sets with

$$\{(x,y)\in X\times Y : y\in B, f(y)=x\} = \mathcal{A}((C_s)_{s\in\mathbb{N}^{<\mathbb{N}}}).$$

First let assume without loss of generality that $C_t \subset C_s$ whenever $s\subset t$. Now for $s\in\mathbb{N}^n$ define $Pr_0(s), Pr_1(s), Pr_2(s)\in\mathbb{N}^{[n/3]}$, where $[n/3]$ is the largest integer $\leq n/3$, by

$$(Pr_0(s))(k) = s(3k)$$
$$(Pr_1(s))(k) = s(3k+1)$$
$$(Pr_2(s))(k) = s(3k+2).$$

Let d_0 be a complete metric on Y, d_1 on X. For $U\subset Y$ and $V\subset X$ we let $d_0(U)$ denote the supremum of $\{d_0(y_0,y_1) : y_0,y_1\in U\}$ and $d_1(V)$ denote the supremum of $\{d_1(x_0,x_1) : x_0,x_1\in V\}$. Choose $(U_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ open subsets of Y, $(V_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ open subsets of X, so that in both cases the empty function labels the whole space (i.e. $U_\emptyset = Y$, $V_\emptyset = X$), and for all $n\in\mathbb{N}$ and $s:\{0,1,...,n-1\}\to\mathbb{N}$

$$d_0(U_s) < 2^{-n},$$
$$U_s = \bigcup_{k\in\mathbb{N}} U_{s^\frown k},$$
$$d_1(V_s) < 2^{-n},$$
$$V_s = \bigcup_{k\in\mathbb{N}} V_{s^\frown k},$$

(where here for $s\in\mathbb{N}^n$ we define $s^\frown k\in\mathbb{N}^{n+1}$ by $s^\frown k|_n = s$ but $s^\frown k(n) = k$). For $n\in\mathbb{N}$ and $s:\{0,1,...,n-1\}\to\mathbb{N}$ let A_s be the set of $x\in X$ for which

(i) $x\in\overline{V_{Pr_1(s)}}$ (where $\overline{V_{Pr_1(s)}}$ indicates the closure of $V_{Pr_1(s)}$);

(ii) there exists x', y' with

(iia) $(x',y')\in C_{Pr_0(s)}$;

(iib) $x'\in\overline{V_{Pr_1(s)}}$;

(iic) $y'\in\overline{U_{Pr_2(s)}}$.

Thus if $x\in\mathcal{A}((A_s)_{s\in\mathbb{N}^{<\mathbb{N}}})$ we may find

$$x_i \to x,$$

$w\in\mathbb{N}^\mathbb{N}$, and a Cauchy sequence $(y_i)_{i\in\mathbb{N}}$ in Y with

$$(x_i,y_i)\in C_{w|_{\{0,1,\ldots,j\}}} =_{df} C_{w|_{j+1}}$$

for all $j<i$; so if y_∞ is the limit of $(y_i)_{i\in\mathbb{N}}$ then for all j

$$(x,y_\infty)\in C_{w|_{\{0,1,\ldots,j\}}}.$$

Hence $y_\infty\in B$ and $x = f(y_\infty)$ will be in D.

Conversely if $x\in D$ and $x = f(y)$, then by the assumption on the $(C_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ we can first choose $w\in\mathbb{N}^\mathbb{N}$ such that

$$(x,y)\in\bigcap_{n\in\mathbb{N}} C_{w|_n}.$$

Then a direct appeal to the definitions enables us to choose $h\in\mathbb{N}^\mathbb{N}$ with

$$x\in\bigcap_{n\in\mathbb{N}} A_{h|_n}.$$

(Claim(II)$\square$)

Returning to the theorem, fix closed sets A_s for $s \in \mathbb{N}^{<\mathbb{N}}$ such that

$$A = \mathcal{A}((A_s)_{s \in \mathbb{N}^{<\mathbb{N}}})$$

and attempt to show A has the Baire property . Assume that $A_s \supset A_t$ for $s \subset t$. For finite sequences s and t let $s^\frown t$ be the sequence obtained by following s by t – so that $s^\frown t(i) = s(i)$ for i less than the length of i, and $s^\frown t(l+i) = t(i)$ for l equal to the length of s. Note then that $\mathcal{A}((A_{s^\frown t})_{t \in \mathbb{N}^{<\mathbb{N}}})$ is in some sense equal to the points we can obtain in A given that we have decided to work below s.

Fix a countable basis for X. Let O_s equal the union of all the basic open sets U such that $\mathcal{A}((A_{s^\frown t})_{t \in \mathbb{N}^{<\mathbb{N}}}) \cap U$ meager, and let $C_s = X \setminus O_s$. (For the measure version of this theorem, we produce Borel $C_s \supset \mathcal{A}((A_{s^\frown t})_{t \in \mathbb{N}^{<\mathbb{N}}})$ with the same outer measure as $\mathcal{A}((A_{s^\frown t})_{t \in \mathbb{N}^{<\mathbb{N}}})$ and $C_s \subset A_s$. This in particular implies that any measurable $B \subset C_s$ disjoint from $\mathcal{A}((A_{s^\frown t})_{t \in \mathbb{N}^{<\mathbb{N}}})$ is null.)

Since each A_s is closed, we have $C_s \subset A_s$ at each $s \in \mathbb{N}^{<\mathbb{N}}$.

Let

$$M_0 = \bigcup_{n \in \mathbb{N}} \bigcup_{s \in \mathbb{N}^n} C_s \setminus (\bigcup \{C_t : t \in \mathbb{N}^{n+1}, t \supset s\}).$$

Claim(III): M_0 is meager.

Proof of claim: Suppose instead that some

$$C_s \setminus (\bigcup \{C_t : t \in \mathbb{N}^{n+1}, t \supset s\})$$

is relatively comeager in some basic non-empty open $U \subset X$. Then at each $k \in \mathbb{N}$ we have from the definition of $C_{s^\frown k}$ that

$$U \cap \mathcal{A}((A_{s^\frown k^\frown t})_{t \in \mathbb{N}^{<\mathbb{N}}})$$

is meager in U, and hence

$$\mathcal{A}((A_{s^\frown t})_{t \in \mathbb{N}^{<\mathbb{N}}}) = \bigcup_{k \in \mathbb{N}} \mathcal{A}((A_{s^\frown k^\frown t})_{t \in \mathbb{N}^{<\mathbb{N}}})$$

is meager in U, and thus $U \cap C_s$ is empty, as required for a contradiction. (Claim(III)□)

Let

$$M_1 = \bigcup_{s \in \mathbb{N}^{<\mathbb{N}}} \mathcal{A}((A_{s^\frown t})_{t \in \mathbb{N}^{<\mathbb{N}}}) \setminus C_s.$$

Claim(IV): M_1 is meager.

Proof of claim: From the definition of C_s we have that it is a countable union of meager sets of the form $\mathcal{A}((A_{s^\frown t})_{t \in \mathbb{N}^{<\mathbb{N}}}) \cap U$, and hence meager. (Claim(IV)□)

Claim(V): $C_\emptyset \Delta A =_{df} (A \setminus C_\emptyset) \cup (C_\emptyset \setminus A)$ is a subset of $M_0 \cup M_1$.

Proof of claim: If $x \in C_\emptyset$ and $x \notin M_0$, then invoking the definition of M_0 allows us to successively choose increasing finite sequences $s_0 \subset s_1 .... \subset s_n \subset s_{n+1} ...$ such that at each n

$$x \in C_{s_n}$$

$$\therefore x \in A_{s_n}.$$

Conversely, if $x \in A = \mathcal{A}((A_s)_{s \in \mathbb{N}^{<\mathbb{N}}})$ but not in M_1 we have $x \in C_\emptyset$ by the definition of M_1. (Claim(V)□)

Claim(V) implies that since $C_\emptyset$ has the the Baire property so too does A.

□

THEOREM 2.54. *(Jankov, von Neumann) Let X, Y be Polish and $A \subset X \times Y$ in $\utilde{\Sigma}^1_1$. Then A can be uniformized by a function that is both Baire and universally measurable, in the sense that for some $h : X \to Y$ we have that for all x and y*

$$(x, y) \in A \Rightarrow (x, h(x)) \in A$$

with $h^{-1}[U]$ having the Baire property and being measurable with respect to any Borel measure for all open $U \subset X$.

PROOF. The idea of the proof is to define for each x a canonical $h(x)$ and $f(x)$ such that

$$(x, h(x)) \in \bigcap_{n \in \mathbb{N}} A_{f(x)|_n}.$$

This will be defined using what is in some sense the "least" possible choice for $h(x)$ and each $(f(x))(n)$.

Suppose $A = \mathcal{A}((A_s)_{s \in \mathbb{N}^{<\mathbb{N}}})$, each A_s closed. Let $\{U_l : l \in \mathbb{N}\}$ be a basis for Y and d a complete metric. For $x \in X$ in the projection of A define $N_{x,n}, M_{x,n} \in \mathbb{N}$ so that at each $n \in \mathbb{N}$ we choose $N_{x,n}$ least so that there exists $f \in \mathbb{N}^{\mathbb{N}}$ and

$$y \in \bigcap_{m<n} U_{M_{x,m}}$$

with

$$(x, y) \in \bigcap_{k \in \mathbb{N}} A_{f|_k}$$

(where here $f|_k$ stands for $f|_{\{0,1,\ldots,k-1\}}$, the restriction of f to its first k values) and such that for all $i \leq n$

$$f(i) = N_{x,i};$$

if there is no y with $(x, y) \in A$ just let $N_{x,n}$ and $M_{x,n}$ always assume the default value 0. Then let $M_{x,n}$ be least such that $d(U_{M_{x,n}}) < 2^{-n}$, and $\overline{U_{M_x,n}} \subset U_{M_x,n-1}$, and there is some $f \supset (N_{x,0}, N_{x,1}, \ldots N_{x,n})$ (that is to say f extends the finite sequence whose first term is $N_{x,0}$, whose second term is $N_{x,1}$, ..., and so on, with final term $N_{x,n}$) and there is some

$$y \in \bigcap_{m \leq n} U_{M_{x,m}}$$

with

$$(x, y) \in \bigcap_{k \in \mathbb{N}} A_{f|_k}.$$

By induction on n we see that for each $s, t \in \mathbb{N}^{\{0,1,\ldots,n-1\}}$ the set $\{x : \forall i < n(M_{x,i} = s(i), N_{x,i} = t(i))\}$ is in the σ-algebra generated by the $\utilde{\Sigma}^1_1$ sets. Thus if we let $h(x) \in \mathbb{N}^{\mathbb{N}}$ be given by

$$\{h(x)\} = \bigcap_{m \in \mathbb{N}} U_{M_{x,m}}$$

then we obtain that for any open U the pullback $h^{-1}[U]$ is again in the least σ-algebra containing the $\utilde{\Sigma}^1_1$ sets and therefore is measurable and has the Baire property .

For any x for which there exists y with $(x, y) \in A$ we obtain by induction on n that at each stage

$$\exists y_n \in \bigcap_{m<n} U_{M_{x,m}}$$

with
$$(x, y_n) \in \bigcap_{k \leq n+1} A_{(N_{x,0}, N_{x,1}, \ldots N_{x,k})}.$$
Then since
$$y_n \to h(x)$$
and each A_s is closed we obtain
$$(x, h(x)) \in \bigcap_{k \in \mathbb{N}} A_{(N_{x,0}, N_{x,1}, \ldots, N_{x,k})}.$$
□

THEOREM 2.55. *(Novikov) Let $A \subset X \times Y$ be $\underset{\sim}{\Sigma}^1_1$, X and Y Polish. Then the sets $\{x \in X : \forall^* y \in Y((x, y) \in A)\}$ and $\{x : \exists^* y \in Y((x, y) \in A)\}$ are $\underset{\sim}{\Sigma}^1_1$.*

PROOF. Recalling claim(II) from the proof of 2.53, suppose $A = \mathcal{A}((A_s)_{s \in \mathbb{N}^{<\mathbb{N}}})$, each A_s closed, $A_t \subset A_s$ whenever $t \supset s$. Assume $A_\emptyset = X \times Y$. Following 2.54 and 2.49, for $x \in X$,
$$\forall^* y \in Y((x, y) \in A)$$
if and only if there is a comeager $C \subset Y$ and continuous $f : C \to \mathbb{N}^{\mathbb{N}}$ so that for all $y \in C$ and $n \in \mathbb{N}$
$$(x, y) \in A_{f(y)|_n}.$$
With further manipulation of the definitions we show the set of x for which such C and f exist to be $\underset{\sim}{\Sigma}^1_1$:

Fix enumerations $\{U_n : n \in \mathbb{N}\}$ and $\{W_n : n \in \mathbb{N}\}$ of the non-empty basic open sets in Y and X. Let d be a complete metric on Y. Let B be the set of all $(x, h) \in X \times (\mathbb{N}^{<\mathbb{N}})^{\mathbb{N}}$ such that

(i) $h(l) = s$ implies that for all W_n containing x there exists
$$y' \in U_l$$
and
$$x' \in W_n$$
with
$$(x', y') \in A_s;$$
(ii) for all U_k nonempty and $m \in \mathbb{N}$ there exists some $l \in \mathbb{N}$, $n > m$, and $s : \{0, 1, ..., n\} \to \mathbb{N}$ with
$$d(U_l) < 2^{-m},$$
$$U_l \subset U_k$$
and
$$h(l) = s;$$
(iii) $U_l \subset U_k$ and $h(l) = s$, $h(k) = t$ imply $s \supset t$.

This set B is Borel; in fact, (ii) is no worse than requiring (x, h) be in a G_δ set while (i) and (iii) are closed.

Claim: $\exists h((x, h) \in B)$ if and only if $\forall^* y \in Y((x, y) \in A)$.

Proof of claim: First suppose $(x, h) \in B$. For each m let $\mathcal{O}_m$ be the union of open sets U_l for which
$$d(U_l) < 2^{-m}$$
and there exists s: $\{0, 1, ..., n\} \to \mathbb{N}$ for some $n > m$ with
$$h(l) = s.$$

Let
$$C_h = \bigcap_{m\in\mathbb{N}} \mathcal{O}_m.$$
Condition (ii) indicates that this set must be comeager. In virtue of (iii) we have that h defines a continuous $f_h : C_h \to \mathbb{N}^{\mathbb{N}}$ by
$$f_h(y) \supset s \Leftrightarrow \exists l(y \in U_l \wedge h(l) \supset s).$$
With (i) we have that whenever $y \in C_h$ we obtain $y_i \to y$ and $x_i \to x$ with each
$$(x_n, y_n) \in A_{f_h(y)|_n}$$
and hence
$$\bigcap_{n\in\mathbb{N}} (x, y) \in A_{f_h(y)|_n}.$$

Conversely if $\forall^* y \in Y((x, y) \in A)$ then we may by 2.54 and 2.49 find a comeager $C \subset Y$ and continuous $f : C \to \mathbb{N}^{\mathbb{N}}$ such that for all $y \in C$
$$(x, y) \in \bigcap_{n\in\mathbb{N}} A_{f_h(y)|_n}.$$
Then we define h by the specification that $h(l) = s$ if $s \in \mathbb{N}^{\leq l}$ is as long as possible subject to the requirement that $f(y) \supset s$ all $y \in U_l \cap C$. (□Claim)

Thus for $\pi_0 : X \times (\mathbb{N}^{<\mathbb{N}})^{\mathbb{N}} \to X$ given by $(x, h) \mapsto x$
$$\{x \in X : \forall^* y \in Y((x, y) \in A)\} = \pi_0[B].$$

The case of $\exists^*$ is similar. □

LEMMA 2.56. *(Pettis) Let G be a Polish group.*

(i) If $A \subset G$ with the property of Baire and non-meager then AA^{-1} contains an open neighborhood of the identity.

(ii) Let H be a Polish group, $\pi : G \to H$ be a Borel homomorphism. Then π is continuous.

(iii) Moreover if π is onto then it is open.

(iv) If $A, B \subset G$ are both non-meager with the Baire property then AB contains an open set.

PROOF. (i): Choose U an open neighborhood of the identity and $g \in G$ with A comeager in Ug, and then W, V open neighborhoods of the identity with
$$W^{-1}V \subset U,$$
which amounts to asserting that for all $\epsilon \in W$
$$V \subset \epsilon U$$
$$\therefore Vg \subset \epsilon Ug.$$

Thus for all $\epsilon \in W$ the translation ϵA is still comeager in Vg. Thus there exists some
$$\delta \in \epsilon A \cap A$$
and
$$\therefore \epsilon \in \delta A^{-1} \subset AA^{-1}.$$

(ii) is a corollary of (i): If U is any non-empty set in H and $Q \subset H$ is a countable dense subset of $\pi[G]$ then
$$Q \cdot U \supset \pi[G].$$

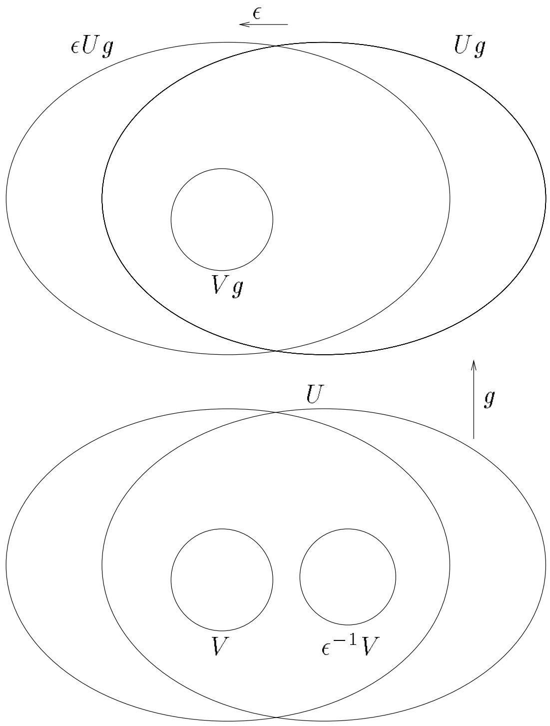

FIGURE 2.1. The shifts of U

Hence for $Q' \subset G$ countable with $\pi[Q'] = Q$ we have

$$Q' \cdot \pi^{-1}[U] \supset G$$

and thus for some $g \in G$ the set $g \cdot \pi^{-1}[U]$ is nonmeager, and hence $\pi^{-1}[U]$ is nonmeager.

For every W open neighborhood of the identity in H we can find U open containing the identity with $U^{-1}U \subset W$, and then since $\pi^{-1}[U]$ nonmeager, $\pi^{-1}[W]$ contains an open set. More generally given an arbitrary open set $W \subset H$ and $g \in \pi^{-1}[W]$, we can find open neighborhood O of 1 in G such that

$$O \subset \pi^{-1}[\pi(g^{-1})W] = g^{-1}\pi^{-1}[W]$$

$$\therefore gO \subset \pi^{-1}[W].$$

(iii) resembles (ii).

(iv): We can find open neighborhood V of the identity and $g, h \in G$ for which gA is relatively comeager in V and Bh is relatively comeager in V. It suffices to show then that $gABh$ includes an open set, which in turn follows by the proof of (i). □

So in particular a Borel group has at most one Polish group topology compatible with its Borel and algebraic structure.

DEFINITION 2.57. A subset P of a Polish space is *perfect* if it is closed, nonempty, and has no isolated points.

For instance, the unit interval is perfect; the Cantor space $2^{\mathbb{N}}$ is perfect.

THEOREM 2.58. *(Kuratowski; Mycielski) Let X be a Polish space without isolated points and suppose that for each $n \in \mathbb{N}$ the relation*

$$R_n \subset X^n$$

is meager as a subset of the n-fold product of X. Then there is a perfect $P \subset X$ such that for all distinct $x_1, ..., x_n \notin P$

$$(x_1, ..., x_n) \notin R_n,$$

and continuous bijection $\pi : 2^{\mathbb{N}} \to P$.

PROOF. Just for the case of a single $R \subset X \times X$; the general case is similar.

Choose a sequence of open dense sets $\{O_n : n \in \mathbb{N}\}$ in $X \times X$ such that

$$R \cap \bigcap_{n \in \mathbb{N}} O_n = \emptyset$$

$$O_{n+1} \subset O_n.$$

Let d be a complete metric on X; in what follows recall the notation that $s \in 2^n$ indicates that $s : \{0, 1, ..., n-1\} \to \{0, 1\}$ is a function from the first n elements of $\mathbb{N}$ to $\{0, 1\}$. We now choose open $\{U_s : s \in 2^{<\mathbb{N}}\}$ in X level by level as follows:

(i) U_s nonempty for all s;

(ii) for $s \subset t$

$$U_t \subset U_s;$$

if moreover $s \neq t$ then $\overline{U_t}$ (the closure of U_t) is included in U_s;

(iii) for $s \neq t \in 2^n$

$$U_s \times U_t \subset O_n;$$

(iv) for $s \in 2^n$

$$d(U_s) < 2^{-n},$$

where $d(U_s)$ is $\sup\{d(x, y) : x, y \in U_s\}$;

(v) $U_s \cap U_t$ nonempty implies $s \subset t$ or $t \subset s$.

Once all this has been done just take

$$P = \bigcup_{f \in 2^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} U_{f|_n}.$$

For any $f \in 2^{\mathbb{N}}$ we can define $\pi(f)$ with

$$\{\pi(f)\} = \bigcap_{n \in \mathbb{N}} U_{f|_n} \neq \emptyset.$$

Thus $\pi : 2^{\mathbb{N}} \to P$ is one to one, continuous, and onto, so P must be perfect. By (iii) $f \neq f'$ implies

$$(\pi(f), \pi(f')) \in \bigcap_{n \in \mathbb{N}} O_n$$

$$\therefore (\pi(f), \pi(f')) \notin R.$$

As for showing that this construction can be completed, suppose inductively we have built $(U_s)_{s \in 2^n}$ for some $n \in \mathbb{N}$. Choose $V_s^0, V_s^1 \subset U_s$ open disjoint nonempty subsets with diameter less than 2^{-n-1} and with closures included in U_s – and this is possible by the assumption that X has no isolated points. Let $(t(i), s(i))_{i<K(n)}$ be an enumeration of all pairs of distinct functions from $\{0, 1, ..., n\}$ to $\{0, 1\}$. We build W_t^i by induction on $i \leq K(n)$, starting with

$W_t^0 = V_s^0$ if $t = s^\frown 0$ for some s

and otherwise

$W^0_t = V^1_s$ if $t = s^\frown 1$ for some s.

Given $(W^i_s)_{s\in 2^{n+1}}$ one finds $W^{i+1}_{s(i)} \subset W^i_{s(i)}$ and $W^{i+1}_{t(i)} \subset W^i_{t(i)}$ so that

$$W^{i+1}_{s(i)} \times W^{i+1}_{t(i)} \subset O_n.$$

This is possible by O_n being dense open. For $t \neq s(i), t(i)$ just set $W^{i+1}_t = W^i_t$. Finally with $(U_t)_{t\in 2^{n+1}} = (W^{K(n)}_t)_{t\in 2^{n+1}}$ we continue the process one further step. $\square$

COROLLARY 2.59. *Let E be a meager equivalence relation on a Polish space X. Then*

$$\mathrm{id}(2^{\mathbb{N}}) \sqsubseteq_c E.$$

PROOF. Since $\mathrm{id}(X)$ must therefore be meager in $X \times X$, the Polish space can have no isolated points and so we can apply 2.58. $\square$

EXERCISE 2.60. Let $f : X \to Y$ be Borel and $A \subset Y$ a $\underset{\sim}{\Sigma}^1_1$ set. Then $f^{-1}[A] \in \underset{\sim}{\Sigma}^1_1$.

EXERCISE 2.61. (i) The collection of $\underset{\sim}{\Sigma}^1_1$ subsets of a Polish space are closed under countable intersections and unions.

(ii) Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of Borel subsets of a product of Polish spaces $X \times Y$. Then the set of $x \in X$ such that

$$\forall n \in \mathbb{N}\, \exists y \in Y((x, y) \in A_n)$$

is $\underset{\sim}{\Sigma}^1_1$.

(iii) If $A \subset X_1 \times X_2 \times .... \times X_n \times X_{n+1}$ is a $\underset{\sim}{\Sigma}^1_1$ subset of a product of Polish spaces, then

$$\{\vec{x} \in X_1 \times ... \times X_n : \exists a \in X_{n+1}(\vec{x}^\frown a \in A)\}$$

is again $\underset{\sim}{\Sigma}^1_1$.

EXERCISE 2.62. If G is a Polish group and $H < G$ a Borel subgroup, then H has countable index (i.e. $|G/H| \leq \aleph_0$) if and only if H is clopen.

EXERCISE 2.63. There is a perfect $P \subset \mathbb{R}$ such that all distinct $x_1, x_2, ..., x_n \in P$ are algebraically independent over $\mathbb{Q}$.

EXERCISE 2.64. Show that if F is an equivalence relation on a Polish space with every equivalence class countable, then

$$F_2 \not\leq_B F.$$

(By analogy with 2.31, for $\theta : B_2 \to X$ a Borel function it suffices to use 2.46, 2.47, and 2.49 to find some comeager $C \subset B_2$ on which the function $\theta|_C$ is continuous, and then to find some $x \in B_2$ so that $\forall^* g \in S_\infty$ and for all finite $F \subset \mathbb{N}$ the set

$$C \cap \{y \in 2^{\mathbb{N}\times\mathbb{N}} : \forall i \in F(y(i, \cdot) = g \cdot x(i, \cdot))\}$$

is relatively comeager in

$$\{y \in 2^{\mathbb{N}\times\mathbb{N}} : \forall i \in F(y(i, \cdot) = g \cdot x(i, \cdot))\},$$

in the sense that if we view $2^{\mathbb{N}\times\mathbb{N}}$ as decomposed into $2^{F\times\mathbb{N}} \times 2^{(\mathbb{N}\setminus F)\times\mathbb{N}}$ then

$$\forall^* z \in 2^{(\mathbb{N}\setminus F)\times\mathbb{N}}(g \cdot x|_{2^{F\times\mathbb{N}}}, z) \in C.$$

This method of approximating Borel reductions by continuous reappears at 3.14 and 3.18.)

EXERCISE 2.65. Let $E \subset X \times X$ be a $\underset{\sim}{\Sigma}_1^1$ equivalence relation.

(i) E is meager if and only if every equivalence class is meager.

(ii) If E has only countably many classes then for all $U \subset X$ open and non-empty there is some $V \subset U$ open and non-empty with E comeager in $V \times V$.

There is a non-trivial notion of the *potential* Borel complexity of an orbit equivalence relation. For the case of G a closed subgroup of S_∞ this was exactly the subject of [**41**].

EXERCISE 2.66. Let G be a Polish group and X a Polish G-space.

(i) Suppose F is a $G_{\delta\sigma}$-equivalence relation and θ is a Borel function witnessing $E_G^X \leq_B F$. Then for $C \subset X$ a dense G_δ on which θ is continuous and $x \in X$ such that $\forall^* g \in G(g \cdot x \in C)$ we have $[x]_G \in G_{\delta\sigma}$.

(ii) Show that c_0 is not $G_{\delta\sigma}$ by observing that if $A \subset c_0$ is G_δ in the product topology on $\mathbb{R}^\mathbb{N}$ then A is nowhere dense in the sup norm topology on c_0, and hence meager as a subset of c_0. (If $A = \bigcap_{i \in \mathbb{N}} U_i$ is a G_δ subset of c_0 not nowhere dense in the topology of c_0, then after possibly translating, we may assuming A is dense inside

$$\{\vec{x} \in c_0 : \sup|x_i| < \epsilon\},$$

for some $\epsilon > 0$. Then we may continually choose $k_0 < k_1 < ... < k_n < k_{n+1} ...$ and bounded open sets $V_n \subset [-\epsilon, \epsilon]^{k_n}$ such that for any

$$(z_0, z_1, ...) = \vec{z} \in \overline{V_{n+1}}$$

we have

$$\vec{z}|_{k_n} \in V_n,$$
$$|z_{k_n}| \geq \frac{\epsilon}{2},$$

and for any $\vec{x} \in \mathbb{R}^\mathbb{N}$ with $\vec{x}|_{k_n} \in V_n$ we have

$$\vec{x} \in U_n.$$

Now compactness allows us to choose $\vec{y}$ so that at every n

$$\vec{y}|_{k_n} \in \overline{V_n}$$
$$\therefore \vec{y}|_{k_{n-1}} \in V_{n-1}.$$

$\vec{y}$ cannot be in c_0 since at each n we have $|y_{k_n}| \geq \epsilon/2$, which contradicts $\vec{y} \in A = \bigcap_{i \in \mathbb{N}} U_i$.)

(iii) Let us say that an orbit equivalence relation is *potentially* $G_{\delta\sigma}$ if it is $\leq_B$-reducible to some $G_{\delta\sigma}$ equivalence relation. From (ii) and (iii) we have that the coset equivalence relation of c_0 in $\mathbb{R}^\mathbb{N}$ is not potentially $G_{\delta\sigma}$.

CHAPTER 3

Turbulence

Here we discuss various sufficient conditions for non-classifiability. §3.1 begins with smoothness, proving that *properly generically ergodic* group actions induce non-smooth orbit equivalence relations. This is the model for §3.2, which introduces the more elaborate notion of *turbulence* and in turn establishes radical non-classifiability results for actions in this class. §3.3 presents some examples and applications. §3.4 consists in historical remarks.

3.1. Generic ergodicity

DEFINITION 3.1. Let G be Polish group and X a Polish G-space. The action is said to be *properly generically ergodic* if:
(i) every orbit is dense; and
(ii) every orbit is meager.
By analogy we may say that the orbit equivalence relation is *properly generically ergodic* if it arises, or can be made to arise, from such an action.

THEOREM 3.2. *(classical) Suppose G is a Polish group, X is a Polish G-space, and every orbit in X is dense. Suppose $\theta : X \to \mathbb{R}$ is Baire measurable and G-invariant (in the sense that $\forall x \in X g \in G(\theta(x) = \theta(g \cdot x))$). Then there is a comeager set $C \subset X$ on which θ is constant.*

PROOF. Let C be a comeager set on which θ is continuous. Let $x \in X$ be such that $\forall^* g \in G(g \cdot x \in C)$; x exists by 2.47. Then $[x]_G \cap C$ is dense in C, so by continuity θ is constant on C. □

Thus we obtain a sufficient condition for an orbit equivalence relation to be non-smooth.

COROLLARY 3.3. *Suppose G is a Polish group and X is a properly generically ergodic Polish G-space. Then $X/G \not\leq_B \mathrm{id}(\mathbb{R})$.*

In the proof of 3.2 we did not need every orbit dense; only a comeager set of dense orbits. But if one orbit is dense then generically so are all.

LEMMA 3.4. *Suppose G is a Polish group and X is a Polish G-space. Suppose some orbit is dense. Then the set of dense orbit is an invariant dense G_δ.*

PROOF. Let $\{U_n : n \in \mathbb{N}\}$ enumerate the non-empty basic open sets in X. Then for each l the set of $x \in X$ such that

$$[x]_G \cap U_l \neq \emptyset$$

is invariant and open. Thus the set of dense orbits is an invariant G_δ.
Therefore as soon as we know this set to be non-empty we are done. □

The next statement foreshadows lemmas from §3.2.

COROLLARY 3.5. *Suppose G is a Polish group and X is a Polish G-space. Suppose some orbit is dense and every orbit is meager.*

Then there is an invariant dense G_δ set $X_0 \subset X$ such that X_0 is a properly generically ergodic Polish G-space. In particular $E_G^X \not\leq_B \mathrm{id}(\mathbb{R})$.

PROOF. Let X_0 be the set of points with dense orbits, and note the notions of categoricity coincide between X and X_0 since X_0 is comeager by 3.4. □

By the same proof we obtain $E_G^X \not\leq_B \mathrm{id}(Y)$ for any Polish space Y.

DEFINITION 3.6. Let E and F be equivalence relations on X and Y. E is said to be *generically F-ergodic* if whenever $\theta : X \to Y$ is Baire measurable with $\forall x_1, x_2 \in X$,

$$x_1 E x_2 \Rightarrow \theta(x_1) F \theta(x_2),$$

then we must have some comeager set $C \subset X$ such that for all $x_1, x_2 \in C$

$$\theta(x_1) F \theta(x_2).$$

So 3.2 states that any properly generically ergodic orbit equivalence relation is $\mathrm{id}(\mathbb{R})$-ergodic, and in particular, since every orbit is meager, the equivalence relation is non-smooth.

EXERCISE 3.7. Let G be a countable infinite group. Show that $2^G/G$ (in the shift action from 2.24) is not smooth. (Hint: The main issue is the existence of some dense orbit. But given finite $F, F' \subset G$ and basic open sets

$$U = \{f \in 2^G : \forall g \in F(f(g) = i_g)\},$$
$$U' = \{f \in 2^G : \forall g \in F'(f(g) = j_g)\},$$

where each $i_g, j_g \in \{0, 1\}$, we may obtain $h \in G$ such that $h \cdot F \cap F' = \emptyset$ and hence $h \cdot U \cap U' \neq \emptyset$. This point granted, a diagonalization argument enables us to produce a sequence of non-empty basic open sets $(U_i)_{i \in \mathbb{N}}$, each

$$\overline{U_{i+1}} \subset U_i,$$

and

$$d(U_i) < 2^{-i},$$

with respect to a complete metric d on 2^G, so that for any basic open set W we have some i and $h \in G$ with $h \cdot U_i \subset W$. For $\{f\} = \bigcap U_i$ we have $[f]_G$ dense.)

EXERCISE 3.8. Show: $\mathrm{id}(\mathbb{R}) <_B E_0$; $\mathrm{id}(\mathbb{R}) <_B \mathbb{R}/\mathbb{Q}$; $\mathrm{id}(\mathbb{R}) <_B 2^{\mathbb{Z}}/\mathbb{Z}$. (3.3 is applicable to E_0, since this equivalence relation arises from an action of the infinite rank $\mathbb{Z}_2$-module on $2^{\mathbb{N}}$; compare for instance 7.9.)

We will see in §7 that if X is a properly generically ergodic Polish G-space then E_0 is continuously reducible to E_G^X.

EXERCISE 3.9. Let G be a Polish group and let X be a Polish G-space. Show that if E_G^X is generically $\mathrm{id}(\mathbb{R})$-ergodic, then

$$\forall^* x \in X([x]_G \text{ is dense}).$$

EXERCISE 3.10. The isomorphism relation on rank one torsion free abelian groups is not smooth. Before proving or even precisely stating this it is necessary to work out some definitions:

(i) (The set of models displaying group structures on $\mathbb{N}$ forms a Polish S_∞-space.) Let $R_\bullet(\cdot, \cdot, \cdot)$ be a three placed relation and let $R_{-1}(\cdot, \cdot)$ be a two placed relation and let $\mathcal{L}$ be the language generated by these two. Let Mod(Grp) $\subset$Mod($\mathcal{L}$) be the set of $\mathcal{M}$ for which:

(a) for all $a, b \in \mathbb{N}$ there exists $c, d \in \mathbb{N}$ so that

$$\mathcal{M} \models R_\bullet(a, b, c) \wedge R_{-1}(a, d)$$

(b) for all $a, b, c, c', d, d' \in \mathbb{N}$

$$\mathcal{M} \models (R_\bullet(a, b, c) \wedge R_\bullet(a, b, c')) \Rightarrow c = c'$$

$$\mathcal{M} \models (R_{-1}(a, d) \wedge R_{-1}(a, d')) \Rightarrow d = d';$$

and if we therefore define $(\bullet(\cdot, \cdot))^{\mathcal{M}} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $((\cdot)^{-1})^{\mathcal{M}} : \mathbb{N} \to \mathbb{N}$ accordingly, so that for all a, b

$$\mathcal{M} \models R_\bullet(a, b, (\bullet(a, b))^{\mathcal{M}})$$

$$\mathcal{M} \models R_{-1}(a, ((a)^{-1})^{\mathcal{M}})$$

then

(c) $(\bullet(\cdot, \cdot))^{\mathcal{M}}$ defines a group operation on $\mathbb{N}$ with inverse $((\cdot)^{-1})^{\mathcal{M}}$ (so that for all a, b, c, d, e, f, g associativity amounts to

$$\mathcal{M} \models (R_\bullet(a, b, d) \wedge R_\bullet(d, c, e) \wedge R_\bullet(b, c, f) \wedge R_\bullet(a, f, g)) \Rightarrow g = e$$

while the defining condition on the inverse for a group comes out as for all a, b, c, d

$$\mathcal{M} \models (R_{-1}(a, b) \wedge R_\bullet(a, b, c)) \Rightarrow (R_\bullet(c, d, d) \wedge R_\bullet(d, c, d)),$$

$$\mathcal{M} \models (R_{-1}(a, d) \Rightarrow R_{-1}(d, a));$$

Show that Mod(Grp) is an invariant G_δ subset of Mod($\mathcal{L}$) and thereby forms a Polish S_∞-space.

(ii) (The torsion free abelian rank 1 groups form a Polish S_∞-space.) Let TFA_1 consist of all $\mathcal{M} \in$ Mod(Grp) such that:

(d) (abelian) for all $a, b, d, d' \in \mathbb{N}$

$$\mathcal{M} \models (R_\bullet(a, b, d) \wedge R_\bullet(b, a, d')) \Rightarrow d = d';$$

(e) (torsion free) for all $a = a_0, a_1, ...a_n \in \mathcal{M}$ such that at each $i < n$

$$\mathcal{M} \models R_\bullet(a, a_i, a_{i+1})$$

$$\mathcal{M} \models R_\bullet(a_n, a_n, a_n)$$

we have

$$\mathcal{M} \models R_\bullet(a, a, a);$$

(f) (rank one) for all $a, b \in \mathbb{N}$ there are $a_1 = a, a_2, ..., a_n$ and $b_1 = b, b_2, ..., b_k$ so that at each $i < n$

$$\mathcal{M} \models R_\bullet(a, a_i, a_{i+1})$$

and for $i < k$

$$\mathcal{M} \models R_\bullet(b, b_i, b_{i+1})$$

and

$$a_n = b_k$$

(in other words $n \cdot a = k \cdot b$).

Show that TFA_1 is an invariant G_δ subset of Mod(Grp).

(iii) Let $(\mathbb{Q}^+, \cdot)$ be the multiplicative group of the positive rationals. Let $2^{\mathbb{Q}}$ be the set of $f : \mathbb{Q} \to \{0,1\}$ with the usual product topology, so that basic open sets have the form

$$\{f \in 2^{\mathbb{Q}} : f(q_1) = i_1, ..., f(q_n) = i_n\}.$$

Let X be the space of $f \in 2^{\mathbb{Q}}$ for which the set A_f of $q \in \mathbb{Q}$ with $f(q) = 1$ forms a subgroup of $(\mathbb{Q}, +)$. Let $(\mathbb{Q}^+, \cdot)$ act on X by

$$\hat{q} \cdot f(q) = f(q\hat{q}).$$

(iv) Show that X is a Polish $(\mathbb{Q}^+, \cdot)$-space.

(v) Use 3.5 to show that $E^X_{(\mathbb{Q}^+,\cdot)}$ is non-smooth.

(vi) Show that $X/(\mathbb{Q}^+, \cdot) \leq_B \mathrm{TFA}_1/S_\infty$ and hence conclude that isomorphism relation on elements of $\mathrm{Mod}(\mathcal{L})$ corresponding to torsion free rank 1 abelian groups is non-smooth.

(vii) On the other hand, show that the action of S_∞ on TFA_1 is not properly generically ergodic since there *is* a dense G_δ orbit.

It is implicit in [**1**] that we conversely obtain $\mathrm{TFA}_1/S_\infty \leq_B E_0$. Since E_0 can be thought of as a kind of Borel instantiation of the subsets of $\mathbb{N}$ considered up to finite difference, this is tantamount to classifying rank one torsion free abelian groups by elements of $\mathcal{P}(\mathbb{N})/\text{Finite}$. The method of proof for this is to take a rank one torsion free abelian group G with underlying set $\mathbb{N}$ and assign $k_G \in \mathbb{N}$ to be the first non-zero element of G. One then defines a function $\theta(G)$ in $2^{\mathbb{N}\times\mathbb{N}}$, where for p_n the nth prime we let $(\theta(G))(m,n) = 1$ if and only if $(p_n)^m$ divides k_G. With some effort one can show that for any two G_1 and G_2 we have $\theta(G_1)E_0(2^{\mathbb{N}\times\mathbb{N}})\theta(G_2)$ if and only if $G_1 \cong G_2$.

Exercise 3.11. The action of S_∞ on itself by conjugation is smooth. (The number of n-cycles for $n \in \{1, 2, 3, ..., \infty\}$ is a complete invariant.)

The next exercise may seem artificial. The point is that this is exactly kind of reduction one obtains by the Ulm invariants (see [**48**]) for countable abelian p-groups. The definition itself is technical, and does not play an important part until chapter 9.

Exercise 3.12. Recall that ω_1 is the first uncountable ordinal, so that $(\omega_1, <)$ is an uncountable well ordered set all of whose proper initial segments are countable. Let $2^{<\omega_1}$ be the set of all bounded subsets of ω_1 – that is to say the $A \subset \omega_1$ for which there is some $\alpha \in \omega_1$ such that for all $\beta \in A$ we have $\beta < \alpha$.

Let us say that a function $\theta : X \to 2^{<\omega_1}$ from a standard Borel space space to the bounded subsets of ω_1 is *UBMC* (short for *universally Baire measurable in the codes*) if we have the following:

whenever $g : Y \to X$ is a Borel function from a Polish space there will be some $\alpha \in \omega_1$ and comeager $C \subset Y$ such that:

(i) for all $y \in C$

$$\theta(g(y)) \subset \{\beta \in \omega_1 : \beta < \alpha\};$$

(ii) if $\pi : \mathbb{N} \to \{\beta \in \omega_1 : \beta < \alpha\}$ is a bijection, then the function

$$C \to 2^{\mathbb{N}}$$

$$y \mapsto \{n \in \mathbb{N} : \pi(n) \in \theta(g(y))\}$$

is Borel. (The issue of this function being Borel does not depend on the choice of π.)

Let us say that an equivalence relation E on a standard Borel space X is *UBMC reducible to* $2^{<\omega_1}$ if there a UBMC function $\theta : X \to 2^{<\omega_1}$ such that for all $x_0, x_1 \in X$

$$x_0 E x_1 \Leftrightarrow \theta(x_0) = \theta(x_1).$$

Show that if $E_0 \leq_B E$ then E is not UBMC reducible to $2^{<\omega_1}$.

One can define a similar notion for functions from a space to the hereditarily countable sets. Thus we can say

$$\theta : X \to \mathrm{HC}$$

is UBMC if whenever $g : Y \to X$ is Borel then there is a comeager set $C \subset Y$ and Borel

$$y \mapsto \mathcal{M}_y$$
$$C \to \mathrm{Mod}(\mathcal{L}),$$

where $\mathcal{L}$ is the language generated by a single binary relation E, such that for all $y \in C$

$$\mathcal{M}_y \cong (\{\theta(g(y))\} \cup \mathrm{TC}(\theta(g(y))), \in);$$

in other words, on the comeager set C we have a Borel manner of supplying representatives of the hereditarily countable set $\theta(g(y))$.

3.2. The definition of turbulence

DEFINITION 3.13. Let G be a Polish group acting continuously on a Polish space X. The action is said to be *turbulent* if:

(i) every orbit is dense;

(ii) every orbit is meager;

(iii) for all $x, y \in X$, $U \subset X$, $V \subset G$ open with $x \in U$, $1 \in V$, there exists $y_0 \in [y]_G$ and $(g_i)_{i\in\mathbb{N}} \subset V$, $(x_i)_{i\in\mathbb{N}} \subset U$ with

$$x_0 = x,$$

$$x_{i+1} = g_i \cdot x_i,$$

and for some subsequence $(x_{n(i)})_{i\in\mathbb{N}} \subset (x_i)_{i\in\mathbb{N}}$

$$x_{n(i)} \to y_0.$$

Part (i) in this definition is implied by (iii), but left in place to indicate the resemblance with 3.2. (i) and (ii) are already familiar from §3.1. Rather it is part (iii) which desires some comment.

Roughly speaking we might view condition (i) as saying that every orbit acts *chaotically* – a point in the space may begin at some specific place, but if we wait long enough the group action has the ability to rub it up arbitrarily close to any other orbit. Condition (iii) on the other hand states that not only do the orbits behave chaotically, but even when we try to restrict them and localize their behavior , they still have the capacity to approach representatives of any other orbit.

Some analogy might be provided by the old standby of a billiard ball bouncing back and forth on a perfectly smooth and frictionless table with perfectly elastic boundaries. Thinking of the position the ball as a function of time we may naturally associate this with an $\mathbb{R}$-action. Given reasonably generic starting conditions, we might expect that the ball will on its travels eventually pass through every part of the surface.

Now suppose we cordon off one section of the table by an array of one way doors. The ball may exit the cordoned section whenever moved to do so, but having done so it can never return. It we the examine the *local orbit* of the billiard ball inside this section we no longer expect the orbit to fill out the space. It should depart to never return after some finite time, and its entire local orbit consists of just finitely many finite line segments, and thus, since this is a compact 1-dimensional set, we may find a subregion of the cordoned section on which it has not had the chance to encroach.

Indeed it is true that the associated action of $\mathbb{R}$ is *not* turbulent, and appropriately modified the above considerations can be turned into a proof that there are no turbulent $\mathbb{R}$-actions. The examples of turbulent actions in §3.3 all arise from infinite dimensional groups, and from 3.14 and 0.4, or the more general construction of §7.4 below, we can conclude that locally compact groups never give rise to turbulent actions. The notion of turbulence does not appear in the dynamics of locally compact groups, but rather is motivated by the general theory of equivalence relations and the much nastier groups which operate in this context.

We will prove that if E_G arises from a turbulent action and E_{S_∞} from a continuous S_∞ action, then E_G is generically E_{S_∞}-ergodic. Before then there are some preliminaries, the first of which states that turbulent orbit equivalence relations are generically ergodic with respect to the equivalence relation on codes for countable subsets of $2^{\mathbb{N}}$. In other words, any Baire measurable in the codes function from a turbulent Polish G-space to the countable subsets of $2^{\mathbb{N}}$ is constant on a comeager set.

LEMMA 3.14. *Let G be a Polish group acting continuously on a Polish space X with 3.13(iii): Namely: for all $x, y \in X$, $U \subset X$, $V \subset G$ open with $x \in U$, $1 \in V$, there exists $y_0 \in [y]_G$ and $(g_i)_{i\in\mathbb{N}} \subset V$, $(x_i)_{i\in\mathbb{N}} \subset U$ with*

$$x_0 = x,$$

$$x_{i+1} = g_i \cdot x_i,$$

and for some subsequence $(x_{n(i)})_{i\in\mathbb{N}} \subset (x_i)_{i\in\mathbb{N}}$

$$x_{n(i)} \to y_0.$$

Then E_G, the orbit equivalence relation of G on X, is generically F_2-ergodic.

Prelude to the proof. As a preliminary to proving this lemma, let us first discuss the proof of the much easier result that if $\theta : X \to B_2$ and various other relevant functions are all *continuous* with

$$\forall x \in X \forall g \in G(\theta(x) F_2 \theta(g \cdot x))$$

then $x \mapsto [\theta(x)]_{F_2}$ is constant on X. Recall that $B_2 = \{z \in 2^{\mathbb{N}\times\mathbb{N}} : \forall k \neq l(z(k,\cdot) \neq z(l,\cdot)\}$. We have already seen at 2.30 that $F_2 \leq_B F_2|_{B_2}$, and it is somewhat more natural to work on the set B_2 where the equivalence relation F_2 is actually induced by an S_∞-action. We fix $x, y \in X$ and $k \in \mathbb{N}$ and attempt to show how one might set about proving $\theta(x)(k,\cdot) \in \{\theta(y)(n,\cdot) : n \in \mathbb{N}\}$. Note that for any z in the space we may cover G by countably many sets $\{A_l^z : l \in \mathbb{N}\}$ such that on each A_l^z we have for all $g \in A_l^z$

$$\theta(g \cdot z)(l,\cdot) = \theta(z)(k,\cdot).$$

This is just at once true by the invariance assumption on θ.

Continuing with the policy of attempting to see how the proof moves forward under the most favorable circumstances, let us suppose that each such A_l^z is always open. Then for each z there will be some A_l^z which is not empty, and so we may find some $z' \in [z]_G$ and V a basic open neighborhood of the identity such that for all $g \in V$

$$\theta(g \cdot z')(l, \cdot) = \theta(z)(k, \cdot).$$

For each basic open V we may let $B(V, l)$ be the set of z' such that for all $g \in V$

$$\theta(g \cdot z')(l, \cdot) = \theta(z')(l, \cdot).$$

Again, let us just optimistically assume that $B(V, l)$ is always open.

Now going back to our original x and y, we will have some $x' \in [x]_G$ and basic open V containing the identity such that $x' \in B(V, l)$ and for all $g \in V$

$$\theta(g \cdot x')(l, \cdot) = \theta(x')(l, \cdot) = \theta(x)(k, \cdot).$$

Let $U \subset B(V, l)$ be some basic open set containing x'. Using the assumption of turbulence applied to x', y, U and V we may find $y_0 \in [y]_G$ and $h_i \in V$ such that if we let $x_0 = x'$ and $x_{i+1} = h_i \cdot x_i$ then y_0 is an accumulation point of $\{x_i : i \in \mathbb{N}\}$ and each $x_i \in U \subset B(V, l)$.

At this point we can use definition of $B(V, l)$ to note that each $\theta(x_{i+1})(l, \cdot) = \theta(x_i)(l, \cdot)$, and then obtain by induction i that each $\theta(x_i)(l, \cdot) = \theta(x)(k, \cdot)$. Thus by continuity of θ we have

$$\theta(x)(k, \cdot) = \theta(y_0)(l, \cdot),$$

as required.

Of course this is all rather wishful. In general these sets will be far from being open. But we will steadily verify that the sets in question have the Baire property, and so there will be some open set which they equal on a comeager subset of X or G; these repeated verifications make the actual proof seem technical at first sight.

PROOF. Choose $\theta : X \to B_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ Borel such that for all $x \in X$ and $g \in G$

$$\{\theta(x)(n, \cdot) : n \in \mathbb{N}\} = \{\theta(g \cdot x)(n, \cdot) : n \in \mathbb{N}\}.$$

Let $G_0 < G$ be a countable dense subgroup, with $(g_i)_{i\in\mathbb{N}}$ some enumeration. Let $(U_n)_{n\in\mathbb{N}}$ be a countable basis of X and $(V_n)_{n\in\mathbb{N}}$ a neighborhood basis for G at the identity. Choose $C \subset X$ a comeager Borel set on which θ is continuous by 2.49, and let $C_0 = \bigcap_{i\in\mathbb{N}} g_i \cdot C$ so that it is G_0 invariant and then $C_1 = \{x \in X : \forall^* g \in G(g \cdot x \in C_0)\}$. By 2.47 C_1 is comeager; by 2.50 Borel.

Claim(1): For all $x \in C_1$ and $k \in \mathbb{N}$ there exists $W \subset G$ open and non-empty and $l \in \mathbb{N}$ so that for all $h \in W$

$$h \cdot x \in C_0 \Rightarrow \theta(x)(k, \cdot) = \theta(h \cdot x)(l, \cdot).$$

Proof of claim: Note that we may cover $S_\infty \cdot \theta(x) =_{df} [\theta(x)]_{S_\infty}$ by countably many closed sets $(A_j)_{j\in\mathbb{N}}$ so that for each j and $y \in A_j$

$$\theta(x)(k, \cdot) = y(j, \cdot).$$

Let D_x be the preimage of C_0 under $g \mapsto g \cdot x$ and note that $g \mapsto \theta(g \cdot x)$ is continuous on D_x; since the preimage of each A_j is relatively closed and D_x is not meager, there must be some $W \subset G$ non-empty and open so that for some j

$$\forall g \in W \cap D_x(\theta(g \cdot x) \in A_j),$$

and $l =$ this j is as required. $(\Box\text{Claim(1)})$

Now define $I : \mathbb{N} \times X \to \mathbb{N} \cup \{\infty\}$ so that $I(l, x)$ equals the least n so that for all g_i and $g_{i'}$ in V_n

$$g_i \cdot x, g_{i'} \cdot x \in C_0 \Rightarrow \theta(g_i \cdot x)(l, \cdot) = \theta(g_{i'} \cdot x)(l, \cdot),$$

and equals ∞ if no such n exists. Note that if $x \in C_0$ and $I(l, x) = j$ then *for all* $g \in V_j$

$$g \cdot x \in C_0 \Rightarrow \theta(g \cdot x)(l, \cdot) = \theta(x)(l, \cdot)$$

by the continuity of θ on C_0 and the density of $\{g_i \cdot x : i \in \mathbb{N}\}$ in $G \cdot x \cap C_0$.

Considering $\mathbb{N} \cup \{\infty\}$ as a discrete space, I is transparently Borel, so choose comeager $C_2 \subset C_0 \cap C_1$ on which $I|_{\mathbb{N} \times C_2}$ is continuous, and then take $C_3 = \{x \in X : \forall^* g \in G(g \cdot x \in C_2)\}$. Note $C_3 \subset C_1$. Claim(1) implies that for all $x \in C_3$ and $k \in \mathbb{N}$ there will exist some $\hat{x} \in [x]_G \cap C_2$ and l for which $I(l, \hat{x}) < \infty$ and $\theta(x)(k, \cdot) = \theta(\hat{x})(l, \cdot)$. (We may assume that the $h \cdot x = \hat{x}$ promised by claim(1) is in C_2, since if not then we replace $\hat{x}$ by $\epsilon \cdot \hat{x}$ for some $\epsilon \in G$ which is close to the identity and has $\epsilon \cdot \hat{x} \in C_2$.)

With all this preparation let us finally suppose that $x, y \in C_3$; fixing k it is enough to show that $\theta(x)(k, \cdot)$ appears in $\{\theta(y)(n, \cdot) : n \in \mathbb{N}\}$. By the above reasoning we can assume that $x \in C_2$ and that

$$I(k, x) = m < \infty.$$

Since I is continuous on $\mathbb{N} \times C_2$ we can find some open set $U \subset X$ containing x so that for all $z \in C_2 \cap U$

$$I(k, z) = m.$$

Choose an open neighborhood W of the identity in G so that $W^3 \subset V_m$ and let $V = W \cap W^{-1}$.

Now apply the definition of turbulence to x, y, U, and V to find $y_0 \in [y]_G$ and $\{h_i : i \in \mathbb{N}\} \subset V$ so that for $x_0 = x$ and $x_{i+1} = h_i \cdot x_i$ we have each $x_i \in V$ and that y_0 is an accumulation point of the set $\{x_i : i \in \mathbb{N}\}$.

Claim(2): There are $\{\delta_i : i \in \mathbb{N}\} \subset V$ so that $d(x_i, \delta_i \cdot x_i) < 2^{-i}$, and $\delta_i \cdot x_i \in U$, and $\delta_i \cdot x_i \in C_2$

Proof of claim: For each i let $W_i \subset V$ be an open neighborhood of the identity such that for all $\delta \in W_i$

$$d(x_i, \delta \cdot x_i) < 2^{-i}$$

and

$$\delta \cdot x_i \in U.$$

By definition of C_3, for each x_i there is a comeager subset of $D_i \subset G$ on which $g \cdot x_i \in C_2$. Choosing δ_i inside $D_i \cap W_i$ suffices. ($\Box$Claim(2))

Let $\hat{x_i} = \delta_i \cdot x_i$ for $i > 0$; $\hat{x}_0 = x_0$. Let $\hat{h}_i = \delta_{i+1} h_i \delta_i^{-1}$ for $i > 0$; at $i = 0$ just let $\hat{h}_i = \delta_1 h_0$. Note that y_0 is still an accumulation point of $\{\hat{x}_i : i \in \mathbb{N}\}$.

Since each $\hat{x}_i \in C_2 \cap U$ we have $I(k, \hat{x}_i) = m$ for each i. Thus by $\hat{h}_i \in W^3 \subset V_m$, the definition of I, and induction on i,

$$\theta(\hat{x}_i)(k, \cdot) = \theta(\hat{x}_0)(k, \cdot) =_{df} \theta(x)(k, \cdot).$$

Claim(3): There is $\delta \in V_m$ such that $\delta \cdot y_0 \in C_0$ and at each i

$$\delta \cdot \hat{x}_i \in C_0,$$

$$\theta(\delta \cdot \hat{x}_i)(k, \cdot) = \theta(\hat{x}_i)(k, \cdot).$$

Proof of claim. The set of δ with $\delta \cdot y_0 \in C_0$ is comeager in G and therefore relatively comeager in V_m by $y \in C_1$, and similarly the set for which $\delta \cdot \hat{x}_i \in C_0$ is comeager at each i. However by the assumption that $\hat{x}_i \in U \cap C_2$ we have that the set of $\delta \in V_m$ with $\theta(\delta \cdot \hat{x}_i)(k, \cdot) = \theta(\hat{x}_i)(k, \cdot)$ is relatively comeager. Choosing δ in the intersection of all these comeager sets suffices. ($\square$Claim(3))

By continuity of θ on C_0,

$$\theta(\delta \cdot y_0)(k, \cdot) = \lim_{i\to\infty}\theta(\delta \cdot \hat{x}_i)(k, \cdot) = \lim_{i\to\infty}\theta(\hat{x}_i)(k, \cdot) = \theta(x)(k, \cdot),$$

as required. $\square$

DEFINITION 3.15. Let G be a Polish group, X a Polish G-space. For $x \in X$, $U \subset X$, $V \subset G$ both open, with $x \in U$ and $1_G \in V$, we define the *local U-V-orbit of x* to be the set of $\hat{x}$ for which there exists some $k \in \mathbb{N}$ and $x = x_0, x_1, x_2, ..., x_k \in U$ and $g_0, g_1, ..., g_{k-1} \in V$

$$x_k = \hat{x}$$

and for all $i < k$

$$x_{i+1} = g_i \cdot x_i.$$

This local orbit is denoted by $\mathcal{O}(x, U, V)$. We then define $\varphi_0(x, U, V)$ to encode $\overline{\mathcal{O}(x, U, V)}$, the closure of $\mathcal{O}(x, U, V)$), with the specification that if $\{U_l : l \in \mathbb{N}\}$ is a basis of X then

$$\varphi_0(x, U, V) = \{l \in \mathbb{N} : U_l \cap \mathcal{O}(x, U, V) \neq \emptyset\}.$$

Under the natural identification of $2^{\mathbb{N}}$ with the collection of all subsets of $\mathbb{N}$ the object $\varphi_0(x, U, V)$ can be calculated in a Borel fashion from x, in the sense that

$$\varphi_0(\cdot, U, V) : X \to 2^{\mathbb{N}}$$

$$x \mapsto \{l : \exists k \exists (x_i)_{i \leq k} \subset U \exists (g_i)_{i<k} \subset V(x_0 = x, x_k \in U_l, \forall i < k(x_{i+1} = g_i \cdot x_i))\}$$

is a Borel function for each open U and V: For each k let $A_k(U, V, U_l)$ be the set of x for which there exist open $U_0', U_1', ...U_k' = U_l \cap U$, and $g_0, ..., g_{k-1} \in V$ with $x \in U_0'$, $g_i \cdot (U_i') \subset U_{i+1}'$; each $A_k(U, V, U_l)$ is itself open, and $l \in \varphi_0(x, U, V)$ if and only if $x \in A_k(U, V, U_l)$ for some k.

With the definition of 3.15 we can express 3.13(iii) compactly: for all $x \in U \subset X$ and $1_G \in V \subset G$

$$[y]_G \cap \overline{\mathcal{O}(x, U, V)} \neq \emptyset.$$

LEMMA 3.16. *Let G be a Polish group and X a Polish G-space, and suppose that E_G^X is generically F_2-ergodic. Then for all $U \subset X$ open, $V \subset G$ open containing the identity there is a comeager $C \subset X$ so that for all $x \in C$ and $x_0 \in [x]_G \cap U$ the set $\overline{\mathcal{O}(x_0, U, V)}$ has non-empty interior.*

PROOF. Let $(g_i)_{i\in\mathbb{N}}$ enumerate a countable dense subset of G. Let W be the interior of the complement of U.

First note that for any closed nowhere dense A and $i \in \mathbb{N}$ the set

$$B_i(A) = \{x \in X : g_i \cdot x \in W\} \cup \{x \in X : g_i \cdot x \in U \setminus A\}$$

is open dense, and so

$$B(A) = \bigcap_{i\in\mathbb{N}} B_i(A)$$

is a dense G_δ.

Using the assumptions of the lemma we may go to a comeager set C on which

$$x \mapsto \{\varphi_0(g_i \cdot x, U, V) : g_i \cdot x \in U\} = \{\varphi_0(x_0, U, V) : x_0 \in [x]_G \cap U\}$$

is constant. And now for any $x \in C$ and $A \in \{\overline{\mathcal{O}(g_i \cdot x, U, V)} : g_i \cdot x \in U\}$ it must be the case that A contains an open set, or else we find $y \in C \cap B(A)$ for which A will not be an element of $\{\overline{\mathcal{O}(g_i \cdot y, U, V)} : g_i \cdot y \in U\}$. □

If X is a Polish G-space with (i) and (ii) from the definition of turbulence, and the conclusion of 3.16 holds for *every* $x \in X$ then the action is turbulent.

LEMMA 3.17. *Let G and H be Polish groups and X and Y Polish G and H-spaces respectively. Suppose that*

$$\theta : X \to Y$$

is a Borel function such that for all $x \in X$ and $g \in G$

$$\theta(x) E_H \theta(g \cdot x).$$

Then for any open neighborhood W of the identity in H there is a comeager set of $x \in X$ for which there is a basic open neighborhood V of the identity in G with

$$\forall^* g \in V \exists h \in W(h \cdot \theta(x) = \theta(g \cdot x)).$$

PROOF. First:

Claim: For all $x \in X$, there is a comeager set of $g_0 \in G$ for which there exists some open neighbourhood V of the identity on which

$$\forall^* g_1 \in V \exists h \in W(h \cdot \theta(g_0 \cdot x) = \theta(g_1 g_0 \cdot x)).$$

Proof of claim: Fix x and W. Choose a smaller open

$$(W')^2 \subset W$$

$$1_H \in W'$$

$$(W')^{-1} = W'.$$

We may cover H by countably many right translates $\{W' h_i : i \in \mathbb{N}\}$, for appropriately chosen $(h_i)_{i \in \mathbb{N}}$. Therefore we may cover

$$\theta[[x]_G] =_{df} \{\theta(x_0) : x_0 \in [x]_G\}$$

by $\{W' h_i \cdot \theta(x) : i \in \mathbb{N}\}$. Each $W' h_i \cdot \theta(x)$ is $\utilde{\Sigma}^1_1$, and thus its pullback under the function

$$g \mapsto \theta(g \cdot x)$$

is $\utilde{\Sigma}^1_1$, and so has the Baire property by 2.53. So find open $(O_i)_{i \in \mathbb{N}}$ with $\bigcup_{\mathbb{N}} O_i$ open dense in G and for all $i \in \mathbb{N}$

$$\forall^* g \in O_i(\theta(g \cdot x) \in W' h_i \cdot \theta(x)).$$

At each i let $C_i \subset O_i$ be this relatively comeager set; $\bigcup_{i \in \mathbb{N}} C_i$ is comeager in G, so consider any i and choose $g_0 \in C_i$. Take V to be a basic open neighbourhood of the identity for which we have $V g_0 \subset O_i$. $C_i (g_0)^{-1}$ is still relatively comeager in V, and for all $g_1 \in C_i \cap C_i (g_0)^{-1} \cap V$ we have some $h \in W'$ with

$$\theta(g_0 \cdot x) = h h_i \cdot \theta(x);$$

since $g_1 g_0 \in C_i$ there is some $h' \in W'$ with

$$\theta(g_1 g_0 \cdot x) = h' h_i \cdot \theta(x);$$

and so for $h^* \in W'(W')^{-1} \subset W$ given by $h^* = h' h^{-1}$ we have

$$h^* \cdot \theta(g_0 \cdot x) = \theta(g_1 g_0 \cdot x).$$

Thus for a comeager set of g_1 in V we do indeed have the conclusion of the claim. ($\square$Claim)

The set of x for which the conclusion of the lemma holds is $\underset{\sim}{\Sigma}^1_1$ by 2.55, 2.61(ii) and hence has the Baire property by 2.53 again. If the lemma fails then for all $g_0 \in G$ we would have that on a non-meager set of $x \in X$ it is the case that for all open neighborhoods V of the identity,

$$\exists^* g_1 \in V \forall h \in W(h \cdot \theta(g_0 \cdot x) \neq \theta(g_1 g_0 \cdot x)).$$

By Kuratowski-Ulam as at 2.46, there is consequently a non-meager set of $x \in X$ and a non-meager set of $g_0 \in G$ such that for all open V containing 1_G

$$\exists^* g_1 \in V \forall h \in W(h \cdot \theta(g_0 \cdot x) \neq \theta(g_1 g_0 \cdot x)),$$

with a contradiction to the claim. $\square$

THEOREM 3.18. *Let G be a Polish group, X a Polish G-space, satisfying (iii) from the definition of turbulence: For all $x, y \in X$, all $U \subset X$ open containing x, all $V \subset G$ open containing 1_G,*

$$\overline{\mathcal{O}(x, U, V)} \cap [y]_G \neq \emptyset.$$

Let Y be a Polish S_∞-space.

Then E^X_G is generically $E^Y_{S_\infty}$- ergodic.

PROOF. Let $\theta : X \to Y$ be a Borel map such that $x_1 E^X_G x_2 \Rightarrow \theta(x_1) E^Y_{S_\infty} \theta(x_2)$.

Let $(U_l)_{l \in \mathbb{N}}$ be a basis of X and $(V_l)_{l \in \mathbb{N}}$ a neighborhood basis of G at the identity. Define $I : X \times \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ by setting $I(x, N)$ to be the least l so that

$$\forall^* g \in V_l \exists h \in S_\infty (\forall i \leq N(h(i) = i) \wedge h \cdot \theta(x) = \theta(g \cdot x)),$$

if such l exists ($= \infty$ if not). If $\mathbb{N} \cup \{\infty\}$ is given the discrete topology then the preimage of a basic open set is in the σ-algebra generated by $\underset{\sim}{\Sigma}^1_1$ sets.

Using lemmas 2.49 let 3.17, let C_0 be a comeager set on which I and θ are continuous and that for all $x \in C_0, N \in \mathbb{N}$

$$I(x, N) < \infty.$$

Following 3.16 find $C_1 \subset X$ comeager and invariant so that for all $l, k \in \mathbb{N}$ and $x \in C_1 \cap U_l$ the closure of the local U_l-V_k-orbit,

$$\overline{\mathcal{O}(x, U_l, V_k)},$$

contains an open set. Now let

$$C_2 = \{x \in C_0 : \forall^* g \in G(g \cdot x \in C_0 \cap C_1)\};$$

by 2.47 this is comeager. Fix $x, y \in C_2$. We will see that there exists $g, h \in S_\infty$ with $g \cdot \theta(x) = h \cdot \theta(y)$, and thereby complete the proof.

We choose successively

$$x_0 = x, x_1, x_2, ...,$$
$$y_0 = y, y_1, y_2, ...,$$

in C_2, and

$$g_0 = 1_{S_\infty}, g_1, g_2 ...,$$
$$h_0 = 1_{S_\infty}, h_1, h_2, ...,$$

in S_∞, and

$$W_0 = X, W_1, ..., W'_0, W'_1, ...,$$

basic open in X,

$$V_{n(0)}, V_{n(1)}, ..., V_{k(0)}, V_{k(1)}, ...,$$

basic open neighborhoods of 1_G so that at each i

(ia) $g_i \cdot \theta(x) = \theta(x_i)$;
(ib) $h_i \cdot \theta(y) = \theta(y_i)$;
(iia) $x_{i+1} \in W_{i+1} \cap C_2 \cap \mathcal{O}(x_i, W_i, V_{n(i)})$;
(iib) $y_{i+1} \in W'_{i+1} \cap C_2 \cap \mathcal{O}(y_i, W'_i, V_{k(i)})$;
(iii) $W_i \supset W'_i \supset W_{i+1}$;
(iv) for d a previously fixed complete metric on Y and $i > 0$

$$d(\theta[W_i \cap C_2]) < 2^{-i};$$

(va) for

$$N_{i+1} =_{df} \sup\{g_{i+1}(l), g_{i+1}^{-1}(l) : l \leq i+1\}$$

we have

$$(W_{i+1} \cap C_2) \times \{N_{i+1}\} \subset I^{-1}[\{n(i+1)\}];$$

(vb) for

$$K_i =_{df} \sup\{h_i(l), h_i^{-1}(l) : l \leq i\}$$

we have

$$(W'_i \cap C_2) \times \{K_i\} \subset I^{-1}[k(i)];$$

(via) for all $l \leq i+1 < j+1$

$$g_{j+1}(l) = g_{i+1}(l)$$
$$g_{j+1}^{-1}(l) = g_{i+1}^{-1}(l);$$

(vib) for all $l \leq i < j$

$$h_j(l) = h_i(l)$$
$$h_j^{-1}(l) = h_i^{-1}(l);$$

(viia) $\mathcal{O}(x_i, W_i, V_{n(i)})$ is dense in W'_i;
(viib) $\mathcal{O}(y_i, W'_i, V_{k(i)})$ is dense in W_{i+1}.

Note that if all this succeeds we may in the end let

$$g_\infty = \lim_{i \in \mathbb{N}} g_i$$

and

$$h_\infty = \lim_{i \in \mathbb{N}} h_i;$$

by (vi) the limits exist. Then

$$g_\infty \cdot \theta(x) = h_\infty \cdot \theta(y)$$

by (ii)-(iv).

We can begin the construction with $x_0 = x$, $W_0 = X$, $V_{n(0)} = G$, $g_0 = 1_{S_\infty}$. Then choose $y_0 = y$, W'_0 containing y with

$$(W'_0 \cap C_2) \times \{0\} \subset I^{-1}[\{k(0)\}],$$

some $k(0) \in \mathbb{N}$, and let $h_0 = 1_{S_\infty}$.

So let us suppose we have built these as above for all $j \leq i$ and concentrate on producing x_{i+1}, g_{i+1}, and W_{i+1}. The further step of producing y_{i+1}, h_{i+1}, and W'_{i+1} is completely symmetrical and can be omitted.

Fix nonempty open $U \subset W'_i$ in which $\mathcal{O}(y_i, W'_i, V_{k(i)})$ is dense and then also demand that $d(\theta[U \cap C_2]) < 2^{-i-1}$. Then choose $x_{i+1} \in \mathcal{O}(x_i, W_i, V_{n(i)}) \cap U$, and let $z_0 = x_i, z_1, ..., z_k \in W_i$ and $g'_0, g'_1, ..., g'_{k-1} \subset V_{n(i)}$ witness this, in the sense that

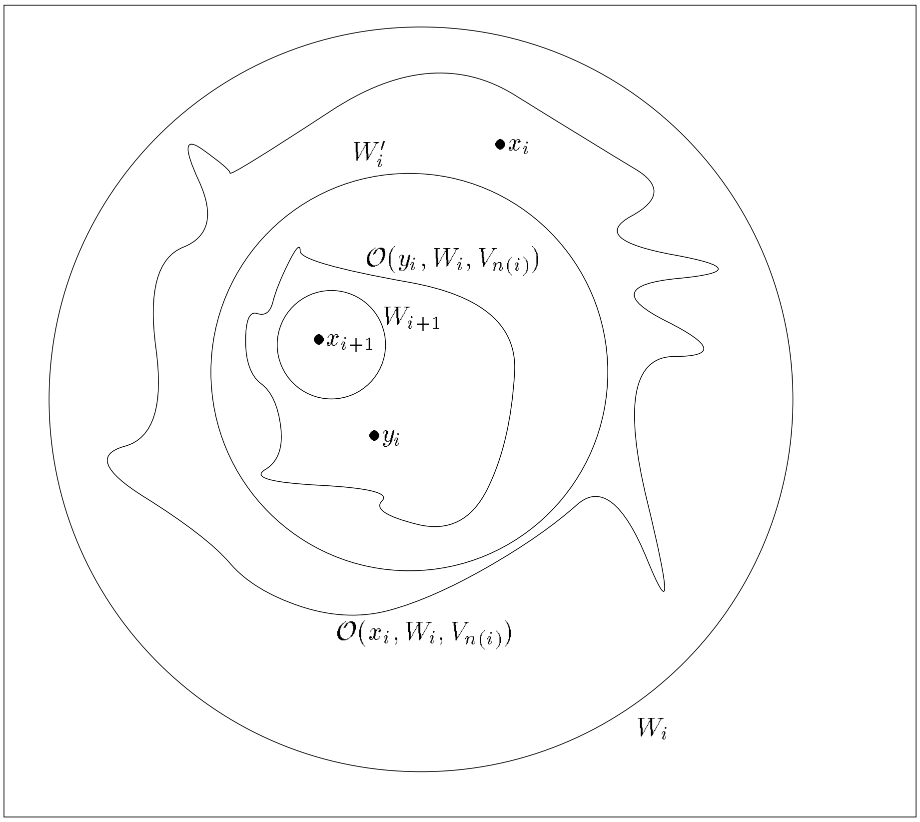

FIGURE 3.1. The local orbits folding in

$z_{i+1} = g_i' \cdot z_i$, $z_0 = x_i$, $z_k = x_{i+1}$. From now on we may as well assume that $i > 0$, or else (via) places no restriction on our choice of g_1.

Claim(I): Without loss of generality each $z_j \in C_2$.

Proof of claim: Choose V_j' open neighborhoods of the identity of G so that $V_j' = (V_j')^{-1}$, $V_{j+1}' \subset V_j'$, and

$$V_j' g_j' V_j' \subset V_{n(i)}.$$

Since

$$\forall^* g \in G(g \cdot x_i \in C_2)$$

and each $z_j \in [x]_G$ we may find $\epsilon_j \in V_j'$ so that

$$\epsilon_j \cdot z_j \in C_2 \cap W_i.$$

If we now replace

$$z_0, z_1, z_2, ..., z_k$$

by

$$z_0, \epsilon_1 \cdot z_1, \epsilon_2 \cdot z_2, ..., \epsilon_k \cdot z_k,$$

and

$$x_{i+1}$$

by

$$\epsilon_k \cdot x_{i+1},$$

and

$$g'_0, g'_1, ..., g'_{k-1}$$

by

$$\epsilon_1 g'_0, \epsilon_2 g'_1(\epsilon_1)^{-1}, \epsilon_3 g'_2(\epsilon_2)^{-1}, ..., \epsilon_k g'_{k-1}(\epsilon_{k-1})^{-1}$$

then we are done. ($\square$Claim(I))

Thus in particular for $i > 0$ $I(z_j, N_i) = n(i)$ for each j. With slight perturbation of the z_j's we can use the assumption

$$I(z_j, N_i) = n(i)$$

as follows:

Claim(II): Without loss of generality for each j there is $f_j \in S_\infty$ with

$$f_j \cdot \theta(z_j) = \theta(z_{j+1})$$
$$\forall l \leq N_i(f_j(l) = l).$$

Proof. First fix a relatively comeager $B(z_0) \subset V_{n(i)}$ so that $\forall g \in B(z_0)$ we have that there exists $f_0 \in S_\infty$ with

$$f_0 \cdot \theta(z_0) = \theta(g \cdot z_0),$$
$$\forall l \leq N_i(f_0(l) = l).$$

This uses $I(z_0, N_i) = n(i)$. Then by the argument of claim(I) we may replace g'_0 by some $\epsilon \cdot g'_0 \in B(z_0)$ and then by the definition of $B(z_0)$ find f_0 as desired. Clearly this can be continued inductively so that each $g'_j \in B(z_j)$, for $B(z_j)$ defined by the requirement that it be relatively comeager and so that $\forall g \in B(z_j)$ we have that there exists $f_j \in S_\infty$ with $f_j \cdot \theta(z_j) = \theta(g \cdot z_j)$ and $\forall l \leq N_i(f_j(l) = l)$.

($\square$Claim(II))

Let $f^*_i = f_{k-1} f_{k-2} ... f_0$ and note then $\theta(z_k) =_{df} \theta(x_{i+1}) = f^*_i \cdot \theta(x_i)$. Each f_j leaves $0, 1, ..., N_i$ fixed, and hence so does f^*_i; then we may let $g_{i+1} = f^*_i g_i$, which indeed gives us (va) above for g_{i+1}. Now take basic non-empty open $W_{i+1} \subset U$ containing x_{i+1} and some $n(i+1)$ so that

$$(W_{i+1} \cap C_2) \times \{N_{i+1}\} \subset I^{-1}[\{n(i+1)\}],$$

where as indicated above N_{i+1} is the sup of $\{g_{i+1}(l), g^{-1}_{i+1}(l) : l \leq i\}$. $\square$

COROLLARY 3.19. *(A) If G is a Polish group and X is a turbulent Polish G-space, then for any Polish S_∞-space Y*

$$E^X_G \not\leq_B E^Y_{S_\infty}.$$

(B) No turbulent orbit equivalence relation admits classification by countable models.

Also following from the proof of 3.18 is that there can be no *Baire measurable* reduction of a turbulent orbit equivalence relation to any $E^Y_{S_\infty}$. Since these arguments were based on Baire category methods it is natural to relax the definitions to the requirement that they hold on a comeager set.

DEFINITION 3.20. Let G be a Polish group and X a Polish G-space. The action is said to be *generically turbulent* if:

(i) $\forall^* x \in X([x]_G$ is dense);

(ii) every orbit is meager;

(iii) $\forall^* x, y \in X$, for all $U \subset X$ and $V \subset G$ basic open with $x \in U$ and V containing the identity, there exists $y_0 \in [y]_G$ with

$$y_0 \in \overline{\mathcal{O}(x, U, V)}.$$

THEOREM 3.21. *Let G be a Polish group, X a Polish G-space. Then the following are equivalent:*

(I) The action is generically turbulent.

(II) The orbit equivalence relation E_G^X is generically F_2-ergodic.

(III) For any countable language $\mathcal{L}$, the orbit equivalence relation is generically $\cong|_{\mathrm{Mod}(\mathcal{L})}$-ergodic.

(IV) Every orbit is meager; for a comeager collection of $x \in X$ the orbit $[x]_G$ is dense and for all $U \subset X$, $V \subset G$ basic open with $x \in U$, $1 \in V$, there exists $U_0 \subset U$ non-empty with $\mathcal{O}(x, U, V)$ dense in U_0.

(V) Every orbit is meager, and for a comeager collection of $x \in X$ it is the case that $[x]_G$ is dense and for all open $U \subset X$, $V \subset G$ with $x \in U$, $1 \in V$, there exists $U_0 \subset U$ non-empty with $\mathcal{O}(x, U, V)$ dense in U_0.

(VI) Some orbit is dense, every orbit is meager, and there exists $x \in X$, such that for all $U \subset X$, $V \subset G$ basic open with $x \in U$, $1 \in V$, there exists open $U_0 \subset U$ non-empty with $\mathcal{O}(x, U, V)$ dense in U_0.

(VII) There is an invariant dense G_δ subspace $X_0 \subset X$ with X_0 a turbulent Polish G-space.

PROOF. (I)$\Leftrightarrow$(II)$\Leftrightarrow$(III)$\Leftrightarrow$(IV): (I)$\Rightarrow$(II) follows exactly as in 3.14, and (II)$\Rightarrow$(III) is as in 3.18, while (III)$\Rightarrow$(II) is trivial from 2.41. (II)$\Rightarrow$(IV) is the proof of 3.16. (IV)$\Rightarrow$(I): since if

$$U_0 \subset \overline{\mathcal{O}(x, U, V)}$$

is open and non-empty and if $[y]_G$ is dense, then we may find $y_0 \in [y]_G \cap \overline{\mathcal{O}(x, U, V)}$, as desired.

(IV)$\Leftrightarrow$(V): It is immediate that (V) entails (IV). As for the converse, if x satisfies the statement of (IV), with its limitation to basic open sets, and U and V are as in (V), then we may find basic open $U' \subset U$ and $V' \subset V$ containing x and 1 respectively; noting that $\mathcal{O}(x, U', V') \subset \mathcal{O}(x, U, V)$ we are done by assumption that $\mathcal{O}(x, U', V')$ is somewhere dense.

So we have the equivalence of (I)-(V) and we just need to tie these to (VI) and (VII).

Trivially (IV)$\Rightarrow$(VI).

(VI)$\Rightarrow$(IV): For this purpose let $(g_i)_{i \in \mathbb{N}}$ be a dense countable subset of G, $\{U_l : l \in \mathbb{N}\}$ a countable basis of X, $\{V_l : l \in \mathbb{N}\}$ a countable neighbourhood basis at 1_G, and let $x \in X$ witness the conclusion of (VI).

Now for each $g_i \cdot x \in U_l$ and V_k let $W_{l,i,k} \subset U_l$ be the interior of $\overline{\mathcal{O}(g_i \cdot x, U_l, V_k)}$. This set will be non-empty by assumption on $[x]_G$.

Note that $[x]_G$ must be dense, since if we choose some $[z]_G$ which is dense, then we may find $z' \in W_{l,i,k} \cap [z]_G$ – and since z' is a limit point of $[x]_G$ and since $[z']_G = [z]_G$ is dense, $[x]_G$ is dense as well.

Let $B_{l,k}$ be the set of $y \in X$ such that either y is not in U_l or there exists some i with

$$\mathcal{O}(y, U_l, V_k) \cap W_{l,i,k} \neq \emptyset.$$

Note that each $B_{l,k}$ is G_δ by direct calculation, and dense since $[x]_G \subset B_{l,k}$. For $V_m \cap W_{l,i,k} \neq 0$ and $g_i \cdot x \in U_l$, let $C_{l,i,k,m}$ be the set of $y \in W_{l,i,k}$ for which

$$\mathcal{O}(y, U_l, V_k) \cap V_m \neq \emptyset.$$

$C_{l,i,k,m}$ is open and dense in $W_{l,i,k}$ whenever $g_i \cdot x \in U_l$ since

$$W_{l,i,k} \cap \mathcal{O}(g_i \cdot x, U_l, V_k) \subset C_{l,i,k,m}.$$

Hence if we let $D_{l,i,k}$ be the set of $y \in Y$ such that

$$\forall m((y \in W_{l,i,k} \wedge (U_m \cap W_{l,i,k} \neq \emptyset)) \Rightarrow \mathcal{O}(y, U_l, V_k) \cap V_m \neq \emptyset)$$

then $D_{l,i,k}$ is a dense G_δ.

Now for any y with $[y]_G$ dense and in the intersection of all the $g_j \cdot B_{l,k}$ and $g_j \cdot D_{l,i,k}$ we will have that the local orbits of y are all somewhere dense. By 3.4 and the above calculations, the set of such y is comeager.

(VII)$\Rightarrow$(IV): At once since the conclusion of (IV) must hold on X_0 by 3.16, and therefore X since X_0 is comeager in X.

For (IV)$\Rightarrow$(VII): Following the notation of (VI)$\Rightarrow$(IV) let X_0 be the set

$$(\bigcap_{i,l,k}(B_{l,k} \cap D_{l,i,k}))^{*G}.$$

$\square$

EXERCISE 3.22. If X is a turbulent Polish G-space, then there is no $\underset{\sim}{\Sigma}^1_1$ set $A \subset X$ meeting every orbit in a countable but non-empty set. (Use 2.54.)

3.3. Examples

EXAMPLE 3.23. Recalling $c_0 = \{\vec{x} \in \mathbb{R}^{\mathbb{N}} : x_i \rightarrow 0\}$ equipped with the topology given by the sup norm, $d(\vec{x}, \vec{y}) = \sup_{i \in \mathbb{N}}|x_i - y_i|$, we may let c_0 act on the ambient space $\mathbb{R}^{\mathbb{N}}$ by translation. The inclusion map $c_0 \hookrightarrow \mathbb{R}^{\mathbb{N}}$ is continuous and so this makes $\mathbb{R}^{\mathbb{N}}$ a Polish c_0-space. This is one of the simplest examples of a turbulent action.

It is clear that every orbit is dense, since c_0 is dense. We obtain every orbit being meager from c_0 being meager, which in turn follows by 2.56(i) and $(-c_0) + c_0 = c_0$ not containing an open set. As for the final condition 3.13(iii) to the effect that every orbit is somehow "locally dense", fix $\vec{x}, \vec{y} \in \mathbb{R}^{\mathbb{N}}$, $U \subset \mathbb{R}^{\mathbb{N}}$ containing $\vec{x}$, $V \subset c_0$ an open neighbourhood of the identity in c_0. We may assume that for some $\epsilon > 0, l \in \mathbb{N}$

$$V = \{\vec{z} \in c_0 : \sup_{i \in \mathbb{N}} z_i < \epsilon\}$$

$$U = \{\vec{z} \in \mathbb{R}^{\mathbb{N}} : \forall i < l(|z_i - x_i| < \epsilon)\},$$

and that $\vec{y} \in U$. Fixing some arbitrary open neighbourhood U_0 of $\vec{y}$ it suffices to find some $\vec{z} \in \mathcal{O}(x, U, V) \cap U_0$.

So using the density of $[x]_{c_0}$ we may find $\vec{g} \in c_0$ with $\vec{g} \cdot x \in U_0 \cap U$. Let $\delta = \sup_{\mathbb{N}}|g_i|$, and choose $k \in \mathbb{N}$ large enough that $\delta/k < \epsilon$. Let $\vec{h} = \vec{g}/k$, in the sense that $h_i = g_i/k$ at each $i \in \mathbb{N}$. Let $\vec{h}^j = \vec{h}$ for each $j < k$, and

$$\vec{x}_0 = \vec{x},$$

$$\vec{x}^{j+1} = \vec{h}^j \cdot \vec{x}^j.$$

Then $\vec{x}^k \in U_0$ and each $\vec{h}^j \in V$.

A similar result holds for $l^1(\mathbb{R}) = \{\vec{x} \in \mathbb{R}^{\mathbb{N}} : \sum |x_i| < \infty\}$ or $l^1(\mathbb{C}) = \{\vec{x} \in \mathbb{C}^{\mathbb{N}} : \sum |x_i| < \infty\}$. There is a completely general result regarding Polishable subgroups of $\mathbb{R}^{\mathbb{N}}$.

DEFINITION 3.24. Let $G \subset \mathbb{R}^{\mathbb{N}}$ be a Polishable subgroup. G is said to be *super dense* if for all $(x_0, x_1, ...x_n) \in \mathbb{R}^{<\mathbb{N}}$ there is some $\vec{y} \in G$ with $y_i = x_i$ for all $i \leq n$.

PROPOSITION 3.25. *Let $G \subset \mathbb{R}^{\mathbb{N}}$, $G \neq \mathbb{R}^{\mathbb{N}}$ be super dense. Then the action by translation of G on $\mathbb{R}^{\mathbb{N}}$ is turbulent.*

PROOF. Let τ be the Polish topology on G. Note that by 2.56

$$\pi_n : G \to \mathbb{R}^n$$

$$\vec{x} \mapsto (x_0, x_1, ..., x_{n-1})$$

is a continuous map; by assumption of super density it is onto and so open.

Now we can simply repeat the proof from 3.23 with suitable changes. The argument that every orbit is dense and meager is as before.

So fix $U \subset \mathbb{R}^{\mathbb{N}}$ open containing $\vec{x}$, and assume that $U = \{\vec{z} \in \mathbb{R}^{\mathbb{N}} : \forall i < l(|z_i - x_i| < \epsilon)\}$. Fix $\vec{y} \in \mathbb{R}^{\mathbb{N}}$, and note that we may find some $\vec{g} \in G$ with $\vec{g} \cdot \vec{y} \in U$ by assumption on G. Let U_0 be an open set containing $g \cdot \vec{y}$ and V an open set containing 1_G. We may assume

$$U_0 = \{\vec{z} \in \mathbb{R}^{\mathbb{N}} : \forall i < n(|z_i - y_i| < \epsilon_0)\}.$$

Then we may find some $(h_0, h_1, ...) \in \mathbb{R}^n$ so that $h_i \cdot x_i = y_i$.

Since π_n is an open map we can find some open $W \subset \mathbb{R}^n$ with $\pi_n[V] = W$. Then for some sufficiently large k we will have $(h_0, h_1, ...h_{n-1})/k \in W$. Choosing $\vec{g} \in V$ with $g_i = h_i/k$ for $i < n$, we may let $\vec{x}^0 = \vec{x}, \vec{x}^{j+1} = \vec{g} \cdot \vec{x}^j$. $\square$

EXAMPLES 3.26. Under the natural identification of $2^{\mathbb{N}}$ with $\mathbb{Z}_2^{\mathbb{N}}$ we may view the Cantor space as a compact Polish group with the operation of pointwise addition. A further identification is obtained by associating every $f \in 2^{\mathbb{N}}$ with the set $A_f \in \mathcal{P}(\mathbb{N})$ (the collection of all subsets of $\mathbb{N}$) defined by

$$n \in A_f \Leftrightarrow f(n) = 1.$$

Under this identification the group operation on $2^{\mathbb{N}}$ corresponds in $\mathcal{P}(\mathbb{N})$ to the operation of symmetric difference:– $A\Delta B = A \setminus B \cup B \setminus A$.

A subset of $\mathcal{P}(\mathbb{N})$ is said to be an *ideal* if closed under finite unions and the process of going to an arbitrary subset. An ideal on $\mathcal{P}(\mathbb{N})$ is said to be *Polishable* if it is Polishable as a group with respect to the operation of symmetric difference.

Two examples of Polishable ideals on $\mathcal{P}(\mathbb{N})$ are:

(i) the *density ideal*:

$$I_d = \{A \in \mathcal{P}(\mathbb{N}) : \frac{|A \cap \{0, 1, .., n\}|}{n+1} \to 0\};$$ [1]

(ii) the *summable ideal*:

$$I_2 = \{A \in \mathcal{P}(\mathbb{N}) : \sum_{n \in A} \frac{1}{n+1} < \infty\}.$$

These groups are Polish in the topologies generated by the metrics

$$d_0(A, B) = \sup_{n \in \mathbb{N}} \frac{|A\Delta B \cap \{0, 1, .., n\}|}{n+1}$$

[1] Where $|A \cap \{0, 1, .., n\}|$ denotes the size of the set $A \cap \{0, 1, .., n\}$.

and

$$d_1(A,B) = \sum_{n \in A \Delta B} \frac{1}{n+1}.$$

Both the actions by translation on the surrounding space $2^{\mathbb{N}}$ are turbulent. This also provides a counterpoint to the examples of 3.25, since here $\mathcal{O}(x,U,V)$ depends as much on V as it does on U – for instance in (i) (or (ii)) above, if we let $U = \mathcal{P}(\mathbb{N})$, A be any element in $\mathcal{P}(\mathbb{N})$ and

$$V_k = \{B \in I_2 : \forall n < k (n \notin B)\}$$

then

$$\overline{\mathcal{O}(A,U,V_k)} = \{C \subset \mathbb{N} : \forall n < k (n \in C \Leftrightarrow n \in A)\}.$$

More recently Alexander Kechris has in [**54**] used the notion of turbulence and work of Slawek Solecki from [**77**] to completely characterize for which Borel ideals $I \subset \mathcal{P}(\mathbb{N})$ we have the equivalence relation E_I given by

$$A E_I B \Leftrightarrow A \Delta B \in I$$

admitting classification by countable models.

A rather different kind of example for a group studied in functional analysis:

THEOREM 3.27. *(Kechris, Sofronidis) The action of U_∞ upon itself by conjugation is generically turbulent.*

[**10**] shows the conjugation action of U_∞ to be meager and to have a dense orbit, but the condition (iii) from turbulence did not follow from this almost classical argument. Kechris and Sofronidis not only obtained (iii) but determined the orbits $x \in U_\infty$ satisfying condition (V) from 3.21.

We will investigate a sequence of examples similar to this one in chapters 4 and 5.

EXAMPLE 3.28. Choose a rapidly increasing sequence of primes, $p_0, p_1, ...$, so that $p_{k+1} > k p_0 p_1 ... p_k$. Let $G = \mathbb{Z}_{p_0} \times \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} ...$, so the elements of G will be functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) < p_n$ and addition is mod p_n on the nth coordinate. Then let

$$H = \{f \in G : \sup_{\mathbb{N}} \frac{|f(n)|_n}{p_n}\} < \infty,$$

where $|f(n)|_n$ should be understood as indicating the least $m \geq 0$ for which $f(n) + m = p_n$ or $f(n) - m = 0$. H is a Polish group with respect to this sup norm and the action of H on G is turbulent.

In some respects this example is one of minimal algebraic complexity. Not only is the group H abelian, but also *monothetic* – the group element $\vec{1} \in H$ given by $\vec{1}(n) = \bar{1}$ for all n generates a dense cyclic subgroup of H.

EXAMPLE 3.29. Finally Kechris has found an application in [**55**] which would initially appear to have no relation to the study of equivalence relations. Here he unearthed a completely new proof of a result of Preiss and Rataj to the effect that there is no analytic maximal set of pairwise orthogonal measures on any uncountable standard Borel space; in fact, this result was not only given a faster proof but considerably strengthened. Turbulence found an entrance by Kechris' embedding of a turbulent orbit equivalence relation into the equivalence relation of absolute continuity of measures, to which he then applied a version of 3.22.

3.4. Historical remarks

3.4.1. Smoothness as a notion of classification. The results of §3.1 are implicit in classical papers. Indeed some version of these results have appeared independently in many different places – for instance [**25**] and [**5**], and in [**47**] one may find a discussion of even earlier proofs that $\mathbb{R}/\mathbb{Q}$ is not smooth.

Smoothness is very commonly taken to be archetypal notion of classifiable. In the writings of some analysts, such as perhaps [**18**], one almost senses a feeling that smoothness might be the only thing that could be *meant* by providing a complete classification of an equivalence relation; that somehow this is the only sensible way one could even think about the problem. The interesting properties are those that can be presented as Borel subsets of a Polish space. A classification theorem for an equivalence relation E on a space X should consist in finding a family of invariant, preferably Borel invariant, subsets of X, and the classification is complete when any two distinct equivalence classes can be separated by a member of our family. In order for the notion of completely classifiable to have any content, and not just apply across the board to every virtually every equivalence relation, we should require that the family of invariant Borel sets be countable. Thus by 2.27(ii) we are inexorably led to the conclusion that an equivalence relation is completely classifiable if and only if it is smooth.

This is also the view point the authors of [**25**] and [**14**] have drawn from [**63**], and despite its plausibility it turns out to be a surprisingly harsh standard. It is a position from which we must conclude that even the rank one torsion free abelian groups are unclassifiable and must reject the K-theoretic classification of [**24**]; any equivalence relation into which we may embed E_0 is by this definition unclassifiable.[2]

On the other hand a completely different culture is ascendant in the branch of mathematical logic known as model theory. For instance one finds implicit in the introductory passages of [**2**] that the classification of countable structures is completely and satisfactorily settled by the Scott analysis, which we will recall at §6.1 below, and that the crucial directions of research are related to which classes of *uncountable, discrete* structures allow some higher level analogue of Scott's original analysis. From this standing it is simply a triviality that all the finite rank, or even infinite rank, torsion free abelian groups can be completely classified.

This book has tried to take from both these points of view. §3.1 recalls a coherent theory of smoothness, and the work of §3.2 came about as an attempt to parallel these classical and folklore results to more permissive notions of classification, such as the assignment of a countable structure considered up to isomorphism as a complete invariant.

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[2] A footnote is irresistible . The perspective of [**21**] is to indeed embrace rank one torsion free abelian groups as classifiable since we may associate to them elements of $\mathcal{P}(\mathbb{N})/\mathrm{Finite}$ as complete invariants – in fact they are considered classifiable precisely because $\cong|_{\mathrm{TFA}_1} \leq_B E_0$. Rather [**21**] seems to take the battle ground to be for rank *two* torsion free abelian groups.

Recently [**36**] showed that for all $n \geq 2$ one has $E_0 <_B \cong|_{\mathrm{TFA}_n}$; the argument used the non-amenability of $\mathrm{SL}_2(\mathbb{Z})$ for rank 2 and Kazhdan's property (T) for $n \geq 3$ to obtain not only $E_0 <_B \cong|_{\mathrm{TFA}_n}$ but also the *non-treeability* of $\cong|_{\mathrm{TFA}_n}$ – which in essence states that $\cong|_{\mathrm{TFA}_n}$ is not $\leq_B$ reducible to any free Borel action of a countable free group. Since then [**56**] obtained the parallel non-treeability result for $n = 2$ through an appeal to deep facts about algebraic groups.

The exact classification difficulty of the $\cong|_{\mathrm{TFA}_n}$'s remains undetermined, but at the time of writing it would appear intertwined with the general ergodic theory of countable group actions.

3.4.2. Alternative proofs. In particular §3.2 and §3.3 show that there are Polish group actions whose orbit equivalence relations are not reducible to orbit equivalence relations induced by S_∞. This question was raised at 3.5.4 and 8.2.4 of [**4**]. As a comment on the viewpoint at the time of posing, almost all the Polish group actions which had been seriously studied were so reducible; most examples of orbit equivalence relations commonly considered bore a resemblance to the logic action of S_∞ on $\mathrm{Mod}(\mathcal{L})$ or the actions of locally compact groups.

During 1996 the perspective began to change. Groups arising from real analysis such as the classical Banach spaces were subjected to increasing scrutiny, and the counterexamples that once seemed exotic became commonplace.

It should not however be suggested that 3.18 was the first demonstration that there are such unclassifiable group actions. Earlier in spring of 1995 the author had a rather different construction, using work of [**77**] to the effect that there are F_σ Polishable ideals whose coset equivalence relation is not reducible to a countable equivalence relation and the analysis in [**38**] of *essentially* F_σ equivalence relations induced by S_∞ all being reducible to countable equivalence relations. The broader results of §3.2 came later.

Indeed even after the development of turbulence, 3.18 was not the first proof of non-reducibility to S_∞-actions. Earlier it was shown by a transfinite induction along the rank of the corresponding *Scott sentence* associated to $\theta(x)$ that for any generically F_2-ergodic E we must in general further obtain that E is generically $E^Y_{S_\infty}$-ergodic. This alternative proof was not only shorter, but also more informative. Its disadvantage was in being less than completely elementary. Moreover the proof presented at 3.18 turns out to be useful in the construction from §9.2.

All well and good. Later an offhand reference was noticed in [**20**] to an unpublished proof of the density ideal's coset equivalence relation not admitting classification by countable models. Trivially this ideal is Polishable, and thus Harvey Friedman and Lee Stanley answered a principal question of [**4**] before it was ever even asked.

In early 1997 Friedman visited Los Angeles and made available the argument, which should eventually be published in [**19**]. In fact it was indeed very close to the afore mentioned induction along the Scott sentence.

3.4.3. Borel dichotomy theorems. A towering influence has been exerted on the theory of Borel equivalence relations by [**30**]:

THEOREM 3.30. *(Harrington, Kechris, Louveau) Let E be a Borel equivalence relation. Then exactly one of the following two things must happen:*

1. $E \leq_B \mathrm{id}(\mathbb{R})$;
2. $E_0 \sqsubseteq_c E$.

This is the paradigm *dichotomy theorem*: Either an equivalence relation is smooth, or its non-smoothness is explained by the embeddability of E_0. The result fails for general $\underset{\sim}{\Sigma}^1_1$ equivalence relations, even those induced by the continuous actions of Polish groups; but in [**37**] it is noted that either $E_0 \sqsubseteq_c E^X_G$ or there is UBMC (see 3.12) reduction of E^X_G to $2^{<\omega_1}$.

In view of 3.30 and a handful of similar discoveries early in the analysis of equivalence relations, one might hope that understanding turbulence, or any notion of non-classifiability, should ultimately rest on establishing the correct battery of dichotomy theorems. A definition helps to formalize this idea.

DEFINITION 3.31. An orbit equivalent relation is *minimum* turbulent if it is turbulent and Borel reducible to every other turbulent orbit equivalent. An orbit equivalent relation E_G^X is *minimal* turbulent if for all

$$E \leq_B E_G^X$$

either:

1. E admits classification by countable models; or
2. $E_G^X \leq_B E$.

[**33**] indicated that the structure here is more complicated than the smooth\non-smooth dichotomy.

THEOREM 3.32. *There is no minimum turbulent orbit equivalence relation.*

The proof of this result required showing $\mathbb{R}^{\mathbb{N}}/l^1$ and $\mathbb{R}^{\mathbb{N}}/c_0$ are $\leq_B$-incomparable, and that the first of these is *minimal.* So at least we obtain:

THEOREM 3.33. $\mathbb{R}^{\mathbb{N}}/l^1$ *(equivalently,* $2^{\mathbb{N}}/I_2$*) is minimal turbulent.*[3]

In view of these developments it seemed natural at the time of the first draft of this manuscript to conjecture that $\mathbb{R}^{\mathbb{N}}/c_0$ is also minimal turbulent. However Mike Oliver in [**70**] showed that the large mass of $\leq_B$ incomparable Borel equivalence relations (all of which are easily seen to be turbulent) from [**61**] may be reduced to $\mathbb{R}^{\mathbb{N}}/c_0$, and thus one has:

THEOREM 3.34. *(Oliver)* $\mathbb{R}^{\mathbb{N}}/c_0$ *(equivalently,* $2^{\mathbb{N}}/I_d$*) is not minimal turbulent.*[4]

Of course given 3.33 and the happy glow of Harrington, Kechris, Louveau we might still hope for a simple basis of turbulence. But in 1998 this was refuted by Ilijas Farah:

THEOREM 3.35. *(Farah; see* [**16**]*) There is no finite or even countable collection of turbulent orbit equivalence relation*

$$E_{G_1}^{X_1}, \ldots E_{G_k}^{X_k}, \ldots$$

such that for any turbulent orbit equivalence relation E_G^X *there is some* $k \in \mathbb{N}$ *with*

$$E_{G_k}^{X_k} \leq_B E_G^X.$$

And then the final stake through the heart:

THEOREM 3.36. *(Farah; see* [**17**]*) There is a turbulent orbit equivalence relation which is not* $\leq_B$ *above any minimal turbulent orbit equivalence relation.*

In short, we will never be able to produce any kind of dichotomy theorem along the lines of [**30**] and this avenue may never lead to any insight regarding when an equivalence relation admits classification by countable models.

Instead of continually revising downwards the goals for dichotomy theorems, this manuscript tries to suggest another approach. Rather than attempting to understand various divisions of classifiable versus non-classifiable in terms of a small or finite basis of unruly examples, we may analyze smoothness and classifiability by countable structures by isolating the dynamic properties necessary for their

[3] As noted in [**33**], Kechris has shown $2^{\mathbb{N}}/I_2 \leq_B \mathbb{R}^{\mathbb{N}}/l^1 \leq_B 2^{\mathbb{N}}/I_2$.

[4] Oliver also observes that $\mathbb{R}^{\mathbb{N}}/c_0 \leq_B 2^{\mathbb{N}}/I_d \leq_B \mathbb{R}^{\mathbb{N}}/c_0$.

failure. To this end chapter 9 shows that the presence of a turbulent action is a necessary condition for an orbit equivalence relation to deny having at least a UBMC assignment of HC sets as complete invariants, just as the role of a properly generically ergodic action is always implicit in a group action that refuses a UBMC classification by bounded subsets of the countable ordinals.

CHAPTER 4

Classifying homeomorphisms

This chapter consists of detailed examples for various kinds of homeomorphism groups.

The classification result at §4.2 for $\mathrm{Hom}^+([0,1])$ is essentially folklore. The non-classification theorem at §4.3 for the unit square I believe to be new. Ultimately we obtain that the natural notion of isomorphism for homeomorphisms of the unit square is strictly more complicated than the isomorphism relation on countable structures.

First there are some preliminaries.

4.1. Definitions and remarks

DEFINITION 4.1. $\mathrm{Hom}([0,1])$ is the group of all homeomorphisms of the unit interval, and more generally for K a compact metric space $\mathrm{Hom}(K)$ denotes the group of homeomorphisms of K. $\mathrm{Hom}^+([0,1])$ denotes the subgroup of orientation preserving homeomorphisms – that is to say, those $\pi \in \mathrm{Hom}([0,1])$ with $\pi(0) = 0, \pi(1) = 1$.

We equip $\mathrm{Hom}(K)$ with a topology by taking a compatible metric d on K and letting $\hat{d}$ be defined on $\mathrm{Hom}(K)$ by

$$\hat{d}(\pi_1, \pi_2) = \sup\{d(\pi_1(x), \pi_2(x)), d(\pi_1^{-1}(x), \pi_2^{-1}(x)) : x \in K\}.$$

By compactness of K the topology generated by $\hat{d}$ does not depend on the choice of d.

LEMMA 4.2. *For K a compact metric space, $\mathrm{Hom}(K)$ is a Polish group.*

PROOF. First: Multiplication is continuous:- Let U be open with $\pi_1 \circ \pi_2 \in U$. Fix $\epsilon > 0$ so that U includes the set of π with $\hat{d}(\pi, \pi_1 \circ \pi_2) < \epsilon$. Then by compactness choose positive $\delta < \epsilon/3$ such that for all $x_1, x_2 \in K$, $d(x_1, x_2) < \delta$ implies

$$d(\pi_1(x_1), \pi_1(x_2)) < \epsilon/3$$

and

$$d(\pi_2^{-1}(x_1), \pi_2^{-1}(x_2)) < \epsilon/3.$$

Then for all σ_1, σ_2 with

$$\hat{d}(\pi_1, \sigma_1), \hat{d}(\pi_2, \sigma_2) < \delta$$

we have for all $x \in K$

$$d(\pi_2(x), \sigma_2(x)) < \delta$$

$$\therefore d(\pi_1 \circ \pi_2(x), \pi_1 \circ \sigma_2(x)) < \epsilon/3$$

$$\therefore d(\pi_1 \circ \pi_2(x), \sigma_1 \circ \sigma_2(x)) \leq d(\pi_1 \circ \pi_2(x), \pi_1 \circ \sigma_2(x)) + d(\pi_1 \circ \sigma_2(x), \sigma_1 \circ \sigma_2(x))$$
$$< \epsilon/3 + \epsilon/3 < \epsilon$$

by the triangle inequality. Similarly we obtain $d(\pi_2^{-1} \circ \pi_1^{-1}(x), \sigma_2^{-1} \circ \sigma_1^{-1}(x)) < 2\epsilon/3$ all x and so $\hat{d}(\pi_1 \circ \pi_2, \sigma_1 \circ \sigma_2) \leq 2\epsilon/3 < \epsilon$.

It is immediate from the definitions that the function $\pi \mapsto \pi^{-1}$ is continuous on $\mathrm{Hom}(K)$.

We obtain completeness of the metric by adapting the usual proof that a uniformly convergent sequence of functions has a continuous limit. Suppose $(\pi_i)_{i\in\mathbb{N}}$ is $\hat{d}$-Cauchy. By definition of the metric we obtain for all x that $(\pi_i(x))_{i\in\mathbb{N}}$ and $(\pi_i^{-1}(x))_{i\in\mathbb{N}}$ are d-Cauchy. So we may define $\pi_\infty, \sigma_\infty : K \to K$ by

$$\pi_\infty(x) = \lim_{i\in\mathbb{N}} \pi_i(x),$$
$$\sigma_\infty(x) = \lim_{i\in\mathbb{N}} \pi_i^{-1}(x).$$

These are both continuous and we just need that they pair as each's inverse for $\pi_\infty \in \mathrm{Hom}(K)$. As $i \to \infty$ we have firstly that

$$\pi_i(x) \to \pi_\infty(x)$$

and secondly, by the uniformity of the convergence, that

$$\sigma_\infty(\pi_i(x)) \to \pi_i^{-1}(\pi_i(x)) = x.$$

Thus $\sigma_\infty(\pi_\infty(x)) = x$ all $x \in K$ and hence $\sigma_\infty = (\pi_\infty)^{-1}$.

Finally. Separability:- For $\epsilon > 0$ and $x \in K$ let $B_\epsilon(x) = \{y \in K : d(x,y) < \epsilon\}$. For $0 < \epsilon_0 < \epsilon_1$ let $H_{\epsilon_0,\epsilon_1}$ be the set of $\pi \in \mathrm{Hom}(K)$ for which all x_1, x_2 in K with $d(x_1, x_2) < \epsilon_0$ has

$$d(\pi(x_1), \pi(x_2)), d(\pi^{-1}(x_1), \pi^{-1}(x_2)) < \epsilon_1.$$

Compactness implies for all ϵ_1 and π there is some ϵ_0 with $\pi \in H_{\epsilon_0,\epsilon_1}$. We only need show for $0 < \epsilon_0 < \epsilon_1 < 2\epsilon_1 < \epsilon_2$ that $H_{\epsilon_0,\epsilon_1}$ has a countable subset that is within $\hat{d}$-distance ϵ_2 of every other element of $H_{\epsilon_0,\epsilon_1}$. But if $k_0, k_1, ..., k_n \in K$ is so that $\bigcup B_{\epsilon_0}(k_i)$ covers K and if $\{\pi_i : i \in \mathbb{N}\}$ is chosen so that

$$\{(\pi_i(k_0), \pi_i(k_1), ..., \pi_i(k_n), \pi_i^{-1}(k_0), ..., \pi_i^{-1}(k_n)) : i \in \mathbb{N}\}$$

is dense (as a subset of K^{2n+2}) in

$$\{(\pi(k_0), \pi(k_1), ..., \pi^{-1}(k_n)) : \pi \in H_{\epsilon_0,\epsilon_1}\}$$

then for any $\pi \in H_{\epsilon_0,\epsilon_1}$ we may choose π_i so for all j

$$d(\pi_i(k_j), \pi(k_j)), d(\pi_i^{-1}(k_j), \pi^{-1}(k_j)) < \epsilon_2 - 2\epsilon_1.$$

Then for all $k \in K$ we may find some k_j with $d(k, k_j) < \epsilon_0$, and conclude that

$$\begin{aligned} d(\pi_i(k), \pi(k)) &\leq d(\pi_i(k), \pi_i(k_j)) + d(\pi_i(k_j), \pi(k_j)) + d(\pi(k_j), \pi(k)) \\ &< \epsilon_1 + (\epsilon_2 - 2\epsilon_1) + \epsilon_1 < \epsilon_2 \end{aligned}$$

and similarly

$$d(\pi_i^{-1}(k), \pi(k)^{-1}) < \epsilon_1 + (\epsilon_2 - 2\epsilon_1) + \epsilon_1 < \epsilon_2$$

as needed. □

EXERCISE 4.3. (a) In the case of $K = [0,1]$ and $\{k_i : i \in \mathbb{N}\} \subset [0,1]$ dense we may take as a basis for $\mathrm{Hom}([0,1])$ sets of the form

$$\{\sigma : |\sigma(k_i) - \pi(k_i)| < \epsilon \text{ for } i = 0, 1, ..., n\}$$

as n ranges over $\mathbb{N}$, π over $\mathrm{Hom}(K)$, ϵ over the positive reals.

(b) For $K = [0,1]^2$ this pointwise topology is already weaker the one than generated by the $\hat{d}$ of 4.1.

(c) For K compact we obtain as a subbasis for the topology of $\mathrm{Hom}(K)$ sets of the form

$$\{\pi : \pi[C] \subset \mathcal{O}\}$$

for $\mathcal{O} \subset K$ open and $C \subset K$ closed.

EXERCISE 4.4. Show that every element of $\mathrm{Hom}^+([0,1])$ is torsion free, and thus S_∞ is not homeomorphically isomorphic to a closed subgroup of $\mathrm{Hom}^+([0,1])$.

4.2. Classification in dimension 1

NOTATION 4.5. $\approx$ denotes the conjugacy equivalence relation on $\mathrm{Hom}^+([0,1])$ – so that for $\pi_1, \pi_2 \in \mathrm{Hom}^+([0,1])$ we set $\pi_1 \approx \pi_2$ if there is some $\sigma \in \mathrm{Hom}^+([0,1])$ with

$$\sigma \circ \pi_1 \circ \sigma^{-1} = \pi_2.$$

Let $\approx'$ denote conjugacy on the full $\mathrm{Hom}([0,1])$ and $\approx''$ on $\mathrm{Hom}([0,1]^2)$.

THEOREM 4.6. *(folklore) There is a countable language $\mathcal{L}$ such that*

$$\approx \leq_B \cong|_{\mathrm{Mod}(\mathcal{L})}.$$

PROOF. Let $\mathcal{L}$ be generated by the symbols $<, P_+, P_-, P_=$, with $<$ a binary relation, $P_+, P_-, P_=$ unary predicates. For $\pi \in \mathrm{Hom}^+([0,1])$ define the underlying set of $\mathcal{M}(\pi)$ to consist of the disjoint union of $(P_+)^{\mathcal{M}(\pi)}$, $(P_-)^{\mathcal{M}(\pi)}$, $(P_=)^{\mathcal{M}(\pi)}$, where:

(i) $(P_+)^{\mathcal{M}(\pi)}$ is the set of all open intervals (x, y) such that x is less than y, both are in the unit interval, $\pi(x) = x$ and $\pi(y) = y$, and for all $z \in (x, y)$ one has $\pi(z) > z$;

(ii) $(P_-)^{\mathcal{M}(\pi)}$ is the set all open intervals (x, y) such that x is less than y, both are in the unit interval, $\pi(x) = x$ and $\pi(y) = y$, and for all $z \in (x, y)$ one has $\pi(z) < z$;

(iii) $(P_=)^{\mathcal{M}(\pi)}$ is the set of all open intervals (x, y) in $[0,1]$ that are *maximal* with respect to the property of $\pi|_{(x,y)} = \mathrm{id}|_{(x,y)}$ – in other words

$$\forall z \in (x, y)(\pi(z) = z),$$

but for all $\epsilon > 0$ there exist $z_0 \in (x - \epsilon, x)$ and $z_1 \in (y, y + \epsilon)$

$$\pi(z_0) \neq z_0,$$
$$\pi(z_1) \neq z_1.$$

For $(x, y), (x', y') \in \mathcal{M}(\pi)$ we set $(x, y) < (x', y')$ if and only if $y \leq x'$. Since the intervals are non-overlapping this defines a strict linear ordering on the countable structure $\mathcal{M}(\pi)$. It is more or less immediate that $\pi_1 \approx \pi_2 \Rightarrow \mathcal{M}(\pi_1) \cong \mathcal{M}(\pi_2)$. Strictly speaking we need to build a structure with $\mathbb{N}$ as its underlying set, but there are no special difficulties in obtaining in a Borel fashion from π a model isomorphic to $\mathcal{M}(\pi)$ in $\mathrm{Mod}(\mathcal{L})$. So this leaves us just with showing that

$$\mathcal{M}(\pi_1) \cong \mathcal{M}(\pi_2) \Rightarrow \pi_1 \approx \pi_2.$$

So fix π_1, $\pi_2 \in \mathrm{Hom}^+([0,1])$, and $\Phi : \mathcal{M}(\pi_1) \cong \mathcal{M}(\pi_2)$ an isomorphism of models.

Claim: For all $(x, y) \in \mathcal{M}(\pi_1)$ there exists an orientation preserving homeomorphism

$$\sigma_x : (x, y) \to \Phi((x, y))$$
$$\pi_1|_{(x,y)} = \sigma_x^{-1} \circ \pi_2|_{\Phi((x,y))} \circ \sigma_x.$$

Proof of claim: Assume that $(x,y) \in (P_+)^{\mathcal{M}(\pi_1)}$; the other case of $(x,y) \in (P_-)^{\mathcal{M}(\pi_1)}$ is exactly similar and that of $(x,y) \in (P_=)^{\mathcal{M}(\pi_1)}$ trivial. π_1 is strictly increasing on (x,y), so π_2 is strictly increasing on $\Phi((x,y)) =_{df} (x',y')$ by the assumption on Φ.

Choose any $z_0 \in (x,y)$ and $w_0 \in (x',y')$ and let $z_i = (\pi_1)^i(z_0)$, $w_i = (\pi_2)^i(w_0)$ for $i \in \mathbb{Z}$. Fix some order preserving homeomorphism $\sigma' : [z_0, z_1] \rightarrow [w_0, w_1]$. Now note that there will be a unique extension $\sigma_x : (x,y) \rightarrow (x',y')$ with $\sigma_x((\pi_1)^i(z)) = (\pi_2)^i(\sigma'(z))$ for any $z \in [z_0, z_1]$. (□Claim)

Now take $\sigma'' = \bigcup\{\sigma_x : (x,y) \in \mathcal{M}(\pi_1)\}$. σ'' is defined on an open dense subset of $[0,1]$, has open dense range, is order preserving, and hence σ'' extends to some $\sigma \in \mathrm{Hom}^+([0,1])$. Since for a dense set of $x \in [0,1]$

$$\sigma^{-1}(\pi_2(\sigma(x))) = \pi_1(x)$$

it follows that that σ conjugates π_1 and π_2. □

Lemma 4.7. *Let G be a Polish group and $H < G$ a clopen subgroup. Let X be a Polish G-space. If E_H^X admits classification by countable models, so does E_G^X.*

Proof. Let $\theta : X \rightarrow \mathrm{Mod}(\mathcal{L})$ be a Borel function witnessing the assumption on E_H^X, and F a fresh binary relation. Take $\mathcal{L}'$ to be the language generated by $\mathcal{L}$ and F. Since G is separable and H open we can find $\{Hg_i : i \in \mathbb{N}\}$ a countable enumeration of the left cosets of H. We will define $\theta'(x) \in \mathrm{Mod}(\mathcal{L}')$ to be something like the disjoint union of $\{\theta(g_i \cdot x) : i \in \mathbb{N}\}$.

Now for any $x \in X$ we can take $\theta'(x)$ to be an $\mathcal{L}'$ model such that: F defines an equivalence relation on $\theta'(x)$; for $(A_i)_{i \in \mathbb{N}}$ some enumeration of the F-equivalence classes we have that each A_i is isomorphic as an $\mathcal{L}$-structure to $\theta(g_i \cdot x)$; for S a relation symbol in $\mathcal{L}$ and $\vec{a} \in A_i^{<\mathbb{N}}$, we have

$$(\theta(x) \models S(\vec{a}, \vec{b})) \Rightarrow \vec{b} \in A_i^{<\mathbb{N}}.$$

□

Lemma 4.8. *Let $\sim'$ denote conjugacy on* $\mathrm{Hom}([0,1])$ *by elements in* $\mathrm{Hom}^+([0,1])$, *so that $\pi_1, \pi_2 \in \mathrm{Hom}([0,1])$ are $\sim'$-equivalent if there is some $\sigma \in \mathrm{Hom}^+([0,1])$ so that*

$$\sigma \pi_1 \sigma^{-1} = \pi_2.$$

Then $\sim'$ admits classification by countable models.

Proof. In light of 4.6 it suffices to assign countable models to $\pi : [0,2] \rightarrow [0,2]$ that are orientation *reversing* and fixing 1 – that is,

$$\pi(0) = 2, \pi(1) = 1, \pi(2) = 0.$$

(The transition here from $\mathrm{Hom}([0,1])$ to $\mathrm{Hom}([0,2])$ is clearly harmless, and any orientation reversing map must have a fixed point, which we may as well assume to be 1.) For such $\pi : [0,2] \rightarrow [0,2]$ orientation reversing we let $\mathcal{N}(\pi) = \mathcal{M}(\pi^2|_{[0,1]})$, where $\mathcal{M}(\pi^2|_{[0,1]})$ is chosen according to 4.6. It suffices to show that if $\mathcal{N}(\pi_1) \cong \mathcal{N}(\pi_2)$ then $\pi_1 \sim' \pi_2$.

So fix π_1, π_2 with $\mathcal{N}(\pi_1) \cong \mathcal{N}(\pi_2)$. By 4.6 find $\sigma_0 \in \mathrm{Hom}^+([0,1])$ with

$$\sigma_0 \circ (\pi_1)^2|_{[0,1]} \circ \sigma_0^{-1} = (\pi_2)^2|_{[0,1]}.$$

Let $\hat{\pi}_1 = \hat{\sigma}_0 \pi_1 \hat{\sigma}_0^{-1}$, where

$$\hat{\sigma}_0|_{[0,1]} = \sigma_0,$$

$$\hat{\sigma}_0|_{[1,2]} = \mathrm{id}|_{[1,2]}.$$

It suffices to show $\hat{\pi}_1$ and π_2 are conjugated by an element of $\mathrm{Hom}^+([0,2])$.

Choose $\sigma_1 \in \mathrm{Hom}^+([1,2])$ such that $\sigma_1 \circ \hat{\pi}_1|_{[0,1]} = \pi_2|_{[0,1]}$; let

$$\hat{\sigma}_1 \in \mathrm{Hom}^+([0,2]),$$
$$\hat{\sigma}_1|_{[0,1]} = \mathrm{id}_{[0,1]},$$
$$\hat{\sigma}_1|_{[1,2]} = \sigma_1.$$

At once

$$\hat{\sigma}_1 \circ \hat{\pi}_1 \circ \hat{\sigma}_1^{-1}|_{[0,1]} = \pi_2|_{[0,1]};$$

so to obtain

$$\hat{\sigma}_1 \circ \hat{\pi}_1 \circ \hat{\sigma}_1^{-1}|_{[1,2]} = \pi_2|_{[1,2]}$$

it suffices to check that

$$(\hat{\sigma}_1 \circ \hat{\pi}_1 \circ \hat{\sigma}_1^{-1})|_{[1,2]} \circ (\hat{\sigma}_1 \circ \hat{\pi}_1 \circ \hat{\sigma}_1^{-1})|_{[0,1]} = (\hat{\sigma}_1 \circ \hat{\pi}_1 \circ \hat{\sigma}_1^{-1})^2|_{[0,1]}$$

is equal to

$$(\pi_2)^2|_{[0,1]} = \pi_2|_{[1,2]} \circ \pi_2|_{[0,1]} = \pi_2|_{[1,2]} \circ \hat{\sigma}_1 \circ \hat{\pi}_1 \circ \hat{\sigma}_1^{-1}|_{[0,1]}.$$

Since $\hat{\sigma}_1|_{[0,1]} = \mathrm{id}|_{[0,1]}$ this would amount to asserting

$$(\hat{\pi}_1)^2|_{[0,1]} = (\pi_2)^2|_{[0,1]}$$

which is exactly

$$(\hat{\sigma}_0 \circ \pi_1 \circ \hat{\sigma}_0^{-1})^2|_{[0,1]} = (\pi_2)^2|_{[0,1]}.$$

□

THEOREM 4.9. *The orbit equivalence relation $\approx'$ induced by the self action of* $\mathrm{Hom}([0,1])$ *by conjugation admits classification by countable models.*

PROOF. By 4.8 and 4.7 since $\mathrm{Hom}^+([0,1])$ clopen in $\mathrm{Hom}([0,1])$. □

On the other hand we obtain a kind of converse using the following from [**20**]:

THEOREM 4.10. *(Friedman, Stanley) Let $<$ be a binary relation, and define the space* Mod(LO) *be the class of $\{<\}$-structures on $\mathbb{N}$ that are linear orderings. Then for any countable language $\mathcal{L}$*

$$\cong|_{\mathrm{Mod}(\mathcal{L})} \leq_B \cong|_{\mathrm{Mod(LO)}}.$$

PROOF. First assume that $\mathcal{L}$ is generated by a single binary relation R. Fixing an $\mathcal{L}$-structure $\mathcal{M}$ we describe a countable linear ordering $L(\mathcal{M})$. The linear ordering will not actually have $\mathbb{N}$ as its underlying set, so we leave to the reader the routine verifications that this may be reorganized to provide in a Borel fashion a linear ordering of $\mathbb{N}$ isomorphic to $L(\mathcal{M})$.

Let η be some transcendental number and let

$$s(\cdot,\cdot) : \mathbb{N}^2 \hookrightarrow \mathbb{N}$$

be a pairing function. Let $(p_i)_{i\in\mathbb{N}}$ enumerate the primes greater than 3. For each $\vec{a} = \langle a_0, a_1, ..., a_n\rangle$ in $\mathcal{M}^{<\mathbb{N}}$ we associate the natural number

$$N(\vec{a}) = 3^n \cdot \prod_{i,j\leq n} p_{s(i,j)}^{\epsilon_{i,j}},$$

where $\epsilon_{i,j} = 3$ if $a_i \neq a_j$ and $\mathcal{M} \models R(a_i, a_j)$, $\epsilon_{i,j} = 2$ if $a_i \neq a_j$ and $\mathcal{M} \models \neg R(a_i, a_j)$, $\epsilon_{i,j} = 1$ if $a_i = a_j$ and $\mathcal{M} \models R(a_i, a_j)$, $\epsilon_{i,j} = 0$ if $a_i = a_j$ and $\mathcal{M} \models \neg R(a_i, a_j)$.

Thus $N(\vec{a})$ is a natural number that encodes the length of $\vec{a}$ and how the various elements of the sequence are related.

We let $B_i = (1,2) \cap (\mathbb{Q} + (\eta^i))$. Let

$$B = \bigcup_{i \in \mathbb{N}} B_i.$$

The relevant features are that the B_i's partition B into countably many dense subsets, B is a linear ordering without endpoints, B is dense in itself (in the sense that any open interval in B contains some point), and that any non-positive integer is smaller than any element in B.

We then let $L(\mathcal{M})$ consist of all finite sequences of the form $\langle b_0, ..., b_{n-1}, -k\rangle$ where

(i) each $b_i \in B$;
(ii) $k \in \mathbb{N}$, and $k \leq N(\vec{a})$ where $\langle a_0, ..., a_{n-1}\rangle$ is defined by having at each j

$$b_j \in B_{a_j}.$$

We give $L(\mathcal{M})$ the lexicographic ordering. Thus

$$\langle b_0, ..., b_{n-1}, -k\rangle < \langle b'_0, ..., b'_{m-1}, -k'\rangle$$

if either

(i) there is $i < \min(n, m)$ with $b_i \neq b'_i$, and for i least such we have $b_i < b'_i$; or
(ii) $n < m$ and for $i < n$ we have $b_i = b'_i$; or
(iii) $\langle b_0, ..., b_{n-1}\rangle = \langle b'_0, ..., b'_{m-1}\rangle$ and $-k < -k'$.

As a rough picture, the idea of this construction is that we begin with a linear ordering consisting of copies of finite sequences from $\mathcal{M}$, so that each sequence appears in some sense $\mathbb{Q}$-often, and to the near right of any appearance of a sequence we have appearances of all its possible extensions. If at each $i \leq n$ we have some $b_i \in B_{a(i)}$ for some $a(i) \in \mathcal{M}$, we should choose to think of the sequence $\langle b_0, ..., b_n\rangle$ as representing the sequence $\langle a(0), a(1), ..., a(n)\rangle$. Since each B_i is dense, there are not only many representatives of each $\langle a(0), a(1), ..., a(n)\rangle$, but moreover given $\langle b_0, ..., b_n\rangle$ and $a \in \mathcal{M}$ there are many choices of $b \in B_a$ with which we can extend our sequence to $\langle b_0, ..., b_n, b\rangle$ so that it now represents $\langle a(0), a(1), ..., a(n), a\rangle$. We have then taken the further step of replacing each point of the form $\langle b_0, ..., b_n\rangle$ with a *finite* discrete linear ordering whose size indicates how the terms in the corresponding $\langle a(0), a(1), ..., a(n)\rangle \in \mathcal{M}^{<\mathbb{N}}$ sequence are related.

Although the ideas are similar in both cases, the easier direction of the proof of the theorem is in showing $\mathcal{M} \cong \mathcal{N}$ implies $L(\mathcal{M})$ and $L(\mathcal{N})$ are isomorphic. So let us concentrate instead on proving the converse and suppose that

$$\pi : L(\mathcal{M}) \cong L(\mathcal{N})$$

is an isomorphism between the linear orderings. Let E_π be the set of all pairs of sequences

$$(\vec{a}, \vec{c}) = (\langle a_0, ..., a_m\rangle, \langle c_0, ..., c_n\rangle) \in \mathcal{M}^{<\mathbb{N}} \times \mathcal{N}^{<\mathbb{N}}$$

such that

$$\langle b_0, ..., b_m, -k\rangle \in L(\mathcal{M})$$

and

$$\langle b'_0, ..., b'_m, -k'\rangle \in L(\mathcal{N})$$

with each $b_i \in B_{a_i}$, each $b'_i \in B_{c_i}$ and

$$\pi(\langle b_0, ..., b_m\rangle) = \langle b'_0, ..., b'_n\rangle.$$

Claim(I) $(\langle a_0, ..., a_m\rangle, \langle c_0, ..., c_n\rangle) \in E_\pi$ implies that

(a) $m = n$; and for all $i, j \leq m$

(b) $\mathcal{M} \models R(a_i, a_j)$ if and only if $\mathcal{N} \models R(c_i, c_j)$;

(c) $a_i = a_j$ if and only $c_i = c_j$.

Proof of claim. Note that $N(\vec{a})$ is equal to the size of the maximal finite interval containing $\langle b_0, ..., b_m, -k\rangle$ in $L(\mathcal{M})$. (That is to say: There are $x, \hat{x} \in L(\mathcal{M})$ with $x \leq \langle b_0, ..., b_m, -k\rangle \leq \hat{x}$ having exactly $N(\vec{a})$ many objects in the closed interval $[x, \hat{x}]^{L(\mathcal{M})}$, and moreover for any $y < x$ and $\hat{y} > \hat{x}$, the intervals $[y, x]^{L(\mathcal{M})}$ and $[\hat{x}, \hat{y}]^{L(\mathcal{M})}$ are both infinite.) Since the same holds on the $\mathcal{N}$ side and π is an isomorphism of linear orderings, it follows that $N(\vec{a}) = N(\vec{c})$, and thus we have the claim by the definition of $N(\vec{a})$. (Claim$\square$)

Claim(II) For all $(\langle a_0, ..., a_m\rangle, \langle c_0, ..., c_m\rangle) \in E_\pi$ and $a \in \mathcal{M}$ there is some $c \in \mathcal{N}$ with

$$(\langle a_0, ..., a_m, a\rangle, \langle c_0, ..., c_m, c\rangle) \in E_\pi.$$

Proof of claim. Fix $\langle b_0, ..., b_m, -k\rangle \in L(\mathcal{M})$ and $\langle b'_0, ..., b'_m, -k'\rangle \in L(\mathcal{N})$ witnessing the definition of $(\langle a_0, ..., a_m\rangle, \langle c_0, ..., c_m\rangle) \in E_\pi$ – that is to say, each $b_i \in B_{a_i}$, $b'_i \in B_{c_i}$, and $\pi(\langle b_0, ..., b_m, -k\rangle) = \langle b'_0, ..., b'_m, -k'\rangle$. Choose $b \in B_a$ and let

$$\pi(\langle b_0, ..., b_m, b, 0\rangle) = \langle \bar{b}'_0, ..., \bar{b}'_m, \bar{b}_{m+1}, -\bar{k}\rangle.$$

Note that there is no maximal finite closed interval strictly between $\langle b_0, ..., b_m, -k\rangle$ and $\langle b_0, ..., b_m, b, 0\rangle$ whose size is indivisible by 3^{m+1}. Therefore since π is an isomorphism there can be no maximal finite closed interval strictly between $\langle b'_0, ..., b'_m, -k'\rangle$ and $\langle \bar{b}_0, ..., \bar{b}'_m, \bar{b}_{m+1}, -\bar{k}\rangle$ whose size is indivisible by 3^{m+1}. Considering the definition of the linear orderings we obtain

$$\langle \bar{b}'_0, ..., \bar{b}'_m, \bar{b}_{m+1}, -\bar{k}\rangle = \langle b'_0, ..., b'_m, b', -\bar{k}\rangle,$$

some b', and then for c such that $b' \in B_c$ we have $\langle c_0, ..., c_m, c\rangle$ as required. (Claim$\square$)

Claim(III) For all $(\langle a_0, ..., a_m\rangle, \langle c_0, ..., c_m\rangle) \in E_\pi$ and $c \in \mathcal{N}$ there is some $a \in \mathcal{M}$ with

$$(\langle a_0, ..., a_m, a\rangle, \langle c_0, ..., c_m, c\rangle) \in E_\pi.$$

Proof of claim. As in (II). (Claim$\square$)

We can now use a technique known in model theory as a *back-and-forth* argument, and dating in some sense to the work of Cantor, to obtain an isomorphism between $\mathcal{M}$ and $\mathcal{N}$.

We first let $(x_i)_{i\in\mathbb{N}}$ enumerate $\mathcal{M}$ and $(y_i)_{i\in\mathbb{N}}$ enumerate $\mathcal{N}$. We then build sequences of $a_0, a_1, ..., a_i, ... \in \mathcal{M}$ and $c_0, c_1, ..., c_i, ... \in \mathcal{N}$ such that

(i) each $(\langle a_0, ...a_i\rangle, \langle c_0, ..., c_i\rangle) \in E_\pi$;

(ii) each $x_n \in \{a_0, ...a_{2n}\}$;

(iii) each $y_n \in \{c_0, ..., c_{2n+1}\}$.

We build this sequence inductively, using claim(II) to propagate (ii), claim(III) for (iii). We then define

$$\rho : \mathcal{M} \to \mathcal{N}$$

by

$$a_i \mapsto c_i.$$

By claim(I)(c) this is well defined. By (iii) it is onto and by (ii) it is everywhere defined. Claim(I)(b) requires it to respect the relation R, and therefore provide an isomorphism between $\mathcal{M}$ and $\mathcal{N}$.

We skipped over the easier direction of the equivalence

$$\mathcal{M} \cong \mathcal{N} \Leftrightarrow L(\mathcal{M}) \cong L(\mathcal{N}),$$

but it resembles the back and forth argument above. The main point is that given any bijection π of $\mathbb{N}$, we may find an isomorphism

$$\rho : (B, <) \cong (B, <)$$

such that at each i

$$\rho[B_i] = B_{\pi(i)}.$$

We are only left to describe the general construction for general $\mathcal{L}$. For $\mathcal{L}$ a countable relational language language, we let $(\mathcal{L}_n)$ be an increasing sequence of finite languages with

$$\mathcal{L} = \bigcup_{n \in \mathbb{N}} \mathcal{L}_n.$$

We then proceed as before, but with the one change that $N(\langle a_0, a_1, ..., a_n\rangle)$ will be a natural number that encodes for us how the various elements of the sequence $\langle a_0, a_1, ..., a_n\rangle$ are related over $\mathcal{L}_n$. $\square$

Corollary 4.11. *(folklore) Let $\mathcal{L}$ be a countable language. Then*

$$\cong|_{\mathrm{Mod}(\mathcal{L})} \leq_B \approx .$$

Proof. Following the proof of 4.6 we may encode any countable linear ordering by an element of $\mathrm{Hom}^+([0,1])$ considered up to conjugacy. Now it follows by 4.10. $\square$

Exercise 4.12. Let $\mathcal{M} \in \mathrm{Mod(LO)}$. Then the automorphism group of $\mathcal{M}$, $\mathrm{Aut}(\mathcal{M})$, can be realized as a closed subgroup of $\mathrm{Hom}^+([0,1])$.

Exercise 4.13. The shift action of $\mathrm{Hom}^+([0,1])$ on $\mathcal{F}([0,1])$, given by

$$\pi \cdot C = \{\pi(x) : x \in C\},$$

admits classification by countable models.

Exercise 4.14. For $\mathcal{L}$ a countable language, $\cong|_{\mathrm{Mod}(\mathcal{L})} \leq_B \approx'$.

4.3. Non-classification in dimension 2

Notation 4.15. For the remainder of this section let $G = \prod_{\mathbb{N}} \mathbb{Z}$, with the product group and topological structure, under which it becomes a Polish group. Define

$$\hat{G} = \{\vec{x} \in G : x_n/n \to 0\};$$

so in some sense this is the subgroup consisting of sequences which do not stray too quickly from 0. Note that $\hat{G}$ is a Polishable group, as witnessed by the complete metric

$$d(\vec{x}, \vec{y}) = \sup_{\mathbb{N}} |(x_n - y_n)/n|.$$

Lemma 4.16. *The coset equivalence relation of $\hat{G}$ on G does not admit classification by countable structures.*

PROOF. The action of $\hat{G}$ on G is turbulent: Clearly every orbit is dense; each orbit is meager by 2.56 since $\hat{G} = \hat{G}^{-1}\hat{G}$ contains no open sets; as for condition (iii) in the definition of turbulence, note that if $\vec{g} \in G$,

$$U \supset \{\vec{h} \in G : \vec{h}|_n = \vec{g}|_n\}$$

and

$$V \supset \{\vec{h} \in \hat{G} : d(\vec{h}, \vec{0}) < 1/n\},$$

(where $\vec{0} = (0, 0, ...)$ is the identity in $\hat{G}$) then

$$\overline{\mathcal{O}(\vec{h}, U, V)} \supset \{\vec{g} \in G : \vec{g}|_{n+1} = \vec{h}|_{n+1}\}.$$

□

THEOREM 4.17. *The conjugacy equivalence relation* $\approx''$ *on* $\mathrm{Hom}([0,1])^2$ *does not admit classification by countable models.*

PROOF. It suffices to show that

$$E^G_{\hat{G}} \leq_B \approx'' .$$

The notation is smoother working with $\mathrm{Hom}([-1,1] \times [0,1])$. We define closed rectangles $I_{n,k}$ inside $[0,1] \times [0,1]$ for $n \in \mathbb{N}, n > 0, k \in \mathbb{Z}$, and for that purpose fix an order preserving homeomorphism $f : \mathbb{R} \to (0,1)$. The closed rectangle $I_{n,k}$ has vertices

$(1/2n, f(k/n))$, $\quad$ $(1/(2n+1), f(k/n))$
and
$(1/2n, f((k+1)/n))$, $\quad$ $(1/(2n+1), f((k+1)/n))$.

Also define closed rectangles J_0, J_1, with J_0 occupying the top half of $[-1,0] \times [0,1]$ and having vertices

$(-1, 1)$, $\quad$ $(0, 1)$,
and
$(-1, 1/2)$, $\quad$ $(0, 1/2)$,

and J_1 occupying the bottom half with vertices

$(-1, 1/2)$, $\quad$ $(0, 1/2)$,
and
$(-1, 0)$, $\quad$ $(0, 0)$.

We then choose $\pi_{n,k} \in \mathrm{Hom}(I_{n,k})$, for all n, k, $\pi_0 \in \mathrm{Hom}(J_0)$, and $\pi_1 \in \mathrm{Hom}(J_1)$ such that:

(i) for all (x, y) in $(I_{n,k})^o$ (the interior of $I_{n,k}$), $\pi_{n,k}(x, y) = (x, y')$ for some $y' > y$;

(ii) as (x, y) approaches $\partial(I_{n,k})$ (the boundary of $I_{n,k}$), the distance $d(\pi_{n,k}(x, y), (x, y))$ (the usual euclidean distance from $\pi_{n,k}(x, y)$ to (x, y)) approaches 0;

(iii) as $(x, y) \to [-1, 0] \times [0, 1]$

$$d(\pi_{n,k}(x, y), (x, y)) \to 0$$

uniformly in n, k, in the sense that for all $\delta > 0$ there will be some $\delta' > 0$ for whom all n, k and $(x, y) \in I_{n,k}$ have

$$d((x, y), [-1, 0] \times [0, 1]) < \delta' \Rightarrow d(\pi_{n,k}(x, y), (x, y)) < \delta;$$

(iv) as (x, y) in J_0 approaches $\partial(J_0)$ (the boundary of J_0 as evaluated in $\mathbb{R}^2$)

$$d(\pi_0(x, y), (x, y)) \to 0;$$

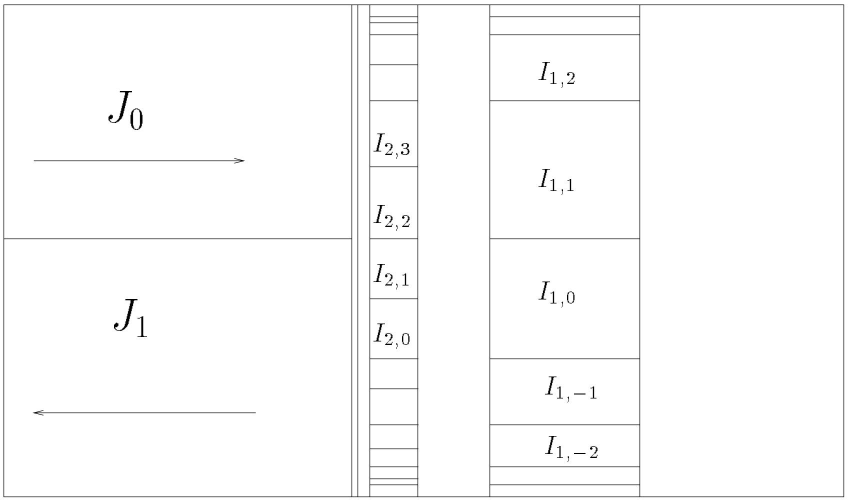

FIGURE 4.1. The homeomorphisms defined region by region

(v) as (x, y) in J_1 approaches $\partial(J_1)$

$$d(\pi_1(x, y), (x, y)) \to 0;$$

(vi) for all $(x, y) \in (J_0)^o$ we have $\pi_0(x, y) = (x', y)$ some $x' > x$;

(vii) for all $(x, y) \in (J_1)^o$ we have $\pi_0(x, y) = (x', y)$ some $x' < x$.

We also need to set $\sigma_{n,k,l} : I_{n,k} \to I_{n,k+l}$ given by

$$(x, f(y)) \mapsto (x, f(y + l/n)),$$

and assume that this is chosen previously in harmony with the $\pi_{n,k}$ so that

$$\pi_{n,k+l} = \sigma_{n,k,l} \circ \pi_{n,k} \circ \sigma_{n,k,l}^{-1}.$$

We further need that the orbits of any point flow as *far* as possible granting the above restrictions. So:

(viii) for all $(x, y), (x, y') \in (I_{n,k})^o$ there exists $i \in \mathbb{Z}$ and $y'' > y'$ such that

$$(\pi_{n,k})^i(x, y) = (x, y'');$$

(ix) for all $(x, y), (x', y) \in (J_j)^o$ $(j = 0, 1)$ there exists $i \in \mathbb{Z}$ and $x'' > x'$ such that

$$(\pi_j)^i(x, y) = (x'', y).$$

Assuming all the above define

$$\rho : G \to \text{Hom}([-1, 1] \times [0, 1]).$$

For $\vec{g} \in G$ define $\rho(\vec{g})$ by the requirement that:

(a) $\rho(\vec{g})|_{[-1,0]\times[0,1]} = \pi_0 \cup \pi_1$;

(bi) for $\vec{g}(n) = k$

$$\rho(\vec{g})|_{I_{n,k}} = \pi_{n,k}^{-1};$$

(bii) for $\vec{g}(n) \neq k$

$$\rho(\vec{g})|_{I_{n,k}} = \pi_{n,k};$$

(c) at all other $(x, y) \in [-1, 1] \times [0, 1]$ we have simply $\rho(\vec{g})(x, y) = (x, y)$.

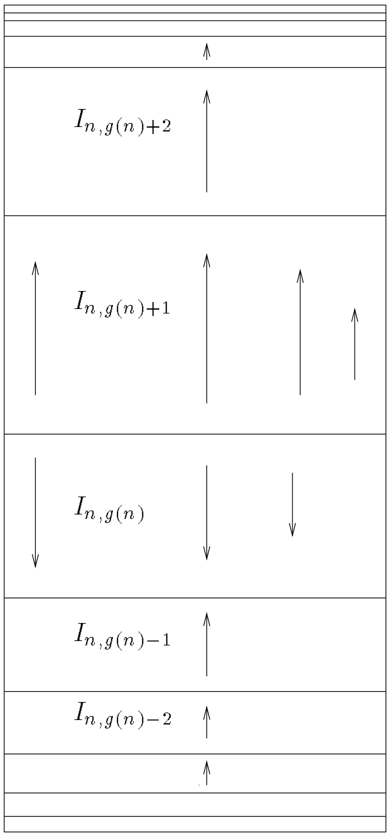

FIGURE 4.2. A closer look

Claim(I): For all $\vec{g}, \vec{h} \in G$ if $\pi \in \mathrm{Hom}([-1,1] \times [0,1])$ with

$$\pi \circ \rho(\vec{g}) \circ \pi^{-1} = \rho(\vec{h})$$

then

$$\pi[[-1,0] \times [0,1]] = [-1,0] \times [0,1],$$
$$\pi[\{0\} \times [0,1]] = \{0\} \times [0,1],$$
$$\pi[\{0\} \times [0,1/2]] = \{0\} \times [0,1/2],$$

and so the point $(0,1/2)$ is fixed by π.

Proof of claim: Let $D_0(\vec{h})$ be the set of all $z \in ([-1,1] \times [0,1])^o$ for which there is some $z' \in \partial([-1,1] \times [0,1])$ with

$$(\rho(\vec{h}))^i(z) \to z'$$

as

$$i \to -\infty.$$

Define $D_1(\vec{h})$ to be the $z \in ([-1,1] \times [0,1])^o$ for which there is some $z' \in \partial([-1,1] \times [0,1])$ with

$$(\rho(\vec{h}))^i(z) \to z'$$

as
$$i \to +\infty.$$
$D_0(\vec{h}) = (J_0)^o$ and $D_1(\vec{h}) = (J_1)^o$ and
$$\{0\} \times (1/2, 1) = \partial(D_0(\vec{h})) \cap ([-1,1] \times [0,1])^o$$
and
$$\{0\} \times (0, 1/2) = \partial(D_1(\vec{h})) \cap ([-1,1] \times [0,1])^o.$$
Similarly
$$\{0\} \times [1/2, 1) = \partial(D_0(\vec{g})) \cap ([-1,1] \times [0,1])^o$$
and
$$\{0\} \times (0, 1/2] = \partial(D_1(\vec{g})) \cap ([-1,1] \times [0,1])^o.$$
By the nature of the definitions $\pi[D_0(\vec{g})] = D_0(\vec{h})$, $\pi[D_1(\vec{g})] = D_1(\vec{h})$, thereby giving the claim. (Claim(I)$\Box$)

Claim(II) If $\vec{g}, \vec{h} \in G$ and $\pi \in \mathrm{Hom}([-1,1] \times [0,1])$ with
$$\pi \circ \rho(\vec{g}) \circ \pi^{-1} = \rho(\vec{h})$$
then for all $n \in \mathbb{N}$
$$\pi[I_{n,g(n)}] = I_{n,h(n)}.$$
Proof of claim. From (I), $[0,1] \times [0,1]$ is a π invariant set. Since the strips of the form
$$(\frac{1}{n}, \frac{1}{n+1}) \times [0,1]$$
are seen to be invariant by a mathematical induction on n, and then since each $(I_{n,m})^o$ is a maximal connected open set in $[0,1]^2$ on which $\rho(\vec{g})$ and $\rho(\vec{h})$ have no fixed points, we obtain $\pi(I_{n,g(n)}) = I_{n,m}$ some m. We can characterize $(I_{n,g(n)})^o$ inside $\{(I_{n,k})^o : k \in \mathbb{Z}\}$ as unique $(I_{n,k})^o$ for which there is some other
$$(I_{n,l})^o \neq (I_{n,k})^o$$
with
$$\{\lim_{i\to+\infty}\rho(\vec{g})^i(z) : z \in (I_{n,l})^o\} =$$
$$\{\lim_{i\to+\infty}\rho(\vec{g})^i(z) : z \in (I_{n,k})^o\}.$$
The same definition works for $\rho(\vec{h})$ to describe $h(n)$, and so we have $m = h(n)$. ($\Box$Claim(II))

Claim(III) If $\vec{g}, \vec{h} \in G$ and $\pi \in \mathrm{Hom}([-1,1] \times [0,1])$ with
$$\pi \circ \rho(\vec{g}) \circ \pi^{-1} = \rho(\vec{h})$$
then for all $n \in \mathbb{N}$ and $l \in \mathbb{Z}$
$$\pi[I_{n,g(n)+l}] = I_{n,h(n)+l}.$$
Proof of Claim: We can specify each $(I_{n,h(n)+1})^o$ inside $\{(I_{n,k})^o : k \in \mathbb{Z}\}$ as the unique $(I_{n,k})^o$ for which
$$\{\lim_{i\to-\infty}\rho(\vec{g})^i(z) : z \in (I_{n,g(n)})^o\} =$$
$$\{\lim_{i\to-\infty}\rho(\vec{g})^i(z) : z \in (I_{n,k})^o\}.$$
Inductively for $l > 0$ we may describe $(I_{n,g(n)+l+1})^o$ as the unique $(I_{n,k})$ with
$$\{\lim_{i\to-\infty}\rho(\vec{g})^i(z) : z \in (I_{n,k})^o\} =$$
$$\{\lim_{i\to+\infty}\rho(\vec{g})^i(z) : z \in (I_{n,g(n)+l})^o\}.$$

A similar characterization holds for the $(I_{n,g(n)-l})^o$'s, $l > 0$, and all these formulations work symmetrically on the $\vec{h}$ side. ($\square$Claim(III))

Then for $\vec{g}, \vec{h}, \pi$ as above, claims (I) and (III) above imply that

$$d(I_{n,0}, I_{n,h(n)-g(n)}) \rightarrow 0$$

$n \rightarrow 0$. Thus

$$|f(\frac{h(n)-g(n)}{n}) - f(0)| \rightarrow 0$$

and hence

$$\frac{h(n)-g(n)}{n} \rightarrow 0$$

which amounts to asserting that $\vec{h} - \vec{g} \in \hat{G}$.

Conversely, if $\vec{h} - \vec{g} \in \hat{G}$ then we may define $\sigma_0 \in \mathrm{Hom}([0,1]^2)$ so that

$$\sigma_0|_{I_{n,l}} = \sigma_{n,l,h(n)-g(n)};$$

on the strips of the form

$$(\frac{1}{2n+2}, \frac{1}{2n+1}) \times [0,1],$$

for $n > 0$, take the "average" of

$$\sigma_0|_{(1/(2n+1),1/(2n))\times[0,1]}$$

and

$$\sigma_0|_{(1/(2n+3),1/(2n+2))\times[0,1]},$$

so that for $z \in \mathbb{R}$, $\lambda \in (0,1)$ let $\sigma_0(\frac{1}{2n+1+\lambda}, f(z))$ equal

$$(\frac{1}{2n+1+\lambda}, f(z + \frac{(h(n)-g(n))(1-\lambda)}{n} + \frac{(h(n+1)-g(n+1))\lambda}{n+1})).$$

On the strip $(1/2, 1) \times [0,1]$ average $\sigma_0|_{(1/4,1/2)\times[0,1]}$ and the identity, so that

$$\sigma_0(\frac{1}{1+\lambda}, f(z)) = (\frac{1}{1+\lambda}, f(z + \frac{h(1)-g(1)}{\lambda})).$$

By the assumption that $|g(n) - h(n)|/n \rightarrow 0$ as $n \rightarrow \infty$

$$d(z, \sigma_0(z)) \rightarrow 0$$

as $z \rightarrow \partial([0,1]^2)$. Hence in particular σ_0 extends to $\sigma \in \mathrm{Hom}([-1,1] \times [0,1])$ with $\sigma|_{[-1,0]\times[0,1]}$ equal to the identity. It follows at once from the definitions of the $\pi_{n,k}$ and $\sigma_{n,k,l}$ that σ conjugates $\rho(\vec{g})$ and $\rho(\vec{h})$. $\square$

Su Gao had previously noted that c_0 is a closed subgroup of $\mathrm{Hom}([0,1]^2)$; the above argument was inspired by his construction.

THEOREM 4.18. $\cong|_{\mathrm{Mod}(\mathcal{L})} <_B \approx''$.

PROOF. Given the preceding theorem and 4.11 it suffices to show that $\approx \leq_B \approx''$ which in turn follows various ways.

For instance comparing the proof of 4.17 we may associate to $\pi \in \mathrm{Hom}^+([0,1])$ a homeomorphism $\sigma(\pi) \in \mathrm{Hom}([-1,2] \times [0,1])$ so that, for $f : \mathbb{R} \rightarrow (0,1)$ a homeomorphism as before,

$$\sigma(\pi)(-1+\lambda, f(y)) = (-1+\lambda, f(y+2\lambda)),$$

$$\sigma(\pi)(-\lambda, f(y)) = (-\lambda, f(y+2\lambda))$$

for $\lambda \in [0, 1/2]$, and

$$\sigma(\pi)|_{[1,2]\times[0,1]} = \mathrm{id}|_{[1,2]\times[0,1]}.$$

Thus this is a homeomorphism that shunts every point in $(-1,0)\times(0,1)$ up towards the top line $(-1,0)\times\{1\}$ and uniformly fades away to the identity at the boundary. On $[0,1]\times[0,1]$ we have

$$(\sigma(\pi))(x,y) = (\pi(x), y).$$

Hence $(\sigma(\pi))|_{[0,1]\times\{0\}}$ is conjugate by a homeomorphism to π.

Note that the boundary $\partial([-1,2]\times[0,1])$ is definable in terms of properties which are internal to $[-1,2]\times[0,1]$ viewed purely as a topological space. $(-1,0)\times\{1\}$ is characterized as the maximal (non-trivial) line segment included in the intersection of

$$\partial([-1,2]\times[0,1])$$

and

$$\{\lim_{i\in\mathbb{N}}(\sigma(\pi))^i(z) : z \in [-1,2]\times[0,1], (\sigma(\pi)) \neq z\}.$$

We then can characterize $(-1,0)\times(0,1)$ as the points whose limit under $\sigma(\pi)$ tend towards the set $(-1,0)\times\{1\}$.

Thus if $\pi_1, \pi_2 \in \mathrm{Hom}^+([0,1])$, $\tau \in \mathrm{Hom}([-1,2]\times[0,1])$,

$$\sigma(\pi_1) = \tau^{-1}\circ\sigma(\pi_2)\circ\tau,$$

then $(-1,0)\times(0,1)$ must be τ invariant. One can then obtain that $(\sigma(\pi_1))|_{[0,1]\times\{0\}}$, $(\sigma(\pi_2))|_{[0,1]\times\{0\}}$ are conjugate by an orientation preserving homeomorphism, and thus π_1, π_2 are conjugate.

The lengthy but ultimately routine verification that

$$\pi \mapsto \sigma(\pi)$$

is Borel can be left as an exercise. □

Definition 4.19. Let $X(M)$ be the set of infinite double nested sequences $(x_{i,j})_{i,j\in\mathbb{N}}$ such that

(i) each $x_{i,j} \in \mathbb{R}^{\geq 0}$;
(ii) $x_{i,j} = x_{j,i}$ all i, j;
(iii) $x_{i,j} \leq x_{i,k} + x_{k,j}$ all i, j, k.

To each $\vec{x} = (x_{i,j})_{i,j\in\mathbb{N}} \in X(M)$ we may naturally associate a complete metric space $S((x_{i,j})_{i,j\in\mathbb{N}})$ with a dense subset $(a_i^{\vec{x}})_{i\in\mathbb{N}}$ with $d(a_i, a_j) = x_{i,j}$. Conversely, if M is a complete separable metric space then by choosing $(a_i)_{i\in\mathbb{N}}$ to be any dense subset we may obtain some $(x_{i,j})_{i,j\in\mathbb{N}} \in X(M)$ with $S((x_{i,j})_{i,j\in\mathbb{N}})$ isometric to M. Note that $X(M)$ is a closed subset of $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}$, and thus is Polish in the subspace topology. We allow the possibility of $x_{i,j} = 0$, in which case we identify $a_i^{\vec{x}}$ and $a_j^{\vec{x}}$; in this way we are able to represent finite metric spaces in $X(M)$.

Now let $X(K)$ be the set of $(x_{i,j})_{i,j\in\mathbb{N}} \in X(M) \in [0,1]^{\mathbb{N}\times\mathbb{N}}$ which code *compact* metric spaces; note I specifically ask that each $x_{i,j}$ be less than or equal to 1, but even with this restriction we may code all compact metric spaces up to homeomorphism. This is an $F_{\sigma\delta}$ subspace of $X(M)$, since compactness of $S((x_{i,j})_{i,j\in\mathbb{N}})$ amounts to the requirement that for all positive rationals q there is a finite set $F \subset \mathbb{N}$ such that for all $j \in \mathbb{N}$ there is some $i \in F$ with $x_{i,j} \leq q$.

For $\vec{x} = (x_{i,j})_{i,j\in\mathbb{N}}, \vec{y} = (y_{i,j})_{i,j\in\mathbb{N}} \in X(K)$ write $\vec{x} \cong^c \vec{y}$ if $S(\vec{x})$ and $S(\vec{y})$ are homeomorphic, $\vec{x} \cong^i \vec{y}$ if isometric.

Lemma 4.20. $\cong^c$ *is* $\underset{\sim}{\Sigma}^1_1$.

PROOF. Let $(z_{i,j})_{i,j\in\mathbb{N}}$ be a sequence of non-negative reals. We need to first argue that the set of such $\vec{z}$ for which there is some $f_{\vec{z}} : \{a_i^{\vec{x}} : i \in \mathbb{N}\} \to S(\vec{y})$ such that

(i) $d_{S(\vec{y})}(a_i^{\vec{y}}, f_{\vec{z}}(a_j^{\vec{x}})) = z_{i,j}$ each i, j;

(ii) $f_{\vec{z}}$ extends to a continuous function $F_{\vec{z}} : S(\vec{x}) \to S(\vec{y})$.

is Borel.

However the existence of an $f_{\vec{z}}$ as indicated by (i) is transparently Borel, since it amounts to verifying certain triangle inequalities along with $\inf_{i\in\mathbb{N}}(z_{i,j}) = 0$ all j and that for all $\epsilon > 0$ there is some i with $z_{i,j} < \epsilon$, and hence that the supposed point $f_{\vec{z}}(a_j^{\vec{x}})$ is in the completion of metric space provided by $\vec{y}$. The further requirement that it extend to a continuous function on $S(\vec{x})$ amounts to only asserting that for all positive rational q there is some other positive rational r such that $(a_i^{\vec{x}}, a_j^{\vec{x}}) < r \Rightarrow (f_{\vec{x}}(a_i^{\vec{x}}), f_{\vec{x}}(a_j^{\vec{x}})) < q$.

Similar calculations show that the statement that $\vec{z}$ and and $\vec{w}$ code up inverse functions $F_{\vec{z}} : S(\vec{x}) \to S(\vec{y})$ and $F_{\vec{w}} : S(\vec{y}) \to S(\vec{x})$ is again Borel. □

EXERCISE 4.21. (i) Use the proof of 4.17 to associate to each $\vec{g} \in G$ a compact metric space $K(\vec{g})$ so that

$$\vec{g} - \vec{h} \in \hat{G}$$

if and only if

$$K(\vec{g}) \cong K(\vec{h}).$$

(Hint: Build $K(\vec{g})$ like this: Begin with the compact metric space $[0,1]^2$ in $\mathbb{R}^2$. Add the the closed line segment from $(0, 1/2)$ to $(-1, 1/2)$ with the idea, as in the proof of 4.17, to distinguish the point $(0, 1/2)$. At each $I_{n,k}$ attach an isometric copy of $[0, 2^{-n-|k|}]^{|g(n)-k|+2} \times I_{n,k}$ with $[0, 2^{-n-|k|}]^{|g(n)-k|+1} \times \{0\} \times I_{n,k}$ identified with our original rectangle $I_{n,k}$; so above each $I_{n,k}$ rectangle there is a hypercube with dimension indicated by $\vec{g}$.)

(ii) Conclude that

$$G/\hat{G} \leq_B \cong^c .$$

EXERCISE 4.22. Let $\mathcal{L}$ be the language with a single binary relation $<$. Let Mod(LO) be the models in which $<$ defines a linear ordering. Show that

$$\cong|_{\mathrm{Mod(LO)}} \leq_B \cong^c .$$

(Given $\mathcal{M}$ in which $<$ defines a linear ordering, choose open subsets of the open unit interval $(0,1)$ arranged in the order type indicated by $\mathcal{M}$ and so that their union $\mathcal{U}(\mathcal{M})$ is dense in $[0,1]$. Then start defining $K(\mathcal{M})$ by beginning with the compact metric space $[0,1]^2 \setminus (\mathcal{U}(\mathcal{M}) \times (0,1))$ but fastening around the boundary of the unit square a further compact metric space which will ensure that whenever $\pi : K(\mathcal{M}) \cong^c K(\mathcal{N})$ we must have $\pi(0,0) = (0,0)$, $\pi(1,0) = (1,0)$ and $\pi[[0,1] \times \{0\}] = [0,1] \times \{0\}$.)

Thus by the last two exercises, 4.16, and 4.10,

$$\cong|_{\mathrm{Mod}(\mathcal{L})} <_B \cong^c$$

for any countable language $\mathcal{L}$.

4.4. Remarks and connections

Perhaps we should be not be surprised that there can be no result in the style of **[24]** for the classification of homeomorphisms of an arbitrary compact metric space. On the other hand, the homeomorphism group of the Cantor space, $\mathrm{Hom}(2^{\mathbb{N}})$, can be realized as a closed subgroup of S_∞, and thus by 2.39 all its orbit equivalence relations admit classification by countable models. In particular then the the classification problem for homeomorphisms of $2^{\mathbb{N}}$ allows reduction to isomorphism on countable models.

Recently Riccardo Camerlo and Su Gao in **[9]** have showed the converse: For any countable language $\mathcal{L}$ we have that $\cong|_{\mathrm{Mod}(\mathcal{L})}$ is $\leq_B$ reducible to the orbit equivalence relation of $\mathrm{Hom}(2^{\mathbb{N}})$'s self action by conjugation.

Although the equivalence relation of homeomorphism on compact metric spaces would seem to be hopelessly complicated, there is some very narrow regularity result. Kechris and Solecki have recently observed that the results of §2.5 of **[6]** imply $\cong^c$ is Borel reducible to the orbit equivalence relation resulting from a continuous action of some Polish group on a Polish space – namely the shift action of the homeomorphism group of the Hilbert cube, $\mathrm{Hom}([0,1]^{\mathbb{N}})$, on $\mathcal{F}([0,1]^{\mathbb{N}})$.

While the classification problem for compact metric spaces up to homeomorphism verges on the intolerable, the classification up to isometry is nearly trivial. It follows from the Gromov-Hausdorff metric in **[27]** that $\cong^i$ is smooth.

The method of proof is to place a pseudo metric on the collection $\mathcal{K}$ of all metric spaces whose underlying set is included in some specific set such as the reals. Firstly one defines the Hausdorff topology on the compact subsets of some specific compact metric space (K,d) – for $C_1, C_2 \subset K$ one can let

$$d_H^K(C_1,C_2) = \sup\{d(x_1,C_2) : x_1 \in C_1\} + \sup\{d(C_1,x_2) : x_2 \in C_2\}$$

where $d(x,C)$ is (as usual) the infimum of $\{d(x,y) : y \in C\}$. For $K_1, K_2 \in \mathcal{K}$ one then defines the Gromov-Hausdorff "distance" between them, $d_{GH}(K_1,K_2)$, to be the infimum of

$$d_H^K(C_1,C_2)$$

where K ranges over all possible compact metric spaces, and C_1 and C_2 range over compact subsets of K that are isometric to K_1 and K_2 respectively. It can be shown here that

$$d_{GH}(K_1,K_2) = 0$$

if and only if K_1 and K_2 are isometric, and the space $\mathcal{K}/\sim$ obtained from $\mathcal{K}$ – our space of compact spaces – by quotienting out by the equivalence relation

$$K_1 \sim K_2 \Leftrightarrow d_{GH}(K_1,K_2) = 0$$

is in fact a compact metric space. All the key calculations can be conducted in the Borel context, and so by 2.27 one has that $\cong^i$ is smooth.

CHAPTER 5

Infinite dimensional group representations

This chapter shows that the infinite dimensional representations of the free group on $\aleph_0$ many generators are generically turbulent under the natural action of U_∞. The irreducible representations of this group are seen to refuse classification by countable structures.

§2.1 defined the infinite unitary group, U_∞, to be the group of isomorphisms of Hilbert space and noted it to be Polish in the natural topology.

DEFINITION 5.1. For G be a countable group, let $\mathrm{Rep}(G, U_\infty)$ be the space of all homomorphisms from G to U_∞. We refer to elements of this space as *infinite dimensional representations*. We view this as a subspace of the G-fold product of U_∞,

$$\prod_G U_\infty.$$

We let U_∞ act in the following way: For $\rho \in \mathrm{Rep}(G, U_\infty)$ and $T \in U_\infty$ we define $T \cdot \rho$ by the requirement that at each $g \in G$

$$(T \cdot \rho)(g) = T \circ \rho(g) \circ T^{-1}.$$

I will denote the corresponding orbit equivalence relation by $\sim_r$ – so that $\rho_1 \sim_r \rho_2$ if there is some T with $T \cdot \rho_1 = \rho_2$. When context makes clear, I will just write ρ_g for $\rho(g)$.

The notion of equivalence, $\sim_r$, expressed in this definition is the natural one: We want to say that two representations are essentially the same if they become literally the same after some unitary transformation of the Hilbert space.

In the next lemma it is convenient to note that sets of the form

$$\{\rho : ||\rho_g(e) - f|| < \delta\}$$

form a subbasis for $\mathrm{Rep}(G, U_\infty)$ as e, f range over the unit vectors of the Hilbert space, g ranges over G, δ ranges over positive reals. By elementary manipulations one can then obtain a subbasis whose members are of the form

$$\{\rho : |\langle \rho_g(e), f\rangle - 1| < \delta\},$$

e, f unit vectors, $g \in G$, $\delta \in \mathbb{R}^+$.

LEMMA 5.2. *$\mathrm{Rep}(G, U_\infty)$ is a Polish U_∞-space.*

PROOF. It is a closed subset of the product space, $\prod_G U_\infty$, and so since the class of Polish spaces is preserved under countable products and the formation of closed subspaces, $\mathrm{Rep}(G, U_\infty)$ is Polish. Moreover the action of U_∞ on this space is continuous, since given a subbasic open set of the form

$$U = \{\rho : |\langle \rho_g(e), f\rangle - 1| < \delta\},$$

for elements e, f of the Hilbert space, and given $T_0 \in U_\infty$, and given τ such that $T_0 \cdot \tau \in U$ it is clear that for any T and ρ with $T^{-1}(e)$, $T^{-1}(f)$, and $\rho_g \circ T_0^{-1}(e)$ sufficiently close to $T_0^{-1}(e)$, $T_0^{-1}(f)$, and $\tau_g \circ T_0^{-1}(e)$ respectively we have

$$\langle((T \cdot \rho)(g))(e), f\rangle = \langle\rho_g(T^{-1}(e)), T^{-1}(f)\rangle$$

very close to

$$\langle\tau_g(T_0^{-1}(e)), T_0^{-1}(f)\rangle = \langle((T_0 \cdot \tau)(g))(e), f\rangle,$$

since ρ_g is an isometry. □

EXAMPLES 5.3. Let $\{e_i : i \in \mathbb{N}\}$ be an orthonormal basis for the Hilbert space.

(a) Let $G = \mathbb{Z}$, $s \in \mathbb{R}$, and $i \neq j \in \mathbb{N}$. Then we may define $\rho \in \mathrm{Rep}(\mathbb{Z}, U_\infty)$ by

$$\rho_n(e_i) = e_i \cos(ns) + e_j \sin(ns),$$

$$\rho_n(e_j) = -e_i \sin(ns) + e_j \cos(ns),$$

and

$$\rho_n(e_k) = e_k$$

for $k \neq i, j$. Each ρ_n extends uniquely to the rest of the Hilbert space after being so defined on the basis.

(b) Now suppose G is a countable subgroup of S_∞. Then we could define $\rho : G \to U_\infty$ by

$$\rho_g(e_i) = e_{g(j)},$$

and extend as above. Indeed, this is a proof that S_∞ can be realized as a closed subgroup of the infinite unitary group.

(c) Suppose G is any countable group and $(\tau^i)_{i\in\mathbb{N}}$ is some infinite sequence of representations. Then we may form a kind of direct product of these representations as follows: Given a partition of the basis $\{e_k : k \in \mathbb{N}\}$ into infinite sets $A_i = \{e_{i,j} : i \in \mathbb{N}\}$ we may define

$$\rho^\infty : G \to U_\infty$$

by

$$\rho_g^\infty(e_{i,j}) = \sum_{k\in\mathbb{N}} \lambda_{i,j,k} e_{i,k}$$

where

$$\rho_g^i(e_j) = \sum_{k\in\mathbb{N}} \lambda_{i,j,k} e_k.$$

In this way we obtain a new representation that mimics each of the old representations on a sequence of closed orthogonal subspaces of the Hilbert space.

DEFINITION 5.4. $\rho \in \mathrm{Rep}(G, U_\infty)$ is *irreducible* if there is no non-trivial Hilbert subspace which is invariant under $\{\rho_g : g \in G\}$.

So neither the examples in 5.3(a) or (c) are irreducible. In the first case the space $\{ce_i + de_j : c, d \in \mathbb{C}\}$ is a closed invariant subspace, and in the second the closure of the span of any A_j provides an invariant subspace.

NOTATION 5.5. Let $\mathbb{F}_\infty$ denote the free group on infinitely many generators. In future let us denote the generators by $\{v_i : i \in \mathbb{N}\}$.

For this group we obtain that there is a comeager set of irreducibles.

LEMMA 5.6. $\forall^* \rho \in \mathrm{Rep}(\mathbb{F}_\infty, U_\infty)$(*$\rho$ is irreducible).*

PROOF. Note first of all that for any two unit elements, e and f, of the Hilbert space, with $||e|| = ||f|| = 1$ we may find a unitary operator T with $T(e) = f$. To see this, we may suppose that $f = \alpha e + \alpha' e'$ some e' orthogonal to e where $|\alpha|^2 + |\alpha'|^2 = 1$. Now we can extend $\{e, e'\}$ to an orthonormal basis $\{e, e', f_0, f_1, ...\}$ and then following 5.3(a) let

$$T(e) = \alpha e + \alpha' e',$$

$$T(e') = -\alpha' e + \alpha e',$$

and

$$T(f_k) = f_k.$$

The point is that for any $\rho \in \mathrm{Rep}(\mathbb{F}_\infty, U_\infty)$ and $k \in \mathbb{N}$ we have some $\rho' \in \mathrm{Rep}(\mathbb{F}_\infty, U_\infty)$ with $\rho'_{v_n} = \rho_{v_n}$ all $n \leq k$ but $\rho'_{v_{k+1}} = T$.

Thus if $\{\hat{e}_i : i \in \mathbb{N}\}$ is a dense subset of the unit elements, for each i, j and $q \in \mathbb{Q}^+$ we have that the set of $\rho \in \mathrm{Rep}(\mathbb{F}_\infty, U_\infty)$ such that there exists n with

$$|\langle \rho_{v_n}(\hat{e}_i), \hat{e}_j \rangle - 1| < q$$

is open and dense in the space of representations. Any representation in the intersection of all these dense open sets will be irreducible. □

LEMMA 5.7. *Every orbit in* $\mathrm{Rep}(\mathbb{F}_\infty, U_\infty)$ *is meager.*

PROOF. Fix τ, and we attempt to show $[\tau]_{U_\infty}$ meager. It suffices to show that we may cover U_∞ by open sets $\{V_i : i \in \mathbb{N}\}$ such that for all i and for all open $U_0 \subset \mathrm{Rep}(\mathbb{F}_\infty, U_\infty)$ there is some non-empty open $U_1 \subset U_0$ such that for all $\rho \in U_1$ and for all $T \in V_i$

$$T \cdot \rho \neq \tau.$$

Let $\{f_i : i \in \mathbb{N}\}$ be a relatively dense subset of the unit ball in our Hilbert space. Let V_i be the set

$$\{T \in U_\infty : |\langle T(e_0), f_i \rangle - 1| < \surd(99/100)\}.$$

Let U_0 be a non-empty open set. After perhaps shrinking to a basic subset we may assume it has the form

$$U_0 = \{\rho : |\langle \rho_{v_i}(e_j), f_{n(i,j)} \rangle - 1| < \delta_{i,j} \forall i, j \leq k\}.$$

Let $i \in \mathbb{N}$.

Case (A) $|\langle \tau_{v_{k+1}}(f_i), f_i \rangle| > 1/2$.

Then let $U_1 = \{\rho \in U_0 : |\langle \rho_{v_{k+1}}(e_0), e_0 \rangle| < 1/10\}$.

Claim(1) U_1 is not empty.

Proof of claim. Let $\sigma \in U_0$; this much is okay since we assumed U_0 is non-empty. We next define ρ_{v_i} for each generator $v_i \in \mathbb{F}_\infty$. $\rho_{v_i} = \sigma_{v_i}$ for $i \neq k+1$, but $\rho_{v_{k+1}}(e_0) = e_1$, and $\rho_{v_{k+1}}(e_1) = e_0$, and $\rho_{v_{k+1}}(e_n) = e_n$ for $n \neq 0, 1$. Since $\{e_n : n \in \mathbb{N}\}$ is a basis, $\rho_{v_{k+1}}$ extends to a unitary operator. By the defining properties of the free group ρ extends to a homomorphism from $\mathbb{F}_\infty$ to U_∞. Since ρ agrees with σ on the set $\{v_0, v_1, ..., v_k\}$, we indeed have $\rho \in U_0$. (Claim(1)□)

Claim(2) If $T \in V_i$ and $\rho \in U_1$ then $T \cdot \rho \neq \tau$.

Proof of claim. Express $T^{-1}(f_i)$ as $T^{-1}(f_i) = c_1 e_0 + c_2 f$ where e_0 and f are orthogonal. Note then that $|c_1|^2 + |c_2|^2 = 1$, and from $T \in V_i$ we have $|c_1|^2 > 99/100$, and hence $|c_2| < 1/10$. We then calculate that

$$|\langle (T \cdot \rho)_{v_{k+1}}(f_i), f_i \rangle| = |\langle T \circ \rho_{v_{k+1}} \circ T^{-1}(f_i), f_i \rangle|$$

$$= |\langle \rho_{v_{k+1}}(T^{-1}(f_i)), T^{-1}(f_i)\rangle| = |\langle \rho_{v_{k+1}}(c_1e_0 + c_2f), c_1e_0 + c_2f\rangle|$$
$$\leq |\langle \rho_{v_{k+1}}(c_1e_0), c_1e_0\rangle| + |\langle \rho_{v_{k+1}}(c_1e_0), c_2f\rangle| + |\langle \rho_{v_{k+1}}(c_2f), c_1e_0\rangle| + |\langle \rho_{v_{k+1}}(c_2f), c_2f\rangle|$$
$$< 1/10 + 1/10 + 1/10 + 1/100 < 1/2,$$

and thus $T \cdot \rho \neq \tau$, as required. (Claim(2)$\square$)

Case (B) $|\langle \tau_{v_{k+1}}(f_i), f_i\rangle| \leq 1/2$.

Then let $U_1 = \{\rho \in U_0 : |\langle \rho_{v_{k+1}}(e_0), e_0\rangle| > 9/10\}$.

Claim(3) U_1 is not empty.

Proof of claim. This is parallel to the proof of claim (1). We first choose $\sigma \in U_0$ and then define $\rho \in \mathrm{Rep}(\mathbb{F}_\infty, U_\infty)$ so that $\rho_{v_i} = \sigma_{v_i}$ for $i \neq k+1$, but $\rho_{v_{k+1}}$ is just left to be the identity – $\rho_{v_{k+1}}(e_n) = e_n$ all n. (Claim(3)$\square$)

Claim(4) If $T \in V_i$ and $\rho \in U_1$ then $T \cdot \rho \neq \tau$.

Proof of claim. Again write $T^{-1}(f_i) = c_1e_0 + c_2f$ where e_0 and f are orthogonal. We then calculate that

$$|\langle (T\cdot\rho)_{v_{k+1}}(f_i), f_i\rangle| = |\langle T \circ \rho_{v_{k+1}} \circ T^{-1}(f_i), f_i\rangle|$$
$$= |\langle \rho_{v_{k+1}}(T^{-1}(f_i)), T^{-1}(f_i)\rangle| = |\langle \rho_{v_{k+1}}(c_1e_0 + c_2f), c_1e_0 + c_2f\rangle|$$
$$\geq |\langle \rho_{v_{k+1}}(c_1e_0), c_1e_0\rangle| - |\langle \rho_{v_{k+1}}(c_1e_0), c_2f\rangle| - |\langle \rho_{v_{k+1}}(c_2f), c_1e_0\rangle| - |\langle \rho_{v_{k+1}}(c_2f), c_2f\rangle|$$
$$> 99/100 - 1/10 - 1/10 - 1/100 > 9/10 - 3/10 > 1/2,$$

as required. (Claim(4)$\square$)

To summarize, we have shown that for each i there is an open dense set $\mathcal{O}_i \subset \mathrm{Rep}(\mathbb{F}_\infty, U_\infty)$ such that for all $\rho \in \mathcal{O}_i$ and $T \in V_i$

$$T \cdot \rho \neq \tau.$$

For C the comeager set

$$\bigcap_{i\in\mathbb{N}} \mathcal{O}_i$$

we have that for all $T \in U_\infty = \bigcup_{i\in\mathbb{N}} V_i$

$$T \cdot \rho \neq \tau.$$

$\square$

We are left with verifying part (iii) in the definition of generic turbulence.

LEMMA 5.8. *There is some $\rho \in \mathrm{Rep}(\mathbb{F}_\infty, U_\infty)$ whose orbit is dense and whose local orbits are each somewhere dense – that is to say, for each $\tau \in [\rho]_{U_\infty}$ and open U containing τ and each open neighbourhood V of the identity in the unitary group, $\mathcal{O}(\tau, U, V)$ contains an open set in its closure.*

PROOF. Let $\{\rho^i : i \in \mathbb{N}\}$ be a countable dense subset of $\mathrm{Rep}(\mathbb{F}_\infty, U_\infty)$. Let $\{A_i : i \in \mathbb{N}\}$ be a partition of the basis of the Hilbert space into infinitely many infinite disjoint sets, and let H_i be the closed subspace generated by A_i, and let $\rho|_{H_i}$ be naturally similar to ρ^i – that is to say, if $A_i = \{e_{i,j} : j \in \mathbb{N}\}$, then

$$\rho_{v_k}(e_{i,j}) = \sum_{n\in\mathbb{N}} \lambda_n e_{i,n}$$

where

$$\rho^i_{v_k}(e_j) = \sum_{n\in\mathbb{N}} \lambda_n e_n.$$

It should be clear that the orbit of ρ is dense, since given any open $U \subset \mathrm{Rep}(\mathbb{F}_\infty, U_\infty)$ and $\tau^i \in U$ we obtain T with $T \cdot \tau \in U$ by interchanging e_j and $e_{i,j}$

for $l \leq N$ for some sufficiently large N. The point is that ρ^i falls inside the basic open U in virtue of its behavior on $\{e_j : j \leq N\}$ for some *finite* N.

Now suppose that $\tau \in [\rho]_{U_\infty}$. Let U be an open neighborhood of τ and let V be an open neighbourhood of the identity operator; by possibly restricting to some subsets we may assume that U has the form

$$U = \{\sigma \in \mathrm{Rep}(\mathbb{F}_\infty, U_\infty) : \forall i, j, k \leq K(|\langle \sigma_{v_i}(e_j), e_k\rangle - \langle \tau_{v_i}(e_j), e_k\rangle| < \delta)\},$$

and that V has the form

$$V = \{T \in U_\infty : \forall j \leq K(|\langle T(e_j), e_j\rangle - 1| < \delta)\}$$

for the same finite K and small $\delta > 0$. Let U_1 be an open non-empty subset of U. It suffices to show that $\mathcal{O}(\tau, U, V) \cap U_1$ is non-empty. By restricting we may assume that there is some finite $L \geq K$ such that for any two σ, σ' we have that if

$$\langle \sigma_{v_i}(e_j), e_k\rangle = \langle \sigma'_{v_i}(e_j), e_k\rangle$$

all $i, j, k \leq L$ then

$$\sigma \in U_1 \Leftrightarrow \sigma' \in U_1.$$

In other words, membership in U_1 is a function of a representation's behavior with respect to v_i, e_j just for $i, j \leq L$.

We may assume that there is some large, but finite N, with

$$\langle \tau_{v_i}(e_j), e_{n,l}\rangle = 0$$

all $i, j \leq L$, $n \in \mathbb{N}$, and $l > N$. In fact we may assume $\tau = S \cdot \rho$ where $S|_{H_l} = \mathrm{id}|_{H_l}$ all sufficiently large l, since

$$\{T \in U_\infty : \exists M \forall l > M(T|_{H_l} = \mathrm{id}|_{H_l})\}$$

is a dense subgroup of U_∞.

Now by the assumption on the $\{\rho^i : i \in \mathbb{N}\}$ we may choose some large m with $A_m \subset \{e_n : n > N\}$ and $\rho^m \in U_1$. Note then that if we define $T_{\pi/2} \in U_\infty$ by the requirement that $T_{\pi/2}(e_{m,i}) = e_i$ and $T_{\pi/2}(e_i) = -e_{m,i}$ at $i \leq L$ but $T_{\pi/2}(e_j) = e_j$ for e_j not among $\{e_0, e_1, ..., e_L\} \cup \{e_{m,0}, e_{m,1}, ..., e_{m,L}\}$ then at all $i, j, k \leq L$ we have

$$\begin{aligned}\langle (T_{\pi/2} \cdot \tau)_{v_i}(e_j), e_k\rangle &= \langle (T_{\pi/2} \cdot (S \cdot \rho))_{v_i}(e_j), e_k\rangle \\ &= \langle \rho_{v_i}((T_{\pi/2} \circ S)^{-1}(e_j)), (T_{\pi/2} \circ S)^{-1}(e_k)\rangle \\ &= \langle \rho_{v_i}(S^{-1}(T^{-1}_{\pi/2}(e_j))), S^{-1}(T^{-1}_{\pi/2}(e_k))\rangle \\ &= \langle \rho_{v_i}(S^{-1}(e_{m,j})), S^{-1}(e_{m,k})\rangle = \langle \rho_{v_i}(e_{m,j}), e_{m,k}\rangle,\end{aligned}$$

by assumption on S, which in turn equals

$$\langle \rho^m_{v_i}(e_j), e_k\rangle$$

and therefore we find that $T_{\pi/2} \cdot \tau \in U_1$.

This much indicates our destination. If we can just show there is some way to get there staying inside U and V we will be done; we need show

$$T_{\pi/2} \cdot \tau \in \mathcal{O}(\tau, U, V).$$

Now for each $\varphi \in [0, \pi/2]$ we define T_φ by

$$T_\varphi(e_{m,i}) = e_i \sin(\varphi) + e_{m,i} \cos(\varphi),$$

$$T_\varphi(e_i) = -e_{i,m} \sin(\varphi) + e_i \cos(\varphi),$$

for $i \leq L$, and for e_n not in $\{e_0, e_1, ..., e_L\} \cup \{e_{m,0}, e_{m,1}, ..., e_{m,L}\}$ set

$$T_\varphi(e_n) = e_n.$$

In the case $\varphi = \pi/2$ this agrees with our definition of $T_{\pi/2}$ above.

Claim. For all $\varphi \in [0, 2\pi]$

$$T_\varphi \cdot \rho \in U.$$

Proof of claim. We calculate. Fix $i, j, k \leq K$. Then

$$\begin{aligned} \langle (T_\varphi \cdot \tau_{v_i})(e_j), e_k\rangle &= \langle \tau_{v_i}(T_\varphi^{-1}(e_j)), T_\varphi^{-1}(e_k)\rangle \\ &= \langle \tau_{v_i}(e_j \cos(\varphi) + e_{m,j}\sin(\varphi)), e_k \cos(\varphi) + e_{m,k}\sin(\varphi)\rangle \\ &= \langle \tau_{v_i}(e_j \cos(\varphi)) + \tau_{v_i}(e_{m,j}\sin(\varphi)), e_k\cos(\varphi) + e_{m,k}\sin(\varphi)\rangle; \end{aligned}$$

by assumption that $S|_{H_m} = \mathrm{id}$ and $S \cdot \rho = \tau$ we have

$$\langle \tau_{v_i}(e_j), e_{m,k}\rangle = 0$$

and

$$\langle \tau_{v_i}(e_{m,j}), e_k\rangle = \langle e_{m,j}, \tau_{(v_i)^{-1}}(e_k)\rangle = 0,$$

and thus

$$\begin{aligned} &\langle \tau_{v_i}(e_j \cos(\varphi)) + \tau_{v_i}(e_{m,j}\sin(\varphi)), e_k\cos(\varphi) + e_{m,k}\sin(\varphi)\rangle \\ &= \langle \tau_{v_i}(e_j \cos(\varphi)), e_k \cos(\varphi)\rangle + \langle \tau_{v_i}(e_{m,j}\sin(\varphi)), e_{m,k}\sin(\varphi)\rangle \\ &= \cos^2(\varphi)\langle \tau_{v_i}(e_j), e_k \cos\rangle + \langle \sin^2(\varphi)\langle \rho^m_{v_i}(e_j), e_k\rangle. \end{aligned}$$

Thus

$$\begin{aligned} &|\langle (T_\varphi \cdot \tau_{v_i})(e_j), e_k\rangle - \langle \tau_{v_i}(e_j), e_k\rangle| \\ &= |\cos^2(\varphi)\langle \tau_{v_i}(e_j), e_k\rangle + \sin^2(\varphi)\langle \rho^m_{v_i}(e_j), e_k\rangle - \langle \tau_{v_i}(e_j), e_k\rangle| \\ &\leq \sin^2(\varphi)|\langle \rho^m_{v_i}(e_j), e_k\rangle - \langle \tau_{v_i}(e_j), e_k\rangle| < \sin^2(\varphi)\delta, \end{aligned}$$

as required. Since this holds for all $i, j, k \leq K$ we have membership of $T_\varphi \cdot \tau$ in U. (Claim□)

But now if we choose some sufficiently large M we will have that $T_{\pi/2M} \in V$. Then letting $\tau^0 = \tau$, $\tau^{i+1} = T_{\pi/2M} \cdot \tau^i$, we obtain witnesses that $\tau^M = T_{\pi/2} \cdot \tau \in \mathcal{O}(\tau, U, V)$. □

THEOREM 5.9. *The irreducible representations of $\mathbb{F}_\infty$ considered up to $\sim_r$ do not admit classification by countable structures.*

PROOF. From 5.6 there is an invariant dense G_δ, call it C_0, on which all the representations are irreducible. We obtain from 5.7 and 5.8 applied to 3.21 that there is an invariant dense G_δ set $C_1 \subset \mathrm{Rep}(\mathbb{F}_\infty, U_\infty)$ on which the action is turbulent.

Then taking $X = C_0 \cap C_1$ we have a Polish U_∞-space refusing classification by countable structures whose orbits consist of irreducible representations considered up to $\sim_r$. □

Of course 5.9 also gives a proof that $\sim_r$ for the irreducibles is not smooth for $G = \mathbb{F}_\infty$, but actually much stronger results of this form are known. Thoma in [**79**] shows that the countable groups for which the space of irreducible representations considered up to $\sim_r$ is smooth are exactly those with an abelian subgroup of finite index.

CHAPTER 6

A generalized Scott analysis

§6.1 reworks the Scott analysis and presents applications to the analysis of the Borel complexity of orbits. In particular we use the make over of Scott to develop some simple cases of the Hjorth-Kechris-Louveau theory of Borel equivalence relations induced by continuous S_∞ actions and to obtain that every orbit in a Polish S_∞-space is Borel. This fact actually holds in complete generality for Polish group actions, and in the exercises the reader can find a unrelated proof of the general case as well as a demonstration that even though every individual orbit must be Borel the orbit equivalence relation itself need not be a Borel subset of the product space.

§6.2 provides the heart of this chapter and generalizes Scott's method to arbitrary Polish groups; the new idea is to embark on a transfinite analysis of the local orbits, as defined in §3. We define the class of *GE-groups*, which in chapter 7 will be proved to include all nilpotent and invariantly metrizable Polish groups, as well as all countable products of locally compact Polish groups, and show that the generalized Scott analysis leads to a characterization of when the action of a GE group allows classification by countable models; among other things we see that such a classification is possible if and only if there is no continuous reduction of a turbulent orbit equivalence relation and moreover when such classification is possible we can in fact find a reduction that admits a Borel inverse up to orbit equivalence and complete HC invariants can be calculated in a canonical manner.

§6.3 is nothing other than a counterexample: For non-GE groups Borel reduction to isomorphism on countable models and complete classification by HC invariants may possible even when the canonical invariants of §6.2 fail to be complete.

§6.4 is a tentative step towards trying to find complete invariants that lie outside the class of hereditarily countable sets. In some ways it represents a side line to the main direction of the book, and it is hard to escape the feeling that a more penetrating approach should be possible. On the other hand it does seem to constitute the best tool we currently have for the "fine analysis" of the Borel structure of orbits; in particular one can show that if every orbit is no worse than $\mathbf{\Pi}^0_\alpha$ then the overall equivalence relation can be no worse than $\mathbf{\Pi}^0_{\alpha+\omega}$.

There is a sense in which §6.1 is superfluous. The analysis of §6.2 is more general and the exercises at the end of §6.1 never resurface later in the text. However this preliminary section is important in giving a perspective; it underscores how S_∞ actions are far better understood than those of arbitrary Polish groups.

6.1. A preliminary discussion of a specific case

The original Scott analysis was for the *logic action* at 2.37 of S_∞ on $\mathrm{Mod}(\mathcal{L})$. This allows a harmless variation for general actions of S_∞, which while less intuitive than the original suggests the modifications needed for arbitrary Polish groups. I will only phrase the results for S_∞, but they hold in the context of arbitrary closed subgroups of the symmetric group. In any case, glancing ahead to 7.18 one sees that the orbit equivalence relation induced by a closed subgroup of the symmetric group will be $\leq_B$ reducible to an appropriately chosen $E^X_{S_\infty}$.

DEFINITION 6.1. Let X be a Polish S_∞-space. Let $(U_l)_{l\in\mathbb{N}}$ be a basis of X, and let

$$V_n = \{g \in S_\infty : \forall i < n(g(i) = i)\},$$

so that $(V_n)_{n\in\mathbb{N}}$ is a neighbourhood basis at the identity consisting of clopen subgroups. For $x \in X$ we then define $\varphi_\alpha(x, V_n)$ by simultaneous induction on the ordinal α.

$$\begin{aligned}&\varphi_0(x, V_n) = \{l : \exists \hat{x} \in U_l \bigcap V_n \cdot x\};\\ &\varphi_{\alpha+1}(x, V_n) = \{\langle \varphi_\alpha(\hat{x}, V_m), m\rangle : \hat{x} \in V_n \cdot x, m \geq n\};\\ &\text{for } \lambda \text{ a limit, } \varphi_\lambda(x, V_n) = \{\langle \alpha, \varphi_\alpha(x, V_n)\rangle : \alpha < \lambda\}.\end{aligned}$$

The definition is not quite precise, since $\varphi_0(x, V_n)$ depends not just on x and V_n, but the entire apparatus of the group action and the choice of the sequence $(U_l)_{l\in\mathbb{N}}$. However a cure such as $\varphi_0(x, V_n, (U_l)_{l\in\mathbb{N}}, a, X)$ would hardly be a happy one.

Noting $(V_n)^k = V_n$ for all $k \in \mathbb{Z}$ we have $\{l : \exists \hat{x} \in U_l \bigcap V_n \cdot x\} = \varphi_0(x, X, V_n)$, and hence $\varphi_0(x, V_n) = \varphi_0(x, X, V_n)$, in the sense of §3.2. The existence of neighbourhood basis at the identity consisting of open subgroups is a key property of closed subgroups of S_∞ and this hints at the direction that the generalized analysis must follow in §6.2.

The next lemma expresses that $\varphi_\alpha(x, V_n)$ indicates at least as much as $\varphi_\beta(x, V_n)$ for all $\beta \leq \alpha$

LEMMA 6.2. *For all $x_1, x_2 \in X$, $\beta \leq \alpha$*

$$\varphi_\alpha(x_1, V_n) = \varphi_\alpha(x_2, V_n) \Rightarrow \varphi_\beta(x_1, V_n) = \varphi_\beta(x_2, V_n).$$

PROOF. By transfinite induction on α starting with the base case $\beta = \alpha$. Lifting the induction through limit stages is trivial, and so this just leaves us with showing that given the lemma at α we may ratchet up to $\alpha + 1$.

Suppose then that $\varphi_{\alpha+1}(x_1, V_n) = \varphi_{\alpha+1}(x_2, V_n)$. In particular we may find $\hat{x}_2 \in V_n \cdot x_2$ with

$$\varphi_\alpha(x_1, V_n) = \varphi_\alpha(\hat{x}_2, V_n).$$

Now since $\varphi_\alpha(\hat{x}_2, V_n) = \varphi_\alpha(x_2, V_n)$ we can apply the inductive assumption. □

LEMMA 6.3. *For all $x \in X$ there is some $\gamma^*(x) < \omega_1$ such that for all $x_1, x_2 \in [x]_{S_\infty}, n \in \mathbb{N}$*

$$\exists\alpha < \omega_1(\varphi_\alpha(x_1, V_n) \neq \varphi_\alpha(x_2, V_n)) \Leftrightarrow \varphi_{\gamma^*(x)}(x_1, V_n) \neq \varphi_{\gamma^*(x)}(x_2, V_n).$$

PROOF. For $x_1, x_2 \in [x]_{S_\infty}$ and $n \in \mathbb{N}$ set

$$x_1 \approx^*_n x_2$$

if $x_1 \in V_n \cdot x_2$. Note that this is an equivalence relation, $[x]_{S_\infty}$ has only countably many $\approx^*_n$ equivalence classes, and for $\alpha < \omega_1$

$$\forall x_1, x_2 \in [x]_{S_\infty}(x_1 \approx^*_n x_2 \Rightarrow \varphi_\alpha(x_1, V_n) = \varphi_\alpha(x_2, V_n)).$$

So by cardinality considerations there is some $\gamma_n(x) < \omega_1$ such that for all $\alpha < \omega_1$

$$\varphi_\alpha(x_1, V_n) \neq \varphi_\alpha(x_2, V_n) \Rightarrow \varphi_{\gamma_n(x)}(x_1, V_n) \neq \varphi_{\gamma_n(x)}(x_2, V_n).$$

Then $\gamma^*(x) = \sup\{\gamma_n(x) : n \in \mathbb{N}\}$ is as desired. $\square$

In fact $\gamma^*(x)$ above has the property that *for any* ordinal α and x_1, x_2 in the orbit of x

$$\varphi_\alpha(x_1, V_n) \neq \varphi_\alpha(x_2, V_n) \Rightarrow \varphi_{\gamma^*(x)}(x_1, V_n) \neq \varphi_{\gamma^*(x)}(x_2, V_n).$$

Definition 6.4. For $x \in X$ let $\gamma^*(x)$ be the least countable ordinal satisfying the conclusion of 6.3. Let $\varphi^*_x = \varphi_{\gamma^*(x)+2}(x, V_0)$.

$\gamma^*(x)$ and φ^*_x are invariants of the orbit equivalence relation, in that if $g \cdot x_1 = x_2$ for some $g \in S_\infty$ then

$$\gamma^*(x_1) = \gamma^*(x_2),$$
$$\varphi^*_{x_1} = \varphi^*_{x_2}.$$

The second of these equalities arises from the choice of $V_0 = S_\infty$.

We also have that $\gamma^*(x)$ is indicated by φ^*_x, in the following sense:

Lemma 6.5. *If $\varphi_{\gamma^*(x)+2}(y, V_0) = \varphi^*_x$ then for all $y_1, y_2 \in [y]_{S_\infty} (= V_0 \cdot y)$ and $n \in \mathbb{N}$ $\beta < \omega_1$*

$$\varphi_{\gamma^*(x)}(y_1, V_n) = \varphi_{\gamma^*(x)}(y_2, V_n) \Rightarrow \varphi_\beta(y_1, V_n) = \varphi_\beta(y_2, V_n).$$

Proof. By 6.2 we have this for $\beta \leq \gamma^*(x)$ and it is trivial to show that if λ is a limit and 6.5 holds for all $\beta = \alpha < \lambda$ then it holds for $\beta = \lambda$. So instead let us assume that the lemma is true for $\beta = \alpha$ and attempt to lever up to the case $\beta = \alpha + 1$. For this purpose, fix $y_1, y_2 \in [y]_{S_\infty}$ with

$$\varphi_{\gamma^*(x)}(y_1, V_n) = \varphi_{\gamma^*(x)}(y_2, V_n).$$

By the assumption on y we may find $x_1, x_2 \in [x]_{S_\infty}$ with

$$\varphi_{\gamma^*(x)+1}(y_1, V_n) = \varphi_{\gamma^*(x)+1}(x_1, V_n),$$
$$\varphi_{\gamma^*(x)+1}(y_2, V_n) = \varphi_{\gamma^*(x)+1}(x_2, V_n).$$

Then since

$$\varphi_{\gamma^*(x)}(x_1, V_n) = \varphi_{\gamma^*(x)}(x_2, V_n)$$

it follows by definition of $\gamma^*(x)$ that

$$\varphi_{\gamma^*(x)+1}(x_1, V_n) = \varphi_{\gamma^*(x)+1}(x_2, V_n),$$
$$\therefore \varphi_{\gamma^*(x)+1}(y_1, V_n) = \varphi_{\gamma^*(x)+1}(y_2, V_n).$$

So at once by definition

$$\{\langle \varphi_{\gamma^*(x)}(\hat{y}, V_m), m\rangle : \hat{y} \in V_n \cdot y_1, m \geq n\} = \{\langle \varphi_{\gamma^*(x)}(\hat{y}, V_m), m\rangle : \hat{y} \in V_n \cdot y_2, m \geq n\},$$

on which the inductive assumption produces

$$\{\langle \varphi_\alpha(\hat{y}, V_m), m\rangle : \hat{y} \in V_n \cdot y_1, m \geq n\} = \{\langle \varphi_\alpha(\hat{y}, V_m), m\rangle : \hat{y} \in V_n \cdot y_2, m \geq n\},$$
$$\therefore \varphi_{\alpha+1}(y_1, V_n) = \varphi_{\alpha+1}(y_2, V_n).$$

$\square$

It is important to observe that each of these $\varphi_\alpha(x, V_n)$ objects is an hereditarily countable set (assuming here the Kuratowski device for coding the pairing function – $\langle a, b\rangle = \{\{a, b\}, b\}$). Moreover the invariants naturally level off with an appropriate number of iterations of the operation "a countable subset of".

$\varphi_0(x, V_n)$ *is* a point in $2^{\mathbb{N}}$; $\varphi_1(x, V_n)$ may be coded by a countable subset of $2^{\mathbb{N}}$, since if we have some one to one $s(\cdot, \cdot, ...) : \mathbb{N}^{<\mathbb{N}} \hookrightarrow \mathbb{N}$ then we can represent $\varphi_1(x, V_n)$ by

$$\{\{s(l, \hat{n}) : l \in \varphi_0(\hat{x}, V_{\hat{n}}), \hat{n} \geq n\} : \hat{x} \in V_n \cdot x\};$$

$\varphi_2(x, V_n)$ can be coded by a countable set of countable subsets of $2^{\mathbb{N}}$, and so on. One usually defines the *hereditarily countable sets* to be the sets a for which the transitive closure

$$a \cup \bigcup a \cup \bigcup\bigcup a \cup ...$$

is countable. Under this definition set theorists understand each natural number to be hereditarily countable, with the identification $0 = \emptyset$, $1 = \{\emptyset\}$, $2 = \{0, 1\}$, and onwards. Alternatively we could just choose to define the hereditarily countable sets to be the smallest class of sets containing the natural numbers and closed under the taking of countable subsets. In any case, the $\varphi_\alpha(x, V_n)$ invariants are all elements of HC, the class of hereditarily countable sets.

The next theorem is morally due to Dana Scott, who proved this much for $X = \mathrm{Mod}(\mathcal{L})$, some countable language $\mathcal{L}$.

THEOREM 6.6. *For* $x, y \in X$

$$x E_{S_\infty} y \Leftrightarrow \varphi^*_x = \varphi^*_y.$$

PROOF. As above, the forward direction $\Rightarrow$ is trivial; so instead suppose that $\varphi^*_x = \varphi^*_y$.

We will find sequences of points

$$x_0 = x, x_1, x_2, ... \in [x]_{S_\infty},$$

$$y_0 = y, y_1, y_2, ... \in [y]_{S_\infty},$$

group elements

$$g_0 = 1_{S_\infty}, g_1, g_2, ...$$
$$h_0 = 1_{S_\infty}, h_1, h_2, ...$$

and natural numbers

$$l(0), l(1), l(2), ..., n(0), n(1), n(2), ...$$

such that at each i

(1) $x_i = g_i \cdot x$, $y_i = h_i \cdot y$;
(2) $x_i, y_i \in U_{l(i)}$, $d(U_{l(i)}) < 2^{-i}$ for $i > 0$, $U_{l(i+1)} \subset U_{l(i)}$;
(3) $\varphi_{\gamma^*(x)}(x_i, V_{n(i)}) = \varphi_{\gamma^*(x)}(y_i, V_{n(i)})$;
(4a) $g_{i+1}g_i^{-1} \in V_{n(i)}$;
(4b) $h_{i+1}h_i^{-1} \in V_{n(i)}$;
(5a) if i odd, then for all $k < i$ we have $g_i(k), g_i^{-1}(k) < n(i)$;
(5b) if i even, then for all $k < i$ we have $h_i(k), h_i^{-1}(k) < n(i)$;
(6) $n(i) < n(i + 1)$.

Supposing for the moment that this is all done then we finish abruptly. By (4)-(6) we have that

$$g_i \longrightarrow g_\infty$$

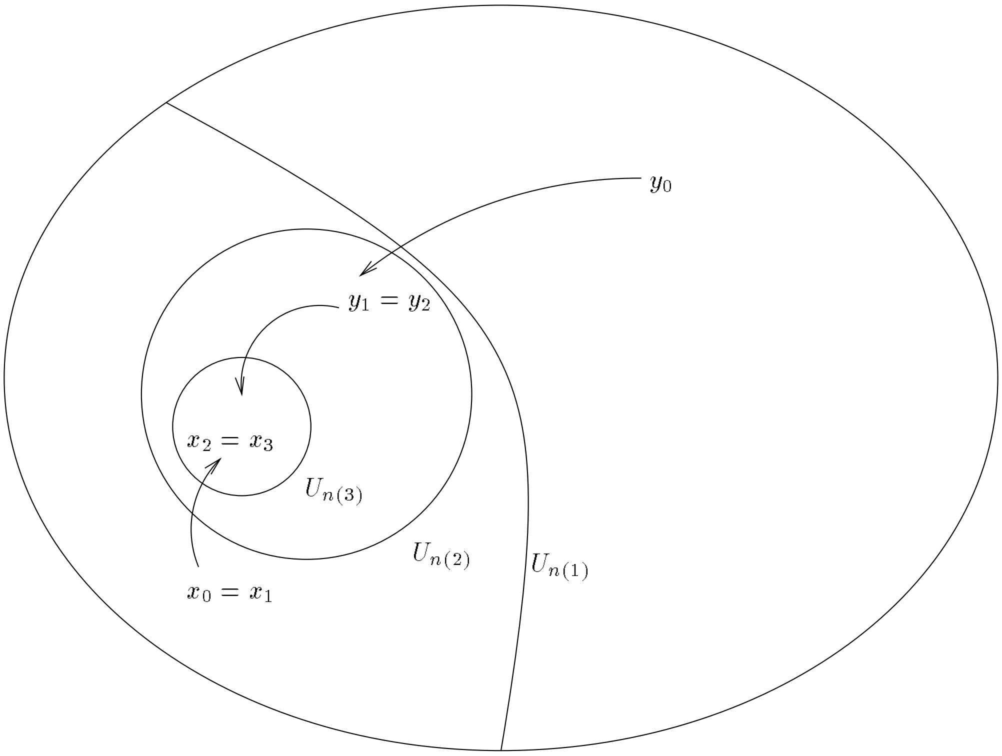

FIGURE 6.1. The x_i's and y_i's drawing closer together

and

$$h_i \to h_\infty$$

for some $g_\infty, h_\infty \in S_\infty$. By (1) and (2) we have

$$x_i \to g_\infty \cdot x,$$

$$y_i \to h_\infty \cdot y,$$

and so

$$g_\infty \cdot x = h_\infty \cdot y,$$

thereby establishing $xE_{S_\infty}y$.

Now working to show that all this can be produced suppose inductively that we have x_i, y_i, g_i, h_i, $l(i)$, $n(i)$ for $i < j$, and seek x_j, y_j, g_j, h_j, $l(j)$, $n(j)$. For simplicity assume that j is even; the odd case is exactly symmetrical. First simply choose $n(j)$ to be the sup of $\{n(j-1)+1, h_{j-1}(k), h_{j-1}^{-1}(k) : k < j\}$, let $h_j = h_{j-1}$, and let $y_j = y_{j-1}$. We take $l(j)$ such that $y_j (=_{df}\ y_{j-1}) \in U_{l(j)} \subset U_{l(j-1)}$ and $d(U_{l(j)}) < 2^{-j}$

On the x side there is more work.

Claim: $\varphi_{\gamma^*(x)+1}(x_{j-1}, V_{n(j-1)}) = \varphi_{\gamma^*(x)+1}(y_{j-1}, V_{n(j-1)})$.

Proof of claim: Suppose otherwise. Then since

$$\{\varphi_{\gamma^*(x)+1}(x', V_n) : x' \in [x]_{S_\infty}, n \in \mathbb{N}\} = \{\varphi_{\gamma^*(x)+1}(y', V_n) : y' \in [y]_{S_\infty} n \in \mathbb{N}\}$$

there will be some $x' \in [x]_{S_\infty}$ with

$$\varphi_{\gamma^*(x)+1}(x', V_{n(j-1)}) = \varphi_{\gamma^*(x)+1}(y_{j-1}, V_{n(j-1)})$$
$$\neq \varphi_{\gamma^*(x)+1}(x_{j-1}, V_{n(j-1)})$$

Then it follows from

$$\varphi_{\gamma^*(x)}(x_{j-1}, V_{n(j-1)}) = \varphi_{\gamma^*(x)}(y_{j-1}, V_{n(j-1)})$$

that

$$\varphi_{\gamma^*(x)}(x', V_{n(j-1)}) = \varphi_{\gamma^*(x)}(x_{j-1}, V_{n(j-1)}),$$

with a contradiction to the definition of $\gamma^*(x)$. (Claim□)

Thus we may choose $x_j \in V_{n(j-1)} \cdot x_{j-1} \cap U_{l(j)}$ such that

$$\varphi_{\gamma^*(x)}(x_j, V_{n(j)}) = \varphi_{\gamma^*(x)}(y_j, V_{n(j)})(=_{df}\ \varphi_{\gamma^*(x)}(y_{j-1}, V_{n(j)}).$$

Then fix $g \in V_{n(j-1)}$ with $g \cdot x_{j-1} = x_j$ and let $g_j = gg_{j-1}$.

And thus the construction continues it way through all $j \in \mathbb{N}$. □

We then have that for continuous S_∞ actions complete invariants can be assigned in HC, the collection of all hereditarily countable sets. The precise same argument works for any Polish group with a neighbourhood basis at the identity consisting of clopen subgroups; it was recorded in [**4**] that these are up to isomorphism exactly the closed subgroups of S_∞.

The Scott invariants can be used in the sharp analysis of S_∞ orbits. To mine into those details we first need some observations regarding the Vaught transforms from 2.51.

LEMMA 6.7. *Let H be a Polish group, Y a Polish H-space.*

(a) If $g \in H$ and $y \in Y$ then

$$g \cdot y \in B^{\Delta V} \Leftrightarrow y \in B^{\Delta V g}.$$

(b) If $g \in H$ and $y \in Y$ then

$$y \in (g^{-1} \cdot B)^{\Delta V} \Leftrightarrow y \in B^{g \Delta V}.$$

(c) If $B \subset Y$ is invariant, in the sense that $h \cdot B = B$ for all $h \in H$ then $B^{\Delta H} = B^{*H} = B$.

PROOF. For (a), note that if there is a comeager subset $C \subset W$ where $W \subset V$ open with

$$\forall h \in C(h \cdot (g \cdot y) \in B),$$

then

$$\forall h \in Cg(h \cdot y \in B),$$

and Cg is comeager in $Wg \subset Vg$ since H acts by homeomorphism on itself. Thus we have $g \cdot y \in B^{\Delta V}$ implies $y \in B^{\Delta V g}$.

By the same argument $(g^{-1} \cdot (g \cdot y)) \in B^{\Delta V g}$ implies $g \cdot y \in B^{\Delta V g g^{-1}} = B^{\Delta V}$.

The proof of (b) resembles (a). Then (c) follows from (b). □

LEMMA 6.8. *H, Y as in 6.7. For $\{W_n : n \in \mathbb{N}\}$ a basis of H, W open in H, and $\{B_n : n \in \mathbb{N}\}$ a sequence of Borel sets in Y,*

$$(\bigcup_{\mathbb{N}} B_n)^{\Delta W} = \bigcup\{(B_n)^{*W_m} : m, n \in \mathbb{N}, W_m \subset W, W_m \neq \emptyset\},$$

$$(\bigcap_{\mathbb{N}} B_n)^{*W} = \bigcap\{(B_n)^{\Delta W_m} : m, n \in \mathbb{N}, W_m \subset W, W_m \neq \emptyset\}.$$

PROOF. First for $(\bigcup_{\mathbb{N}} B_n)^{\Delta W}$. Since for B_n is Borel, $y \in Y$, and the set

$$\{h \in W : h \cdot y \in B_n\}$$

is Borel. Hence

$$\{h \in W : h \cdot y \in \bigcup_{n \in \mathbb{N}} B_n\}$$

is non-meager if and only if there is some n with $\{h \in W : h \cdot y \in B_n\}$ non-meager, if and only if there is some $W_m \subset W$ open and non-empty with $\{h \in W_m : h \cdot y \in B_n\}$ relatively comeager in W_m.

The lemma for $(\bigcap_{\mathbb{N}} B_n)^{*W}$ follows by appealing to the first part of the proof to obtain

$$Y \setminus ((\bigcap_{\mathbb{N}} B_n)^{*W}) = (\bigcup_{\mathbb{N}} Y \setminus B_n)^{\Delta W} =$$
$$\bigcup \{(Y \setminus B_n)^{*W_m} : m, n \in \mathbb{N}, W_m \subset W, W_m \neq \emptyset\}.$$

□

LEMMA 6.9. *H, Y, W as in 6.6. O open in Y implies $O^{\Delta W}$ open; C closed in Y implies C^{*W} closed; $D \subset Y$ a G_δ implies $D^{*W} \in G_\delta$; $F \subset Y$ an F_σ implies that $F^{\Delta W} \in F_\sigma$; D in $G_{\delta\sigma}$ implies $D^{\Delta W} \in G_{\delta\sigma}$ (and more generally $B \subset Y$ in $\utilde{\Sigma}^0_\alpha$ implies $B^{\Delta W} \in \utilde{\Sigma}^0_\alpha$ while $B \in \utilde{\Pi}^0_\alpha$ implies $B^{*W} \in \utilde{\Pi}^0_\alpha$).*

PROOF. Continuity of the action at once implies that $O^{\Delta W}$ is open for any open $W \subset H$. Then the rest of the lemma follows by applying 6.8 as many times as necessary. □

LEMMA 6.10. *Let X, $(U_l)_{l \in \mathbb{N}}$, $(V_l)_{l \in \mathbb{N}}$ be as in 6.1, $x, y \in X$.*
(i) If $O \subset X$ open, $n \in \mathbb{N}$, $\varphi_0(x, V_n) = \varphi_0(y, V_n)$, then

$$x \in O^{\Delta V_n} \Leftrightarrow y \in O^{\Delta V_n}.$$

(ii) If $D \subset X$ an F_σ, $\varphi_1(x, V_n) = \varphi_1(y, V_n)$, then

$$x \in D^{\Delta V_n} \Leftrightarrow y \in D^{\Delta V_n}.$$

(iii) More generally $B \in \utilde{\Sigma}^0_{m+1}$, $\varphi_m(x, V_n) = \varphi_m(y, V_n)$, implies

$$x \in B^{\Delta V_n} \Leftrightarrow y \in B^{\Delta V_n}.$$

PROOF. (i) is a consequence of the definitions and the continuity of the action. For (ii) write $B = \bigcup_{\mathbb{N}} F_i$, each F_i closed. Note then that there is some open $V \subset V_n$, $i \in \mathbb{N}$, $V \neq \emptyset$ with $x \in F_i^{\Delta V}$ if and only if there is some $\hat{x} \in V_n \cdot x$ and $V_{\hat{n}} \subset V_n$ with $\hat{x} \in F_i^{*V_{\hat{n}}}$, and so (ii) follows from (i). The general case at (iii) marches up the natural numbers by an appeal to mathematical induction. □

With some minor notational regearing at the first limit this continues for all countable ordinals: If

$$B \in \utilde{\Sigma}^0_\alpha,$$

$\alpha \geq \omega$ and $\varphi_\alpha(x, V_n) = \varphi_\alpha(y, V_n)$, then

$$x \in B^{\Delta V_n} \Leftrightarrow y \in B^{\Delta V_n}.$$

Note that if $E^X_{S_\infty}$ is say $F_{\sigma\delta}$ in $X \times X$, and so that in particular for every $x \in X$

$$[x]_{S_\infty} = ([x]_{S_\infty})^{*S_\infty} \in F_{\sigma\delta},$$

then $x \mapsto \varphi_2(x, S_\infty)$ is a complete invariant for the orbit equivalence relation. So from the coding mentioned after 6.5, $E^X_{S_\infty}$ is immediately seen to be reducible to F_3 on $2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$, isomorphism on codes for countable sets of countable subsets of $2^{\mathbb{N}}$. Actually by [**41**]:

THEOREM 6.11. *(Hjorth, Kechris, Louveau) Let X be a Polish S_∞ space with the orbit equivalence relation E_{S_∞} an $F_{\sigma\delta}$ subset of $X \times X$. Then $E_{S_\infty} \leq_B F_2$.*

PROOF. Fix $(U_l)_{l\in\mathbb{N}}$, $(V_l)_{l\in\mathbb{N}}$ as before. Fix $x \in X$. Let τ_x be the topology generated by taking the union of $\{\{z : l \notin \varphi_0(z, V_n)\} : l, n \in \mathbb{N}\}$, $\{U_l : l \in \mathbb{N}\}$, and $\{\{z : \varphi_0(z, V_n) = \varphi_0(\hat{x}, V_n)\} : n \in \mathbb{N}, \hat{x} \in [x]_{S_\infty}\}$ as a subbasis. Consider the space (X, τ_x) – by which I mean the space X equipped with τ_x as a topology.

Claim(I): (X, τ_x) is a Polish space.

Proof of claim. First let $(C_i)_{i\in\mathbb{N}}$ enumerate the closed sets of the form

$$\{z : l \notin \varphi_0(z, V_n)\}$$

for various $l, n \in \mathbb{N}$. We then form a sequence of topologies starting with $\tau_0 = \tau$ and then taking

$$\tau_{i+1} = \tau_i \bigcup \{U \cap C_i : U \in \tau_i\}.$$

Each τ_i is Polish by 2.3(ii). Then let τ_∞ be the topology with basis $\bigcup_{i\in\mathbb{N}} \tau_i$. By 2.3(iii) this is Polish.

We can go for another round, observing that each set of the form

$$\{z : \varphi_0(z, V_n) = \varphi_0(\hat{x}, V_n)\}$$

is now τ_∞-closed. Let $(C_{\infty+i})_{i\in\mathbb{N}}$ enumerate these sets. Once more set

$$\tau_{\infty+i+1} = \tau_{\infty+i} \cup (\bigcup \{U \cap C_{\infty+i} : U \in \tau_{\infty+i}\})$$

and once more use 2.3(ii) to see that this is Polish. Again we can use 2.3(iii) to obtain that $\tau_{\infty+\infty}$ – the topology we want – is Polish. (Claim(I)$\square$)

Let C_x be the closure of $[x]_{S_\infty}$ in τ_x. The pair (τ_x, C_x) is an invariant of the orbit $[x]_{S_\infty}$. The next claim shows it to be a complete invariant.

Claim(II): $\forall^* y \in (C_x, \tau_x)(y E_{S_\infty} x)$.

Proof of Claim: Let

$$[x]_{S_\infty} = \bigcap_{i\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} B_{i,j}$$

with each $B_{i,j}$ closed. Note then that

$$[x]_{S_\infty} = (\bigcap_{i\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} B_{i,j})^{*S_\infty}$$

$$= \bigcap \{\bigcup \{B^{*W}_{i,j} : j \in \mathbb{N}, W \subset V, \text{basic open} \neq \emptyset\} : i \in \mathbb{N}, V \subset S_\infty, \text{basic open} \neq \emptyset\}.$$

Then for each i and $V \subset S_\infty$ basic open non-empty we have that the set $O_{V,i}$ of $y \in C_x$ such that there exists $j \in \mathbb{N}$ and $n \in \mathbb{N}$ and $g \in S_\infty$ and $\hat{x} \in [x]_{S_\infty}$ such that

$$g \cdot \hat{x} \in B^{*V_n}_{i,j},$$

$$V_n g \subset V,$$

and

$$\varphi_0(y, V_n) = \varphi_0(\hat{x}, V_n)$$

is dense open in (C_x, τ_x). Moreover for any $y \in O_{V,i}$ and g, j, n as above

$$g \cdot y \in B_{i,j}^{*V_n}$$

by 6.10 and hence

$$y \in B_{i,j}^{*V_n g},$$

with $V_n g \subset V$. But then for y meeting all these open dense $O_{V,i}$

$$y \in \bigcap \{ \bigcup \{ B_{i,j}^{*W} : j \in \mathbb{N}, W \subset V, \text{basic open} \neq \emptyset \} : i \in \mathbb{N}, V \subset S_\infty, \text{ basic open} \neq \emptyset \}$$

as required. (Claim(II)□)

Claim(III): For $x, y \in X$

$$\varphi_1(x, V_0) = \varphi_1(y, V_0) \Rightarrow (C_x, \tau_x) = (C_y, \tau_y).$$

Proof of claim: Clearly $\varphi_1(x, V_0) = \varphi_1(y, V_0)$ implies $\tau_x = \tau_y$. Note that for any $x_0 \in [x]_{S_\infty}$ a neighborhood basis of the point is given by $\{\varphi_0(x_0, V_n) : n \in \mathbb{N}\}$. The assumption $\varphi_1(x, V_0) = \varphi_1(y, V_0)$ then gives that the orbits meet the same τ_x-open sets. (Claim(III)□)

Since the $\varphi_1(x, V_0)$ may be thought of as a countable subset of $2^{\mathbb{N}}$ we are done. □

In fact this is precise, in the sense that $E_{S_\infty}^X \leq_B F_2$ if and only if there is some $F_{\sigma\delta}$ equivalence relation F with $E_{S_\infty}^X \leq_B F$. Thus we can characterize the equivalence relations induced by S_∞ that are potentially $F_{\sigma\delta}$. More generally:

Theorem 6.12. *(Hjorth, Kechris, Louveau) Let X be a Polish-S_∞ space. Then there is a $\underset{\sim}{\Pi}_{n+1}^0$ equivalence relation F with $E_{S_\infty}^X \leq_B F$ if and only if $E_{S_\infty}^X \leq_B F_n$.*

Most of the further ideas needed for the proof are captured in:

Lemma 6.13. *There is a $\underset{\sim}{\Pi}_4^0$ equivalence relation F with $F_3 \leq_B F$.*

Proof. To each $x \in 2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ associate $y_x \in 2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ with the provision that

$$\{y_x(n, m, \cdot) : n, m \in \mathbb{N}\} = \{x(n, m, \cdot) : n, m \in \mathbb{N}\}$$

and that for all $x(n, m, \cdot)$ and $\hat{n} \in \mathbb{N}$

$$x(n, m, \cdot) \in \{y_x(\hat{n}, 2\hat{m}, \cdot) : \hat{m} \in \mathbb{N}\}$$

if and only if

$$x(n, m, \cdot) \in 2^{\mathbb{N}} \setminus \{y_x(\hat{n}, 2\hat{m} + 1, \cdot) : \hat{m} \in \mathbb{N}\}$$

if and only if

$$x(n, m, \cdot) \in \{x(\hat{n}, \hat{m}, \cdot) : \hat{m} \in \mathbb{N}\}.$$

(Here make the minor supposition that for all $\hat{n}$ there exists some n and m with $x(n, m, \cdot)$ not an element of $\{x(\hat{n}, \hat{m}, \cdot) : \hat{m} \in \mathbb{N}\}$.) So on even coordinates $y_x(\hat{n}, 2\hat{m}, \cdot)$ encodes the elements of $\{x(\hat{n}, m, \cdot) : m \in \mathbb{N}\}$ whilst on the odd $y_x(\hat{n}, 2\hat{m} + 1, \cdot)$ indicates the non-members of $\{x(\hat{n}, m, \cdot) : m \in \mathbb{N}\}$.

This granted,

$$\{y_x(n, m, \cdot) : m \in \mathbb{N}\} = \{y_{\hat{x}}(\hat{n}, m, \cdot) : m \in \mathbb{N}\}$$

is $\underset{\sim}{\Pi}_2^0$ in the presence of the assumption

$$\{x(n, m, \cdot) : n, m \in \mathbb{N}\} = \{\hat{x}(n, m, \cdot) : n, m \in \mathbb{N}\},$$

since

$$\{y_x(n, m, \cdot) : m \in \mathbb{N}\} = \{y_{\hat{x}}(\hat{n}, m, \cdot) : m \in \mathbb{N}\}$$

if and only if for all $m, \hat{m} \in \mathbb{N}$

$$y_x(n, 2m, \cdot) \neq y_{\hat{x}}(\hat{n}, 2\hat{m}+1, \cdot).$$

$\square$

EXERCISE 6.14. Let X be a Polish S_∞-space. Show by transfinite induction on $\alpha < \omega_1$ that for all $x_0 \in X$, and $n \in \mathbb{N}$ the set

$$\{x \in X : \varphi_\alpha(x, V_n) = \varphi_\alpha(x_0, V_n)\}$$

is Borel. Use 6.6 to conclude that $[x_0]_{S_\infty}$ is Borel.

The same conclusion is true in complete generality.

EXERCISE 6.15. (Compare [**37**]; $G^\alpha_{x,x}$ below coincides with G^α_x there.) Let G be an arbitrary Polish group and X a Polish G-space. Fix d a complete metric on G and $\mathcal{B}$ a countable basis of G. For $x, y \in X$ let

$$G_{x,y} = \{g \in G : g \cdot x = y\};$$

so this is nonempty if and only if xE_Gy. Let $G^0_{x,y}$ be the set of $V \in \mathcal{B}$ for which there exists $g_i \in V$

$$g_i \cdot x \to y;$$

then continue the definitions through the countable ordinals, with $G^{\alpha+1}_{x,y}$ being the set of $V \in G^\alpha_{x,y}$ such that for all $n \in \mathbb{N}$ there exists some $W \in G^\alpha_{x,y}$ with

$$\overline{W} \subset V$$
$$d(W) < 2^{-n};$$

at λ a limit take

$$G^\lambda_{x,y} = \bigcap_{\alpha<\lambda} G^\alpha_{x,y}.$$

(i) For any x, y there is some $\delta(x,y) < \omega_1$ for which $G^{\delta(x,y)}_{x,y} = G^{\delta(x,y)+1}_{x,y}$.

(ii) For all $V \in \mathcal{B}$ we have $G_{x,y} \cap V \neq \emptyset \Leftrightarrow V \in G^{\delta(x,y)}_{x,y}$.

(iii) For each $\alpha < \omega_1$ and $V \in \mathcal{B}$ the set $\{(x,y) \in X \times X : V \in G^\alpha_{x,y}\}$ is Borel.

(iv) For each $V \in \mathcal{B}$ and $x, y \in X$ there exists a countable ordinal $\delta(V, x)$ so that for any $\alpha > \delta(V,x)$ there is a comeager set of $g \in G$ with $(V \in G^\alpha_{x,g\cdot y} \Rightarrow V \in G^{\alpha+1}_{x,g\cdot y})$.

(v) Let $\delta(x) = \sup\{\delta(V,x) : V \in \mathcal{B}\}$. For all $x, y \in X$

$$xE_Gy$$

if and only if

$$\forall^* g \in G(G^{\delta(x)}_{x,g\cdot y} \neq \emptyset).$$

(vi) Thus $[x]_G$ is Borel.

EXERCISE 6.16. Show by induction on α that there is a Polish S_∞ space such that for all $\alpha < \omega_1$ there are $x, y \in X$ for which

$$\varphi_\alpha(x, V_0) = \varphi_\alpha(y, V_0)$$

without $xE_{S_\infty}y$. Part (i) addresses the successor case of this induction.

(i) Let $\mathcal{M}_0, \mathcal{M}_1 \in \mathrm{Mod}(\mathcal{L})$ and suppose

$$\varphi_\alpha(\mathcal{M}_0, V_0) = \varphi_\alpha(\mathcal{M}_1, V_0)$$

but

$$\mathcal{M}_0 \ncong \mathcal{M}_1.$$

Let $\leq$ be a new binary relation and $\mathcal{L}'$ be the language generated by $\leq$ and $\mathcal{L}$. For each $x \in 2^{\mathbb{Z}}$ let $\mathcal{N}_x \in \mathrm{Mod}(\mathcal{L}')$ be a model that arranges disjoint copies of $\mathcal{M}_0$ and $\mathcal{M}_1$ in a $\mathbb{Z}$ chain as indicated by x:

(a) $\leq^{\mathcal{N}_x}$ is a prelinear ordering of $\mathcal{N}_x$ (so that $a \leq b \leq c$ implies that $a \leq c$ and any two elements are $\leq$ comparable);

(b) the $\equiv$-equivalence classes of $\mathcal{N}_x$ defined by

$$a \equiv b \Leftrightarrow_{df} a \leq^{\mathcal{N}_x} b \leq^{\mathcal{N}_x} a$$

are arranged in order type $\mathbb{Z}$;

(c) for $i \in \mathbb{Z}$, the ith $\equiv$-equivalence class is isomorphic to $\mathcal{M}_0$ if $x(i) = 0$ and $\mathcal{M}_1$ if $x(i) = 1$.

If $x, y \in 2^{\mathbb{Z}}$ have $[x]_{\mathbb{Z}}$ and $[y]_{\mathbb{Z}}$ dense in the shift action of $\mathbb{Z}$ on $2^{\mathbb{Z}}$ then

$$\varphi_{\alpha+1}(\mathcal{N}_x, V_0) = \varphi_{\alpha+1}(\mathcal{N}_y, V_0).$$

(ii) For each $\alpha < \omega_1$ we can find a countable language $\mathcal{L}$ and $\mathcal{M}, \mathcal{M}' \in \mathrm{Mod}(\mathcal{L})$ with

$$\varphi_\alpha(\mathcal{M}, V_0) = \varphi_\alpha(\mathcal{M}', V_0)$$

but

$$\mathcal{M} \ncong \mathcal{M}'.$$

(iii) These examples may be conjoined into a single countable language $\mathcal{L}_\infty$ so that for each $\alpha < \omega_1$ there is $\mathcal{M} \in \mathrm{Mod}(\mathcal{L}_\infty)$ with

$$\gamma(\mathcal{M}) > \alpha.$$

The next exercise can be viewed as a topological formulation of an argument in [**69**]; it is related to the omitting types theorem for $\mathcal{L}_{\omega_1\omega}$, as presented in [**59**], but the reformulation assumes no familiarity with infinitary logic.

Exercise 6.17. Suppose that X is a Polish S_∞ space and $x \in X$ with

$$[x]_{S_\infty} \in \underset{\sim}{\Pi}^0_\alpha.$$

Let $(X, \varphi_{\alpha+1}(x, V_0))$ be the set of all $y \in X$ with

$$\varphi_{\alpha+1}(x, V_0) = \varphi_{\alpha+1}(y, V_0)$$

equipped with the topology generated by basic open sets of the form

$$\{z : \varphi_\alpha(x', V_n) = \varphi_\alpha(z, V_n)\}$$

some $x' \in [x]_{S_\infty}$ and $n \in \mathbb{N}$.

(i) The Borel complexity of $[x]_{S_\infty}$ implies that $(X, \varphi_{\alpha+1}(x, V_0)) = [x]_{S_\infty}$.

(ii) In the above topology $(X, \varphi_{\alpha+1}(x, V_0))$ is a Polish S_∞-space.

(iii) For each n let E_n be the orbit equivalence relation on $(X, \varphi_{\alpha+1}(x, V_0))$ induced by the continuous action of the clopen subgroup V_n of S_∞. 2.65 implies for each basic open

$$U = \{z : \varphi_\alpha(x', V_n) = \varphi_\alpha(z, V_n)\}$$

there is some $m > n$ and $x'' \in [x]_{S_\infty}$ so that

$$\forall^* y_0, y_1 \in U \in \{z : \varphi_\alpha(x'', V_m) = \varphi_\alpha(z, V_m)\}$$

we have

$$y_0 E_n y_1.$$

(iv) For a comeager set of y in $(X, \varphi_{\alpha+1}(x, V_0))$

$$\gamma^*(y) \leq \alpha + 1$$

and hence

$$\gamma^*(x) \leq \alpha + 1.$$

(Hint: Applying Kuratowski-Ulam we may find a comeager set $y \in (X, \varphi_{\alpha+1}(x, V_0))$ such that for all n and $\forall^* g \in S_\infty$ there is some x'' and $m \geq n$ witnessing (iii) above for $g \cdot y$ with

$$\varphi_\alpha(g \cdot y, V_m) = \varphi_\alpha(x'', V_m).)$$

(v) Thus by 6.16 it is possible to have a Polish S_∞-space X where the orbits are unbounded in Borel complexity, and in particular $E^X_{S_\infty}$ is non-Borel.

EXERCISE 6.18. Let X be a Polish S_∞-space. Let $\mathcal{L}$ be the language generated by a single binary relation R.

(i) For $\alpha < \omega_1$ there is a Borel map $\theta_\alpha : X \to \mathrm{Mod}(\mathcal{L})$ such that for all $x \in X$ the structure $\theta_\alpha(x)$ is isomorphic to

$$(\mathrm{TC}(\{\varphi_\beta(x_0, V_n) : x_0 \in [x]_{S_\infty}, n \in \mathbb{N}, \beta < \alpha\}), \in).$$

(ii) For each $\alpha < \omega_1$, the set of $x \in X$ with $\gamma^*(x) = \alpha$ is Borel.

(iii) Mimicking the steps (i)-(v) of 6.15 we obtain that for each Borel $f : Y \to X$ there is some $\delta < \omega_1$ such that

$$\forall^* y \in Y(\gamma^*(f(y)) < \delta.$$

(iv) Thus choosing θ_α as in (i) for each $\alpha < \omega_1$, the resulting map

$$X \to \mathrm{Mod}(\mathcal{L})$$

$$x \mapsto \theta_{\gamma^*(x)+2}(x)$$

is a universally Baire measurable reduction of $E^X_{S_\infty}$ to $\cong|_{\mathrm{Mod}(\mathcal{L})}$.

EXERCISE 6.19. (See §2.7 [**4**]) Let X be a Polish S_∞-space with basis $(U_l)_{l\in\mathbb{N}}$ and let $\mathcal{L}$ be the language with infinitely many relations $(R_{l,k})_{l\in\mathbb{N}}$ of arity k for each $k \geq 1$. That is to say, each $R_{n,1}$ will be a unary predicate for each n, $R_{n,2}$ will be binary relation, $R_{n,3}$ will be relation on triples and so on.

For $n_0, n_1, ..., n_{k-1}$ a finite string of natural numbers, let

$$B_{n_0,n_1,\dots,n_{k-1}} = \{g \in S_\infty : g(n_i) = i \text{ all } i < k\}.$$

For $x \in X$ define $\mathcal{M}(x) \in \mathrm{Mod}(\mathcal{L})$ by the specification

$$\mathcal{M}(x) \models R_{l,k}(n_0, ..., n_{k-1})$$

if and only if

$$\forall g \in B_{n_0,n_1,\dots,n_{k-1}}(g \cdot x \in \overline{U_l}).$$

Then $x \mapsto \mathcal{M}_x$ witnesses that $E^X_{S_\infty} \leq_B \cong|_{\mathrm{Mod}(\mathcal{L})}$.

EXERCISE 6.20. Following the proof of 6.6 we have that if $x_0, x_1 \in [x]_{S_\infty}$ with

$$\varphi_{\gamma^*(x)}(x_0, V_n) = \varphi_{\gamma^*(x)}(x_1, V_n),$$

then there is $g \in V_n$ with

$$g \cdot x_0 = x_1.$$

The aim of the next section is to extend 6.7 to arbitrary Polish groups, though with a certain disappointment, since we now cannot demand that there are *complete invariants* in HC for arbitrary Polish groups. One compromise might be to consider invariants from a class more general than HC. Instead we will forgo the requirement that the invariants be complete, asking only that if any complete invariants in HC can be found then the some analogue of $x \mapsto \varphi_x^*$ will provide the reduction. While this fails for complicated groups, it is true for a class of groups much broader than abelian, and the generalized Scott analysis of §6.2 will ultimately provide a canonical transfinite process that provably reduces an orbit equivalence relation to id(HC) if and only if any such reasonable reduction exists.

6.2. The general case

DEFINITION 6.21. Let G be a Polish group and X be a Polish G-space. Fix $(U_n)_{n\in\mathbb{N}}$ a basis for X with $U_0 = X$. Let $(V_n)_{n\in\mathbb{N}}$ be a neighbourhood basis at 1_G with $V_0 = G$ and each $V_n = V_n^{-1}$. Let $G_0 = \{g_i : i \in \mathbb{N}\}$ be a countable dense subgroup of G and assume that the above bases are suitably invariant, in that for all i, j there are $n, m \in \mathbb{N}$ such that

$$g_i V_j g_i^{-1} = V_m,$$

$$g_i \cdot U_j = U_n.$$

Recalling the definition of the local orbit $\mathcal{O}(x, U, V)$ from 3.15, for $x \in X$, $U \subset X$, $V \subset G$ both open, $\alpha \leq \omega_1$, we define $\varphi_\alpha(x, U, V)$ by induction on α:-

(1) $\varphi_0(x, U, V) = \{l : \mathcal{O}(x, U, V) \cap U_l \neq \emptyset\}$;

(2) $\varphi_{\alpha+1}(x, U, V)$ equals the set of all

$$\langle \varphi_\alpha(\hat{x}, U_n, V_m), n, m\rangle$$

for $\hat{x} \in \mathcal{O}(x, U, V), U_n \subset U, V_m \subset V$;

(3) for λ a limit, set $\varphi_\lambda(x, U, V) = \{\langle \varphi_\alpha(x, U, V), \alpha\rangle : \alpha < \lambda\}$.

LEMMA 6.22. *(X, G, $(U_n)_{n\in\mathbb{N}}$, $(V_n)_{n\in\mathbb{N}}$ as directly above.) For all $x \in X$, $n, m \in \mathbb{N}$ there exists $\gamma(x, n, m) < \omega_1$ such that for all $x_0, x_1 \in [x]_G$*

$$\exists \alpha < \omega_1(\varphi_\alpha(x_0, U_n, V_m) \neq \varphi_\alpha(x_1, U_n, V_m))$$

implies

$$\exists \alpha \leq \gamma(x, n, m)(\varphi_\alpha(x_0, U_n, V_m) \neq \varphi_\alpha(x_1, U_n, V_m)).$$

PROOF. Fixing x, for each $g, \hat{g} \in G$ with $g \cdot x, \hat{g} \cdot x \in U_n$ set $g \approx_{n,m}^x \hat{g}$ if there is some $k \in \mathbb{N}$ and group elements $g_0, g_1, ..., g_{k-1} \in V_m$ and $h_0, h_1, ...h_k \in G$ such that:

$h_0 = g, h_k = \hat{g}$, each $h_{i+1} = g_i h_i$;
each $g_i \in V_m$;
each $h_i \cdot x \in U_n$.

This is an open equivalence relations on G and hence has at most countably many equivalence classes. It is also true that it provides *congruence* with respect to the functions $g \mapsto \varphi_\alpha(g \cdot x, U_n, V_m)$, in the sense that for any $\alpha < \omega_1$ and any $g \approx_{n,m}^x \hat{g}$

$$\varphi_\alpha(g \cdot x, U_n, V_m) = \varphi_\alpha(\hat{g} \cdot x, U_n, V_m).$$

Now for any two equivalence classes $[g]^x_{n,m}(=_{df}\ \{h \in G : h \approx^x_{n,m} g\})$, $[\hat{g}]^x_{n,m}$ we can let $\gamma([g]^x_{n,m}, [\hat{g}]^x_{n,m})$ be the least α such that $\varphi_\alpha(g \cdot x, U_n, V_m) \neq \varphi_\alpha(\hat{g} \cdot x, U_n, V_m)$ if it exists (0 if not). Then

$$\gamma(x, n, m) = \sup\{\gamma([g]^x_{n,m}, [\hat{g}]^x_{n,m}) : g, \hat{g} \in G, g \cdot x, \hat{g} \cdot x \in U_n\}$$

is countable since we are taking the supremum over at most $\aleph_0$ many equivalence classes. $\square$

DEFINITION 6.23. Let $\gamma(x, n, m)$ be least as in the conclusion of the lemma 6.22, and set $\gamma(x) = \sup\{\gamma(x, n, m) : n, m \in \mathbb{N}\}$. Let $\varphi_x = \varphi_{\gamma(x)+2}(x, X, G)$.

LEMMA 6.24. *$x_1 E^X_G x_2$ implies $\gamma(x_1) = \gamma(x_2)$, and $\varphi_{x_1} = \varphi_{x_2}$.*

PROOF. For the first of these statements we note that the structure of the definitions gives at each n and m we have $\gamma(x_1, n, m) = \gamma(x_2, n, m)$. The second then follows since $\mathcal{O}(x_1, X, G) = \mathcal{O}(x_2, X, G)$. $\square$

LEMMA 6.25. *For all $\alpha < \omega_1$ we have for all $x, y \in X,\ n, m \in \mathbb{N}$ that*

$$\varphi_{\alpha+1}(x, U_n, V_m) = \varphi_{\alpha+1}(y, U_n, V_m)$$

implies

$$\varphi_\alpha(x, U_n, V_m) = \varphi_\alpha(y, U_n, V_m).$$

PROOF. By transfinite induction: Suppose instead that this fails and let α be least for which there is some x, y, n, m providing a counterexample. First consider the case that α is a successor ordinal, so $\alpha = \beta + 1$ for some $\beta < \omega_1$. Then

$$\begin{aligned}&\{\langle \varphi_\alpha(\hat{x}, U_{\hat{n}}, V_{\hat{m}}), \hat{n}, \hat{m}\rangle : \hat{x} \in \mathcal{O}(x, U_n, V_m), U_{\hat{n}} \subset U_n, V_{\hat{m}} \subset V_m\}\\ =\ &\{\langle \varphi_\alpha(\hat{y}, U_{\hat{n}}, V_{\hat{m}}), \hat{n}, \hat{m}\rangle : \hat{y} \in \mathcal{O}(y, U_n, V_m), U_{\hat{n}} \subset U_n, V_{\hat{m}} \subset V_m\}\end{aligned}$$

by definition of $\varphi_{\alpha+1}$. Hence

$$\begin{aligned}&\{\langle \varphi_\beta(\hat{x}, U_{\hat{n}}, V_{\hat{m}}), \hat{n}, \hat{m}\rangle : \hat{x} \in \mathcal{O}(x, U_n, V_m), U_{\hat{n}} \subset U_n, V_{\hat{m}} \subset V_m\}\\ =\ &\{\langle \varphi_\beta(\hat{y}, U_{\hat{n}}, V_{\hat{m}}), \hat{n}, \hat{m}\rangle : \hat{y} \in \mathcal{O}(y, U_n, V_m), U_{\hat{n}} \subset U_n, V_{\hat{m}} \subset V_m\}\end{aligned}$$

by the assumption that $\alpha = \beta + 1$ is the first place at which the lemma runs afoul; but this last statement amounts to asserting $\varphi_{\beta+1}(x, U_n, V_m) = \varphi_{\beta+1}(y, U_n, V_m)$.

On the other hand if α is a limit and $\varphi_{\alpha+1}(x, U_n, V_m) = \varphi_{\alpha+1}(y, U_n, V_m)$, then again by definition

$$\begin{aligned}&\{\langle \varphi_\alpha(\hat{x}, U_{\hat{n}}, V_{\hat{m}}), \hat{n}, \hat{m}\rangle : \hat{x} \in \mathcal{O}(x, U_n, V_m), U_{\hat{n}} \subset U_n, V_{\hat{m}} \subset V_m\}\\ =\ &\{\langle \varphi_\alpha(\hat{y}, U_{\hat{n}}, V_{\hat{m}}), \hat{n}, \hat{m}\rangle : \hat{y} \in \mathcal{O}(y, U_n, V_m), U_{\hat{n}} \subset U_n, V_{\hat{m}} \subset V_m\},\end{aligned}$$

and by the assumption that α a limit we have

$$\begin{aligned}&\{\langle \varphi_\beta(\hat{x}, U_{\hat{n}}, V_{\hat{m}}), \beta, \hat{n}, \hat{m}\rangle : \hat{x} \in \mathcal{O}(x, U_n, V_m), U_{\hat{n}} \subset U_n, V_{\hat{m}} \subset V_m, \beta < \alpha\}\\ =\ &\{\langle \varphi_\beta(\hat{y}, U_{\hat{n}}, V_{\hat{m}}), \beta, \hat{n}, \hat{m}\rangle : \hat{y} \in \mathcal{O}(y, U_n, V_m), U_{\hat{n}} \subset U_n, V_{\hat{m}} \subset V_m, \beta < \alpha\}.\end{aligned}$$

This implies in particular that

$$\begin{aligned}&\{\langle \varphi_\beta(\hat{x}, U_n, V_m), \beta, n, m\rangle : \hat{x} \in \mathcal{O}(x, U_n, V_m), \beta < \alpha\}\\ =\ &\{\langle \varphi_\beta(\hat{y}, U_n, V_m), \beta, n, m\rangle : \hat{y} \in \mathcal{O}(y, U_n, V_m), \beta < \alpha\}.\end{aligned}$$

Since for $\hat{z} \in \mathcal{O}(z, U, V)$ one always has $\varphi_\beta(z, U, V) = \varphi_\beta(\hat{z}, U, V)$ this entails

$$\{\langle \varphi_\beta(x, U_n, V_m), \beta, n, m\rangle : \beta < \alpha\} = \{\langle \varphi_\beta(y, U_n, V_m), \beta, n, m\rangle : \beta < \alpha\},$$

as desired. $\square$

COROLLARY 6.26. *For* $\alpha < \beta$,
$$\varphi_\beta(x, U_n, V_m) = \varphi_\beta(y, U_n, V_m)$$
implies
$$\varphi_\alpha(x, U_n, V_m) = \varphi_\alpha(y, U_n, V_m).$$

PROOF. By transfinite induction on β. $\square$

LEMMA 6.27. $\varphi_x = \varphi_{\gamma(x)+2}(y, X, G)$ *implies* $\gamma(x) = \gamma(y)$ *and therefore*
$$\varphi_x = \varphi_y.$$

PROOF. Claim(I): For all $y_0, y_1 \in [y]_G$, $n, m \in \mathbb{N}$
$$\varphi_{\gamma(x)}(y_0, U_n, V_m) = \varphi_{\gamma(x)}(y_1, U_n, V_m)$$
implies
$$\varphi_{\gamma(x)+1}(y_0, U_n, V_m) = \varphi_{\gamma(x)+1}(y_1, U_n, V_m).$$
Proof of claim: If not, then from
$$\{\langle \varphi_{\gamma(x)+1}(y', U_n, V_m), n, m\rangle : n, m \in \mathbb{N}, y' \in [y]_G\}$$
equaling
$$\{\langle \varphi_{\gamma(x)+1}(x_0, U_n, V_m), n, m\rangle : n, m \in \mathbb{N}, x_0 \in [x]_G\}$$
we can obtain $x_0, x_1 \in [x]_G$ so that
$$\varphi_{\gamma(x)+1}(x_1, U_n, V_m) = \varphi_{\gamma(x)+1}(y_1, U_n, V_m) \neq$$
$$\varphi_{\gamma(x)+1}(x_0, U_n, V_m) = \varphi_{\gamma(x)+1}(y_0, U_n, V_m).$$
By 6.25
$$\varphi_{\gamma(x)}(x_1, U_n, V_m) = \varphi_{\gamma(x)}(y_1, U_n, V_m) =$$
$$\varphi_{\gamma(x)}(y_0, U_n, V_m) = \varphi_{\gamma(x)}(x_0, U_n, V_m),$$
with a contradiction to the definition of $\gamma(x)$. (Claim(I)$\square$)

Claim(II): For all $\beta < \omega_1$, $y_0, y_1 \in [y]_G$, $n, m \in \mathbb{N}$
$$\varphi_{\gamma(x)}(y_0, U_n, V_m) = \varphi_{\gamma(x)}(y_1, U_n, V_m) \Rightarrow \varphi_\beta(y_0, U_n, V_m) = \varphi_\beta(y_1, U_n, V_m).$$

Proof of claim: By induction on $\beta \leq \gamma(x)$ when it is trivial, as is carrying the induction through limit stages. So let us suppose the claim is true at β and attempt to lift up to $\beta + 1$. For this purpose suppose $y_0, y_1 \in [y]_G$, $n, m \in \mathbb{N}$
$$\varphi_{\gamma(x)}(y_0, U_n, V_m) = \varphi_{\gamma(x)}(y_1, U_n, V_m).$$
Then by claim(I) we have
$$\varphi_{\gamma(x)+1}(y_0, U_n, V_m) = \varphi_{\gamma(x)+1}(y_1, U_n, V_m),$$
and so by definition
$$\{\langle \varphi_{\gamma(x)}(\hat{y}, U_{\hat{n}}, V_{\hat{m}}), \hat{n}, \hat{m}\rangle : \hat{y} \in \mathcal{O}(y_0, U_n, V_m), U_{\hat{n}} \subset U_n, V_{\hat{m}} \subset V_m\} =$$
$$\{\langle \varphi_{\gamma(x)}(\hat{y}, U_{\hat{n}}, V_{\hat{m}}), \hat{n}, \hat{m}\rangle : \hat{y} \in \mathcal{O}(y_1, U_n, V_m), U_{\hat{n}} \subset U_n, V_{\hat{m}} \subset V_m\},$$
and so by the inductive assumption
$$\{\langle \varphi_\beta(\hat{y}, U_{\hat{n}}, V_{\hat{m}}), \hat{n}, \hat{m}\rangle : \hat{y} \in \mathcal{O}(y_1, U_n, V_m), U_{\hat{n}} \subset U_n, V_{\hat{m}} \subset V_m\}$$
$$= \{\langle \varphi_\beta(\hat{y}, U_{\hat{n}}, V_{\hat{m}}), \hat{n}, \hat{m}\rangle : \hat{y} \in \mathcal{O}(y_0, U_n, V_m), U_{\hat{n}} \subset U_n, V_{\hat{m}} \subset V_m\}$$
$$\therefore \varphi_{\beta+1}(y_0, U_n, V_m) = \varphi_{\beta+1}(y_1, U_n, V_m).$$
(Claim(II)$\square$)

Hence $\gamma(y) \leq \gamma(x)$.

For the reverse direction note then that $\varphi_y = \varphi_{\gamma(y)+2}(x, X, G)$ and apply the already proved part of this lemma. $\square$

The next two lemmas are technical in appearance but nearly vacuous in content. The first states that in some sense $\varphi_\alpha(x, U, V)$ provides information about x that is in some sense *isomorphic* to the information about $g \cdot x$ provided by

$$\varphi_\alpha(g \cdot x, g \cdot U, gVg^{-1});$$

this should hardly be surprising, since we can imagine an isomorphism of the whole complex (X, G) which respects the structure provided by the group action and is given by

$$x \mapsto g \cdot x,$$

$$h \mapsto ghg^{-1}.$$

There is a minor issue here that the choice of the bases $\{U_n\}$ and $\{V_m\}$ may be affected by translation and conjugation by g, and so for strictly notational convenience the lemma is stated just for $g \in G_0$. The second lemma states that as we restrict U and V the $\varphi_\alpha(x, U, V)$ invariant provides more information. Again this has an intuitive basis, since $\varphi_\alpha(x, U, V)$ tries to sample information about x by restricting its movement to the local orbit $\mathcal{O}(x, U, V)$; as we restrict the local orbit further the likelihood of confusing x with some other point diminishes.

LEMMA 6.28. *For $g \in G_0$, the countable dense subgroup fixed at 6.21, $U \subset X$, $V \subset G$ open, and $x, y \in U$, $1 \in V$, $V^{-1} = V$, $\alpha < \omega_1$ we have*

$$\varphi_\alpha(x, U, V) = \varphi_\alpha(y, U, V)$$

implies

$$\varphi_\alpha(g \cdot x, g \cdot U, gVg^{-1}) = \varphi_\alpha(g \cdot y, g \cdot U, gVg^{-1}).$$

PROOF. At the case $\alpha = 0$ it is immediate, since for any $l, \hat{l}$ with $g \cdot U_l = U_{\hat{l}}$

$$l \in \varphi_0(x, U, V) \Leftrightarrow \hat{l} \in \varphi_0(g \cdot x, g \cdot U, gVg^{-1}).$$

The rest proceeds by transfinite induction, with the limit case trivial and the successor case resting on

$$\varphi_{\alpha+1}(g \cdot x, g \cdot U, gVg^{-1}) =$$

$$\{\langle \varphi_\alpha(\hat{x}, U_{\hat{n}}, V_{\hat{m}}), \hat{n}, \hat{m}\rangle : \hat{x} \in \mathcal{O}(g \cdot x, g \cdot U, gVg^{-1}), U_{\hat{n}} \subset g \cdot U, V_{\hat{m}} \subset gVg^{-1}\},$$

which in turn equals the set of $\langle \varphi_\alpha(g \cdot \bar{x}, g \cdot U_n, gV_mg^{-1}), \hat{n}, \hat{m}\rangle$ such that

$$\bar{x} \in \mathcal{O}(x, U, V),$$

$$U_n \subset U,$$

$$V_m \subset V,$$

$$U_{\hat{n}} = g \cdot U_n,$$

$$V_{\hat{m}} = gV_mg^{-1}.$$

by the assumption that $\{U_n : n \in \mathbb{N}\}$ and $\{V_m : m \in \mathbb{N}\}$ are invariant sets under translations and conjugation by elements of G_0. $\square$

LEMMA 6.29. *For $x, y \in \hat{U} \subset U$, $1 \in \hat{V} \subset V$, $V = V^{-1}$, $\hat{V} = \hat{V}^{-1}$, $\alpha < \omega_1$ we have*

$$\varphi_\alpha(x, \hat{U}, \hat{V}) = \varphi_\alpha(y, \hat{U}, \hat{V})$$

implies

$$\varphi_\alpha(x, U, V) = \varphi_\alpha(y, U, V).$$

PROOF. For the case $\alpha = 0$ note that the set of

$$O_{l,U,V} =_{df} \{z \in U : l \in \varphi_0(z, U, V)\}$$

is an open set, and $l \in \varphi_0(z, U, V)$ if and only if $\mathcal{O}(z, \hat{U}, \hat{V}) \cap O_{l,U,V} \neq \emptyset$.

The lemma is then proved by transfinite induction. The limit steps are trivial. For the successor step, from assuming inductively that the lemma is true at α,

$$\varphi_{\alpha+1}(x, \hat{U}, \hat{V}) = \varphi_{\alpha+1}(y, \hat{U}, \hat{V}),$$

and $\hat{U} \subset U$, $\hat{V} \subset V$ it follows that

$$\begin{aligned} &\{\varphi_\alpha(\hat{x}, U_n, V_m) : U_n \subset \hat{U}, V_m \subset \hat{V}, \hat{x} \in \mathcal{O}(x, \hat{U}, \hat{V})\} \\ = &\{\varphi_\alpha(\hat{y}, U_n, V_m) : U_n \subset \hat{U}, V_m \subset \hat{V}, \hat{y} \in \mathcal{O}(y, \hat{U}, \hat{V})\}, \end{aligned}$$

and hence

$$\begin{aligned} &\{\varphi_\alpha(\hat{x}, U_n, V_m) : U_n \subset U, V_m \subset V, \hat{x} \in \mathcal{O}(x, U, V)\} \\ = &\{\varphi_\alpha(\hat{y}, U_n, V_m) : U_n \subset U, V_m \subset V, \hat{y} \in \mathcal{O}(y, U, V)\}, \end{aligned}$$

by the inductive assumption, 6.28, and the observation that for each $\hat{x} \in \mathcal{O}(x, U, V)$, $U_n \subset U$ containing $\hat{x}$, and $V_m \subset V$ we can find $g \in G_0$, $\hat{n}, \hat{m} \in \mathbb{N}$ with

$$\begin{aligned} g \cdot \hat{x} &\in \mathcal{O}(x, \hat{U}, \hat{V}), \\ g \cdot \hat{x} &\in U_{\hat{n}} \subset \hat{U} \cap g \cdot U_n, \\ V_{\hat{m}} &\subset \hat{V} \cap g V_m g^{-1}. \end{aligned}$$

□

LEMMA 6.30. *X, G, and so on, as above. There is a countable language $\mathcal{L}$ and a a Borel map $\theta : X \to \mathrm{Mod}(\mathcal{L})$ such that:*

(i) $xE_G y \Rightarrow \theta(x) \cong \theta(y)$;

(ii) $\theta(x) \cong \theta(y) \Rightarrow \varphi_x = \varphi_y$;

(iii) $[\theta[X]]_{S_\infty} =_{df} \{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}) : \exists x \in X (\mathcal{M} \cong \theta(x))\}$ is G_δ;

(iv) moreover there is a Borel map

$$\rho : [\theta[X]]_{S_\infty} \to X$$

such that for all $\mathcal{M} \in [\theta[X]]_{S_\infty}$

$$\theta(\rho(\mathcal{M})) \cong \mathcal{M}.$$

PROOF. Let $\mathcal{L} = \langle (\mathcal{O}_l)_{l \in \mathbb{N}}, (R_{n,m})_{n,m \in \mathbb{N}}, E, (F_h)_{h \in G_0} \rangle$, where each $\mathcal{O}_l$ is a unary predicate, each $R_{n,m}$ is a unary predicate, E is a binary relation, and each F_h is a unary function. Fix $x \in X$. We will go ahead and describe a model $\mathcal{M}(x)$. It is most convenient to first present it as a countable model whose underlying set is the union over m and n of the equivalence classes in G of the open equivalence relations $\approx^x_{m,n}$, as it was defined at 6.22; after the fact it will be easy to see that this can be reorganized with $\theta(x) \cong \mathcal{M}(x)$ a structure on $\mathbb{N}$ with

$$x \mapsto \theta(x)$$

Borel.

As to motivation, the models $\mathcal{M}(x)$ can be thought of as being *almost* formed out of the local orbits for $[x]_G$. The $\mathcal{O}_l$ are intended to describe which open sets U_l intersect the local orbit, while the $R_{n,m}$ indicate the U_n and V_m that define this "almost local orbit". The F_g functions are included so as to enforce that $\mathcal{M}(x)$ has a kind of homogeneity, in turn used to obtain ρ as a Borel function. The E is not an equivalence relation, but corresponds to something like the membership relation, reflecting when $\varphi_{\alpha+1}(x, U_n, V_m)$ has $\langle\varphi_\alpha(\hat{x}, U_{\hat{n}}, V_{\hat{m}}), \hat{n}, \hat{m}\rangle$ as an element for suitable $\hat{n}, \hat{m}, \hat{x}$. Anyway, this is all only motivation and not yet an accurate description of the proof.

For $h \in G$ and $m, n \in \mathbb{N}$ with $h \cdot x \in U_n$, we let $[h]^x_{n,m}$ be the set of $\hat{h} \in G$ for which there exists $k \in \mathbb{N}$, and

$$g_0, g_1, ..., g_{k-1} \in V_m,$$

and for

$$h_0 = h, h_{i+1} = g_i h_i$$

we have

$$h_k = \hat{h}$$

and each

$$h_i \cdot x \in U_n.$$

So $[h]^x_{n,m}$ is similar but not necessarily identical to the pullback to G of $\mathcal{O}(h \cdot x, U_n, V_m)$ under the map $g \mapsto g \cdot x$; it is not identical in the case that the stabilizer of x is non-trivial – for if $g \cdot x = x$ but g not in V_n then

$$g \cdot x \in \mathcal{O}(x, U_n, V_m)$$

without necessarily $g \in [1_G]^x_{n,m}$.

Define $\mathcal{M}(x)$ to be the set $\{\langle[h]^x_{n,m}, n, m\rangle : h \in G, m, n \in \mathbb{N}, h \cdot x \in U_n\}$. Since any open equivalence relation on an open subset of G can have only countably many classes, each $\{\langle[h]^x_{n,m}, n, m\rangle : h \in G, h \cdot x \in U_n\}$ is countable, and so $\mathcal{M}(x)$ is countable.

We specify the behavior of the relations in $\mathcal{M}(x)$ as follows:-

(A) $\mathcal{O}_l$: $\mathcal{M}(x) \models \mathcal{O}_l(\langle[h]^x_{n,m}, n, m\rangle)$ if (and only if) $l \in \varphi_0(h \cdot x, m, n)$;

(B) E: $\mathcal{M}(x) \models \langle[h]^x_{n,m}, n, m\rangle E \langle[\hat{h}]^x_{\hat{n},\hat{m}}, \hat{n}, \hat{m}\rangle$ if

$$U_n \subset U_{\hat{n}}, V_m \subset V_{\hat{m}}, h \in [\hat{h}]^x_{\hat{n},\hat{m}};$$

(C) $R_{n,m}$: $\mathcal{M}(x) \models R_{n,m}(\langle[h]^x_{\hat{n},\hat{m}}, \hat{n}, \hat{m}\rangle)$ if $m = \hat{m}$ and $n = \hat{n}$;

(D) $(F_g(\langle[h]^x_{n,m}, n, m\rangle))^{\mathcal{M}(x)} = \langle[\hat{h}]^x_{\hat{n},\hat{m}}, \hat{n}, \hat{m}\rangle$ if

$$g \cdot U_n = U_{\hat{n}},$$

$$g V_m g^{-1} = V_{\hat{m}},$$

$$gh \in [\hat{h}]^x_{\hat{n},\hat{m}}.$$

Claim(I): If $g \in G, x, y \in X$ with $g \cdot x = y$, then there exists an isomorphism

$$\pi_g : \mathcal{M}(x) \cong \mathcal{M}(y).$$

Proof of Claim: For any $h \in G$ and n, m with $h \cdot x \in U_n$ let

$$\pi_g(\langle[h]^x_{n,m}, n, m\rangle) = \langle[hg^{-1}]^y_{n,m}, n, m\rangle.$$

From the definition of the $\approx^x_{n,m}$ equivalence relations this is well defined. It is immediate that the map respects the predicates of the form $R_{\hat{m},\hat{n}}$. Since

$$\mathcal{O}(h \cdot x, U_n, V_m) = \mathcal{O}(\hat{h} \cdot y, U_n, V_m) = \mathcal{O}(\hat{h}g \cdot x, U_n, V_m),$$

whenever $\pi_g(\langle [h]^x_{n,m}, n, m\rangle) = \langle [\hat{h}]^y_{n,m}, n, m\rangle$, we have preservation of the $\mathcal{O}_l$ predicates and the binary relation E. Finally for the functions of the form $F_{\hat{g}}$, $\hat{g} \in G_0$, given

$$(F_{\hat{g}}(\langle [h]^x_{n,m}, n, m\rangle))^{\mathcal{M}(x)} = \langle [\hat{g}h]^x_{\hat{n},\hat{m}}, \hat{m}, \hat{n}\rangle,$$
$$\pi_g(\langle [h]^x_{n,m}, n, m\rangle) = \langle [hg^{-1}]^y_{n,m}, n, m\rangle,$$

it follows

$$\pi_g(\langle [\hat{g}h]^x_{\hat{n},\hat{m}}, \hat{m}, \hat{n}\rangle) = \langle [\hat{g}hg^{-1}]^y_{\hat{n},\hat{m}}, \hat{m}, \hat{n}\rangle$$
$$= (F_{\hat{g}}(\langle [hg^{-1}]^y_{\hat{n},\hat{m}}, \hat{m}, \hat{n}\rangle))^{\mathcal{M}(y)} = (F_{\hat{g}}(\pi_g(\langle [h]^y_{\hat{n},\hat{m}}, \hat{m}, \hat{n}\rangle))^{\mathcal{M}(y)}.$$

(Claim(I)□)

Claim(II): Let $x, y \in X$ with

$$\pi : \mathcal{M}(x) \cong \mathcal{M}(y)$$

an isomorphism. For all $\alpha < \omega_1$, if

$$\pi(\langle [h]^x_{n,m}, n, m\rangle) = \langle [\hat{h}]^y_{n,m}, n, m\rangle$$

then

$$\varphi_\alpha(h \cdot x, m, n) = \varphi_\alpha(\hat{h} \cdot y, m, n).$$

Proof of Claim: By transfinite induction on α, with the base case $\alpha = 0$ following from the fact that π respects the $\mathcal{O}_l$ predicates and the limit step being being trivial. For the successor step suppose the claim holds at α for all x, y and π, and then fix $x, y \in X$ and

$$\pi : \mathcal{M}(x) \cong \mathcal{M}(y)$$

with

$$\pi(\langle [h]^x_{n,m}, n, m\rangle) = \langle [\hat{h}]^y_{n,m}, n, m\rangle.$$

By symmetry it suffices to show that

$$\varphi_{\alpha+1}(h \cdot x, U_n, V_m)$$
$$=_{df} \{\langle \varphi_\alpha(\hat{x}, U_{\hat{n}}, V_{\hat{m}}), \hat{n}, \hat{m}\rangle : \hat{x} \in \mathcal{O}(h \cdot x, U_n, V_m), U_{\hat{n}} \subset U_n, V_{\hat{m}} \subset V_m\}$$

is a subset of $\varphi_{\alpha+1}(\hat{h} \cdot y, U_n, V_m)$. But for each

$$\hat{x} \in \mathcal{O}(h \cdot x, U_n, V_m)$$

and

$$U_{\hat{n}} \subset U_n, V_{\hat{m}} \subset V_m$$

we may find some $g \in [h]^x_{n,m}$ with $g \cdot x = \hat{x}$. Fixing this g note that

$$\mathcal{M}(x) \models \langle [g]^x_{\hat{m},\hat{n}}, \hat{n}, \hat{m}\rangle E \langle [h]^x_{m,n}, n, m\rangle$$

and hence there will be some $\hat{g} \in G$ with

$$\mathcal{M}(y) \models \langle [\hat{g}]^y_{\hat{m},\hat{n}}, \hat{n}, \hat{m}\rangle E \langle [\hat{h}]^y_{m,n}, n, m\rangle$$

and

$$\pi(\langle [g]^x_{\hat{n},\hat{m}}, \hat{n}, \hat{m}\rangle) = \langle [\hat{g}]^y_{\hat{n},\hat{m}}, \hat{n}, \hat{m}\rangle.$$

From the definitions

$$\langle \varphi_\alpha(\hat{g} \cdot y, U_{\hat{n}}, V_{\hat{m}}), \hat{n}, \hat{m}\rangle \in \varphi_{\alpha+1}(\hat{h} \cdot y, U_n, V_m)$$

while by inductive assumption

$$\varphi_\alpha(\hat{g}\cdot y, U_{\hat{n}}, V_{\hat{m}}) = \varphi_\alpha(g\cdot x, U_{\hat{n}}, V_{\hat{m}}),$$

as required. (Claim(II) $\square$)

It is now routine to find a Borel function $\theta : X \to \mathrm{Mod}(\mathcal{L})$ so that $\theta(x) \cong \mathcal{M}(x)$ for all $x \in X$. To see this note that $\mathcal{M}(x)$ has underlying set

$$\{[g]^x_{n,m} : g \in G_0, n, m \in \mathbb{N}\},$$

since G_0 is dense and each $\approx^x_{n,m}$ is an open equivalence relation. Recalling our fixed enumeration $(g_i)_{i\in\mathbb{N}}$ of G_0, the function

$$X \to 2^{\mathbb{N}}$$

$$x \mapsto A_x = \{s(i,n,m) \in \mathbb{N} : g_i \cdot x \in U_n, \forall j < i(g_i \not\approx^x_{n,m} g_j)\}$$

is Borel, where $s(\cdot,\cdot,...) : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ is some long previously chosen bijection between $\mathbb{N}^{<\mathbb{N}}$ and $\mathbb{N}$. We can now define $\theta(x) \in \mathrm{Mod}(\mathcal{L})$ by: $\theta(x) \models \mathcal{O}_l(k)$ if and only for $s(i,n,m)$ the kth element of A_x we have

$$l \in \varphi_0(g_i \cdot x, n, m);$$

$\theta(x) \models R_{n,m}(k)$ if and only for $s(i,\hat{n},\hat{m})$ the kth element of A_x we have $n = \hat{n}$, $m = \hat{m}$; and so on. Since functions such as

$$X \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to 2^{\mathbb{N}}$$

$$(x,i,n,m) \mapsto \varphi_0(g_i \cdot x, U_n, V_m)$$

are Borel it is not hard to see that the resulting map $\theta : X \to \mathrm{Mod}(\mathcal{L})$ is Borel.

Thus we have our Borel map and by claims (I) and (II) we have (i) and (ii) from the conclusion of the lemma. There is still a debt for (iii), to the effect that the range of θ has Borel saturation, and for (iv), to the effect that there exists an orbit inverse. In order to honor this let Y be the set of all $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ for which (1)-(7) below hold:

(1) the $R_{n,m}$'s uniquely label: for all $a \in \mathbb{N}$ there are *unique* $n, m \in \mathbb{N}$ for which $\mathcal{M} \models R_{n,m}(a)$; for all $a \in \mathbb{N}$, $n, m \in \mathbb{N}$, if $\mathcal{M} \models R_{n,m}(a)$ then $U_n \neq \emptyset$, and moreover if $U_n \subset U_{\hat{n}}$, $V_m \subset V_{\hat{m}}$ then there is a unique $b \in \mathbb{N}$ with

$$\mathcal{M} \models R_{\hat{n},\hat{m}}(b), aEb;$$

if

$$\mathcal{M} \models aEb, R_{n,m}(a), R_{\hat{n},\hat{m}}(b)$$

then $U_n \subset U_{\hat{n}}$, $V_m \subset V_{\hat{m}}$;

(2) the $\mathcal{O}_l$ predicates remember the structure held by $\{U_l : l \in \mathbb{N}\}$: for all $a \in \mathbb{N}$, $n, m, l \in \mathbb{N}$, $\hat{m} \in \mathbb{N}$ with $V_{\hat{m}} \subset V_m$, if $\mathcal{M} \models \mathcal{O}_l(a), R_{n,m}(a)$ then there is some $b \in \mathbb{N}$,

$$\mathcal{M} \models bEa, R_{l,\hat{m}}(b);$$

moreover $\mathcal{M} \models \mathcal{O}_l(a)$ implies that for each $k \in \mathbb{N}$ there exists $\hat{l}$ with $U_{\hat{l}} \neq \emptyset$,

$$d(U_{\hat{l}}) < 2^{-k}$$

(d the complete metric on X),

$$\overline{U_{\hat{l}}} \subset U_l$$

and

$$\mathcal{M} \models \mathcal{O}_{\hat{l}}(a);$$

(3) in pursuit of the idea that $F_g(a) = F_{\hat{g}}(a)$ exactly when some open set U_l with $\mathcal{O}_l(a)$ requires it: for all $a, b \in \mathbb{N}$, $n, m, \in \mathbb{N}$, $g, \hat{g} \in G_0$ with $\mathcal{M} \models aEb, R_{n,m}(b)$ we have

$$\mathcal{M} \models F_g(b) = F_{\hat{g}}(b)$$

if and only if there is some some $l \in \mathbb{N}$ such that

$$\mathcal{M} \models \mathcal{O}_l(a)$$

and there are

$$h_0, h_1, ..., h_k \in V_m \cap G_0$$

so that each

$$h_i h_{i-1} ... h_0 \cdot U_l \subset U_n$$

and

$$\hat{g} h_k h_{k-1} .... h_0 = g;$$

(4) when aEb, the collection of $\mathcal{O}_l$'s met by a determine the signature for b: for all $a, b \in \mathbb{N}$, $n, m, l \in \mathbb{N}$ such that

$$\mathcal{M} \models aEb, R_{n,m}(b)$$

we have

$$\mathcal{M} \models \mathcal{O}_l(b)$$

if and only if there exists $\hat{l} \in \mathbb{N}$ and $h_0, h_1, ..., h_k \in V_m \cap G_0$ so that each

$$h_i h_{i-1} ... h_0 \cdot U_{\hat{l}} \subset U_n,$$

$$h_k h_{k-1} ... h_0 \cdot U_{\hat{l}} \subset U_l$$

and

$$\mathcal{M} \models \mathcal{O}_{\hat{l}}(a);$$

(5) the E relation provides a partial order respected by the F_g functions that in turn enforce a type of homogeneity: for all $a, b \in \mathbb{N}$, $g \in G_0$

$$\mathcal{M} \models aEb \Rightarrow F_g(a) E F_g(b);$$

for all $a, b \in \mathbb{N}$ there exists $g \in G_0$, $c \in \mathbb{N}$ such that

$$\mathcal{M} \models cEa, F_g(c)Eb;$$

and

$$\mathcal{M} \models \forall a, b, c(aEb \wedge bEc \Rightarrow aEc);$$

$$\mathcal{M} \models \forall a, b((aEb \wedge bEa) \Rightarrow a = b);$$

(6) the composition of the functions respects the group operations....:$(F_{1_G})^{\mathcal{M}}$ is the identity; for $g, \hat{g} \in G$,

$$(F_g \circ F_{\hat{g}})^{\mathcal{M}} = (F_{g\hat{g}})^{\mathcal{M}};$$

...and the functions themselves the open sets: for $a \in \mathbb{N}$ $g \in G_0$, $l, \hat{l} \in \mathbb{N}$, $g \cdot U_l = U_{\hat{l}}$ implies

$$\mathcal{M} \models \mathcal{O}_l(a) \Leftrightarrow \mathcal{O}_{\hat{l}}(F_g(a)),$$

and if $n, m, \hat{n}, \hat{m} \in \mathbb{N}$ with $g \cdot U_n = U_{\hat{n}}$, $gV_n g^{-1} = V_{\hat{m}}$ then

$$\mathcal{M} \models R_{n,m}(a) \Leftrightarrow R_{\hat{n},\hat{m}}(F_g(a));$$

(7) the E structure of the $(F_g)_{g\in G_0}$-closure of a point is determined by its open sets: for all $g, \hat{g} \in G_0$, $a, b, c \in \mathbb{N}$, $n, m, \hat{n}, \hat{m} \in \mathbb{N}$ with

$$\mathcal{M} \models aEb, aEc, R_{n,m}(b), R_{\hat{n},\hat{m}}(c)$$

we have that

$$\mathcal{M} \models F_g(b)EF_{\hat{g}}(c)$$

if and only if

$$g \cdot U_n \subset \hat{g} \cdot U_{\hat{n}},$$
$$gV_m g^{-1} \subset \hat{g}V_{\hat{m}}\hat{g}^{-1},$$

and there exists $l \in \mathbb{N}$, $h_0, ..., h_k \in G \cap V_{\hat{m}}$ such that

$$\mathcal{M} \models \mathcal{O}_l(a),$$

each $h_j h_{j-1}...h_0 \cdot U_l \subset U_{\hat{n}}$, and $\hat{g}h_k h_{k-1}....h_0 = g$.

Y is G_δ subset of $\mathrm{Mod}(\mathcal{L})$ since every condition is defined by a countable intersection of Boolean combinations of open sets. It is not hard to check that $Y \subset [\theta[X]]_{S_\infty}$; perhaps the most problematical of these conditions is (3), but that follows from the set $\{x : g_i \approx^x_{n,m} g_j\}$ being open for each i, j, n, m.

We will find for each $\mathcal{M} \in Y$ some $\rho(\mathcal{M}) \in X$ such that $\theta(\rho(\mathcal{M})) \cong \mathcal{M}$. It will be clear from the construction that ρ is Borel, and then (iii) and (iv).

Fixing $\mathcal{M} \in Y$ we build a sequence of $a_i \in \mathcal{M}$, $n(i), m(i) \in \mathbb{N}$, such that for all $i, b, l \in \mathbb{N}$, $g, \hat{g} \in G_0$:

(a)$_i$ $\mathcal{M} \models a_{i+1}Ea_i$;
(b)$_i$ $\mathcal{M} \models R_{n(i),m(i)}(a_i)$;
(c)$_i$ the closure of $U_{n(i+1)}$ is included in $U_{n(i)}$; $V_{m(i+1)} \subset V_i \cap V_{m(i)}$;
(d)$_i$ $d(U_{n(i)}) < 2^{-i}$; $U_{n(i)} \neq \emptyset$;
(e)$_b$ there exists some $g \in G_0$ and $j \in \mathbb{N}$ with

$$\mathcal{M} \models F_g(a_{n(j)})Eb;$$

(f)$_{g,\hat{g},i}$ if

$$\mathcal{M} \models F_g(a_i) = F_{\hat{g}}(a_i)$$

then there exists some $j > i$ and $h_0, h_1, ..., h_k \in V_{m(i)}$ such that for each $p \leq k$

$$h_p h_{p-1}...h_0 \cdot U_{n(j)} \subset U_{n(i)},$$

and

$$gh_k h_{k-1}...h_0 = \hat{g};$$

(g)$_{i,l}$ if

$$\mathcal{M} \models \mathcal{O}_l(a_i)$$

then there is some $j > i$ and

$$h_0, h_1, ..., h_k \in V_m(i) \cap G_0$$

so that each

$$h_i h_{i-1}...h_0 \cdot U_{n(j)} \subset U_{n(i)},$$

and

$$h_k h_{k-1}...h_0 \cdot U_{n(j)} \subset U_l;$$

The main issue is completing (f) and (e), for which we fix some onto

$$f : \mathbb{N} \twoheadrightarrow \mathbb{N} \times \mathbb{N} \cup G_0 \times G_0 \times \mathbb{N}$$

which will remind us to eventually serve each requirement of the form $(\mathrm{e})_{b,l}$, $(\mathrm{f})_{g,\hat{g},i}$, or $(\mathrm{g})_{i,l}$. Assume that f is not only onto but the preimage of each point under f is infinite, (so for $(\mathrm{f})_{g,\hat{g},j}$ and $(\mathrm{g})_{j,l}$ we will be prompted at some *later* $i > j$). Given that we have constructed $a_i, n(i), m(i)$, we use (2) to choose some $a \in \mathbb{N}$ and $\hat{n}, \hat{m} \in \mathbb{N}$ with with $d(U_{\hat{n}}) < 2^{-i-1}$, $\overline{U_{\hat{n}}} \subset U_{n(i)}$,

$$\mathcal{M} \models aEa_i, R_{\hat{n},\hat{m}}(a);$$

if $f(i) = (2b, l)$ some $b \in \mathbb{N}$ then we find $a_{i+1}Ea$ and $g \in G_0$ with

$$\mathcal{M} \models F_g(a_{i+1})Eb$$

by (5); if $f(i) = (g, \hat{g}, j)$, $j < i$, then we can find $a_{i+1}Ea$ suitable for $(\mathrm{f})_{g,\hat{g},i}$ by (3); if $f(i) = (2j+1, l)$, $j < i$ then we find the required a_{i+1} that fulfills $(\mathrm{g})_{i,l}$ using (4) and (2).

Supposing now that all this is put into place, then let $x \in X$ be defined by

$$\{x\} = \bigcap_{\mathbb{N}} U_{n(i)};$$

by (d) this x exists. Note that for each l we have $x \in U_l$ if and only if there is some i with

$$\overline{U_{n(i)}} \subset U_l.$$

Note also by (5) that for each $i < j$ we have

$$\mathcal{M} \models a_j E a_i.$$

Claim(III): For each $i, \hat{n}, \hat{m}, \hat{l} \in \mathbb{N}$, $g \in G_0$ with $g \cdot U_{n(i)} = U_{\hat{n}}$ and $gV_{m(i)}g^{-1} = V_{\hat{m}}$

$$\hat{l} \in \varphi_0(g \cdot x, U_{\hat{n}}, V_{\hat{m}}) \Leftrightarrow \mathcal{M} \models \mathcal{O}_{\hat{l}}(F_g(a_i)).$$

Proof of Claim: First note that for the case $g = 1_G$, $l \in \varphi_0(x, U_{n(i)}, V_{m(i)})$ if and only if

$$\mathcal{M} \models \mathcal{O}_l(a_i),$$

since the 'if' direction holds by $(\mathrm{g})_{i,l}$ and the 'only if' direction follows from (4). Now the claim follows by (6). (Claim(III)□)

Now by (4) again it follows that if $b \in \mathbb{N}$, $n, m, l, i \in \mathbb{N}$,

$$\mathcal{M} \models a_i E b \wedge R_{n,m}(b)$$

then

$$l \in \varphi_0(x, U_n, V_m) \Leftrightarrow \mathcal{M} \models \mathcal{O}_l(b);$$

then more generally by (6) again we have that for all if $b \in \mathbb{N}$, $n, m, l, i \in \mathbb{N}$, $g \in G_0$ such that

$$\mathcal{M} \models F_g(a_i)Eb, R_{n,m}(b)$$

$$l \in \varphi_0(g \cdot x, U_n, V_m) \Leftrightarrow \mathcal{M} \models \mathcal{O}_l(b).$$

Thus we obtain a map π that respects all the $\mathcal{O}_l$ predicates if we go right ahead and define

$$\pi : \mathcal{M}(x) \to \mathcal{M}$$

by the formulation that

$$\pi(\langle [g]^x_{n,m}, n, m \rangle) = b$$

if and only if for some $i \in \mathbb{N}$

$$\mathcal{M} \models R_{n,m}(b) \wedge F_g(a_i)Eb.$$

From (1) and (3) this is well defined and *everywhere defined.* It is one to one by (f) and onto by (e) and (1). From (3) and (6) we have that π respects the functions of the form F_g, in the sense that

$$(F_g)^{\mathcal{M}}(\pi(d)) = \pi((F_g)^{\mathcal{M}(x)}(d))$$

for all $d \in \mathcal{M}(x)$. Finally for the E relation we use (7). □

DEFINITION 6.31. For X, G and so on as above, and $x \in X$, let $X(\varphi_x)$ be the set

$$\{y \in X : \varphi_y = \varphi_x\}$$

which equals

$$\{y \in X : \varphi_{\gamma(x)+2}(y, X, G) = \varphi_x\}$$

by 6.27. Let $\tau(\varphi_x)$ be the topology on $X(\varphi_x)$ with basic open sets of the form

$$U(\varphi_{\gamma(x)}(\hat{x}, n, m), n, m) =$$

$$\{y \in X(\varphi_x) \cap U_n : \varphi_{\gamma(x)}(y, U_n, V_m) = \varphi_{\gamma(x)}(\hat{x}, U_n, V_m)\}$$

for $\hat{x} \in X$.

This *is* a basis by 6.29.

LEMMA 6.32. *For $x \in X$ as above, $(X(\varphi_x), \tau(\varphi_x))$ is a Polish space.*

PROOF. We produce a sequence of Polish topologies on X,

$$(\tau_{i,j,\alpha})_{i \in \mathbb{N}, j \in \{0,1,2\}, \alpha \leq \gamma(x)+2},$$

so that for $i \in \mathbb{N}, j \in \{0, 1, 2\}, \alpha < \beta$ we have $\tau_{i,j,\alpha} \subset \tau_{i',j',\alpha'}$, whenever $(i, j, \alpha) < (i', j', \alpha')$ in the reverse lexicographic ordering (which is to say, $\alpha < \alpha'$, or $\alpha = \alpha', j < j'$, or $\alpha = \alpha', j = j', i < i'$). $\tau_{i,j,\alpha}$ restricted to $X(\varphi_x)$ will be included in $\tau(\varphi_x)$. At the very last stage we will have that $X(\varphi_x)$ is an open subset of $(X, \tau_{0,2,\gamma(x)+2})$.

(0) $\tau_{0,0,0}$ is the original topology on X.

(1) Each $\tau_{i+1,j,\alpha}$ will have a basis of the form

$$\mathcal{B}_{i+1,j,\alpha} = \{C \cap W \cap O : O \in \mathcal{B}_{i,j,\alpha}\} \bigcup \mathcal{B}_{i,j,\alpha}$$

where $\mathcal{B}_{i,j,\alpha}$ is a basis for $\tau_{i,j,\alpha}$ and C is some $\tau_{i,j,\alpha}$-closed set and W is some $\tau_{i,j,\alpha}$-open set. From this it follows inductively that if $(X, \tau_{i,j,\alpha})$ is Polish then by 2.3 so too $(X, \tau_{i+1,j,\alpha})$.

(2) At the limit steps of this construction, we let $\tau_{0,0,\alpha}$ be the topology generated by the union of $\{\tau_{i,j',\beta} : (i, j) \in \mathbb{N} \times \{0, 1, 2\}, \beta < \alpha\}$, $\tau_{0,1,\alpha}$ the topology generated by $\{\tau_{i,0,\alpha} : i \in \mathbb{N}\}$, $\tau_{0,2,\alpha}$ the topology generated by $\{\tau_{i,1,\alpha} : i \in \mathbb{N}\}$, all of which will be Polish again by 2.3 and have as a basis $\bigcup\{\mathcal{B}_{i,j,\beta} : i \in \mathbb{N}, j \in \{0, 1, 2\}, \beta < \alpha\}$, $\bigcup\{\mathcal{B}_{i,0,\alpha} : i \in \mathbb{N}\}$, $\bigcup\{\mathcal{B}_{i,1,\alpha} : i \in \mathbb{N}\}$ respectively.

(3) Finally we will have that for each $n, m, \hat{x} \in [x]_G \cap U_n$ that

$$\{y \in X : \varphi_\alpha(y, U_n, V_m) = \varphi_\alpha(\hat{x}, U_n, V_m)\} \cup (X \setminus U_n)$$

is closed in $\tau_{0,1,\alpha}$,

$$\{y \in U_n : \varphi_\alpha(y, U_n, V_m) = \varphi_\alpha(\hat{x}, U_n, V_m)\}$$

open in $\tau_{0,2,\alpha}$, and that

$$\{y \in U_n : \varphi_\alpha(y, U_n, V_m) \notin \{\varphi_\alpha(g_i \cdot x, U_n, V_m) : g_j \cdot x \in U_n\}\}$$

is open in $\tau_{0,0,\alpha+1}$.

Granted that this may be done then we are certainly finished, since

$$\{y \in X : \varphi_{\gamma(x)+2}(y, X, G) = \varphi_{\gamma(x)+2}(x, X, G)\}$$

will be $\tau_{0,2,\gamma(x)+2}$-open by (3).

The main part of this construction then is clearly in defining the closed sets to be added at each stage. The precise enumeration is given in the next paragraphs, but the idea is that $\mathcal{B}_{0,1,0}$ should have every (formerly closed intersection open) set of the form

$$\{y \in g_j^{-1} \cdot U_n : l \notin \varphi_0(g_j \cdot y, U_n, V_m)\},$$

$\mathcal{B}_{0,2,0}$ should have all sets of the form

$$\{y \in g_j^{-1} \cdot U_n : \varphi_0(g_j \cdot y, U_n, V_m) = \varphi_0(g_i \cdot x, U_n, V_m)\},$$

and $\mathcal{B}_{0,0,1}$ should have all sets of the form

$$\{y \in g_j^{-1} \cdot U_n : \varphi_0(g_j \cdot y, U_n, V_m) \notin \{\varphi_0(g_i \cdot x, U_n, V_m) : g_i \cdot x \in U_n\}\}.$$

Similarly $\mathcal{B}_{0,1,\alpha}$ will have every set of the form

$$\langle \varphi_\beta(g_i \cdot x, U_{\hat{n}}, V_{\hat{m}}), \hat{n}, \hat{m}\rangle \notin \varphi_\alpha(g_j \cdot y, U_n, V_m)$$

for $\beta < \alpha$. $\mathcal{B}_{0,2,\alpha}$ will have every set of the form

$$\varphi_\alpha(g_i \cdot x, U_n, V_m) = \varphi_\alpha(g_j \cdot y, U_n, V_m),$$

and $\mathcal{B}_{0,0,\alpha+1}$ will have every set of the form

$$\varphi_\alpha(g_j \cdot y, U_n, V_m) \notin \{\varphi_\alpha(g_i \cdot x, U_n, V_m) : g_i \cdot x \in U_n\}.$$

Choose bijections $s_4 : \mathbb{N}^4 \to \mathbb{N}$ and $s_3 : \mathbb{N}^3 \to \mathbb{N}$. Begin by defining

$$C_{s_4(l,n,m,j),0} = \{y \in U_n : l \notin \varphi_0(g_j \cdot y, U_n, V_m)\} \bigcup X \setminus U_n$$

and

$$W_{s_4(l,n,m,j),0} = U_n.$$

So each $C_{i,0}$ is closed, each $W_{i,0}$ is open, and we may let $\tau_{0,0,0}$ be the original topology on X, $\mathcal{B}_{0,0,0} = \{U_l : l \in \mathbb{N}\}$, and at each i

$$\mathcal{B}_{i+1,0,0} = \{U \cap U_i \cap C_i : U \in \mathcal{B}_{i,0,0}\} \cup \mathcal{B}_{i,0,0}$$

and take $\tau_{i+1,0,0}$ to be the topology generated by $\mathcal{B}_{i+1,0}$, and when we move on to $\tau_{0,1,0} = \bigcup_{i \in \mathbb{N}} \tau_{i,0,0}$ we have in particular that each

$$\{y \in g_j^{-1} \cdot U_n : \varphi_0(g_j \cdot y, U_n, V_m) = \varphi_0(g_i \cdot x, U_n, V_m)\} \cup (X \setminus U_n)$$

is closed for $i, j \in \mathbb{N}, g_i \cdot x \in [x]_G \cap U_n$.

By the same device define $\tau_{s_4(i,n,m,j)+1,1,0}$ so that

$$\{y \in g_j^{-1} \cdot U_n : \varphi_0(g_{\cdot} y, U_n, V_m) = \varphi_0(g_i \cdot x, U_n, V_m)\}$$

becomes open, since each of these sets were previously relatively closed in $g_j^{-1} \cdot U_n$ in $\tau_{0,1,0}$. We let $\tau_{0,2,0}$ be the union of the topologies of the form $\tau_{i,1,0}$.

Then we define $\tau_{s_3(n,m,j)+1,2,0}$ so that the set

$$\{y \in g_j^{-1} \cdot U_n : \varphi_0(g_j \cdot y, U_n, V_m) \notin \{\varphi_0(g_i \cdot x, U_n, V_m) : g_i \cdot x \in U_n\}\}$$

now becomes open. We let $\tau_{0,0,1}$ be the union of the topologies $\tau_{i,2,0}$.

So we can simply continue ad infinitum, steadily adding in for each α and i the set

$$\{y \in g_j^{-1} \cdot U_n : \varphi_\alpha(g_j \cdot y, U_n, V_m) = \varphi_\alpha(g_i \cdot x, U_n, V_m)\}$$

first as a relatively closed set, and then as an open set, and then adding in each set of the form

$$\{y \in g_j^{-1} \cdot U_n : \varphi_{\alpha+1}(g_j \cdot y, U_n, V_m) \subset \varphi_{\alpha+1}(x, X, G)\}$$

as a relatively closed subset of $g_j^{-1} \cdot U_n$, and then as an open set in the space. Then we obtain sets of the form

$$\{y \in g_j^{-1} \cdot U_n : \varphi_{\alpha+1}(g_j \cdot y, U_n, V_m) = \varphi_{\alpha+1}(g_i \cdot x, U_n, V_m)\}$$

as relatively closed subsets of the $g_j^{-1}U_n$, which we can then proceed to make open.

Note that this never causes us to exceed the topology $\tau(\varphi_x)$ on $X(\varphi_x)$, since for $\alpha > \gamma(x)$

$$\{y \in U_n \cap X(\varphi_x) : \varphi_\alpha(y, U_n, V_m) = \varphi_\alpha(g_i \cdot x, U_n, V_m)\}$$

equals

$$\{y \in U_n \cap X(\varphi_x) : \varphi_{\gamma(x)}(y, U_n, V_m) = \varphi_{\gamma(x)}(g_i \cdot x, U_n, V_m)\}.$$

□

In the induced action of G on $X(\varphi_x)$ obtained by restricting the original action on X:

LEMMA 6.33. *$(X(\varphi_x), \tau(\varphi_x))$ is a Polish G-space.*

PROOF. By consideration of the basis. If $y_0 \in X(\varphi_x)$ and $g_0 \in G$, $g_0 \cdot y_0 \in W$, for W some $\tau(\varphi_x)$-open set, then for the purposes of checking that the action is continuous we may assume that

$$W = \{z \in X(\varphi_x) : \varphi_\alpha(z, U_n, V_m) = \varphi_\alpha(g_0 \cdot y_0, U_n, V_m)\},$$

some U_n containing $g_0 \cdot y_0$ and some $m \in \mathbb{N}$. We can therefore find a small open set U containing y_0, V containing g_0 with $V \cdot U \subset U_n$; and thus for any $y \in g_0^{-1} \cdot U_n$ with

$$\varphi_\alpha(y, g_0^{-1} \cdot U_n, g_0^{-1}V_m g_0) = \varphi_\alpha(y_0, g_0^{-1} \cdot U_n, g_0^{-1}V_m g_0)$$

and for any $g \in V \cap V_m g_0$, we have by 6.28

$$\varphi_\alpha(g \cdot y, U_n, V_m) = \varphi_\alpha(g_0 \cdot y_0, U_n, V_m).$$

By 6.29, the set of $y \in X(\varphi_x)$ with

$$\varphi_\alpha(y, g_0^{-1} \cdot U_n, g_0^{-1}V_m g_0) = \varphi_\alpha(y_0, g_0^{-1} \cdot U_n, g_0^{-1}V_m g_0)$$

is open. □

LEMMA 6.34. *Every orbit in $(X(\varphi_x), \tau(\varphi_x))$ is dense.*

PROOF. Since $X(\varphi_x)$ is essentially defined to be the collection of elements of X that meet exactly the needed open sets. □

LEMMA 6.35. *If every orbit in $(X(\varphi_x), \tau(\varphi_x))$ is meager then the action of G on this space is turbulent.*

PROOF. Since we then have that each $y \in X(\varphi_x) \cap U_n$ will have some $\hat{x} \in [x]_G$ with

$$\varphi_{\gamma(x)}(y, U_n, V_m) = \varphi_{\gamma(x)}(\hat{x}, U_n, V_m).$$

Then

$$\mathcal{O}(y, U(\varphi_{\gamma(x)}(\hat{x}, n, m)), V_m)(= \mathcal{O}(y, U_n, V_m))$$

is dense in $U(\varphi_{\gamma(x)}(\hat{x}, n, m))$. □

DEFINITION 6.36. A Polish group H is a *GE group* if whenever Y is a Polish H-space and $y \in Y$ has $[y]_H$ a G_δ, then for all $z \in Y$

$$\overline{[y]_H} = \overline{[z]_H}$$

implies $zE_H y$.

In other words, a dense G_δ orbit under a GE group takes over the space, allowing no other orbits to share the precise same closure.

The class of GE groups is surprisingly broad. It includes locally compact Polish groups, since in this case every orbit must be F_σ. Less obviously it includes abelian and even nilpotent, as well as invariantly metrizable and $\aleph_0$ products of locally compact.

The following lemma seems to be have been discovered by various people, including David Marker and Ramez Sami. Although I cannot be sure as to its first author, something similar is already implicit in [**14**].

LEMMA 6.37. *Let H be a Polish group and Y a Polish space. If $y \in Y$ with $[y]_H$ non-meager, then $[y]_H$ is G_δ.*

PROOF. After possible pruning down to the space

$$\overline{[y]_H}$$

we may assume that the orbit of y is dense. Since $[y]_H$ is certainly no worse than $\underset{\sim}{\Sigma}^1_1$, it has the Baire property. Thus from it being non-meager, we may find an open set $U \subset Y$, $U \neq \emptyset$, so that $[y]_H$ is comeager in U, and then by the density of $[y]_H$ it is comeager in Y. So fix A a dense G_δ, $A \subset [y]_H$. Then by 2.47 $A^{*H} =_{df} \{y \in Y : \forall^* h \in H(h \cdot x \in A)\}$ is a comeager set, and therefore in particular non-empty.

Note that $A^{*H} \subset [y]_H$, and so by invariance of these two sets and the fact that H acts transitively on $[y]_H$, these are equal. By 6.9 A^{*H} is a G_δ. $\square$

DEFINITION 6.38. For E and F equivalence relations on Polish Y_0 and Y_1, write

$$E \leq_{FB} F$$

to indicate that there is a *faithful* Borel reduction of E to F, so that for some Borel $\theta : Y_0 \to Y_1$ witnessing $E \leq_B F$ we have that for each invariant Borel set $A \subset Y_0$ the saturation of its image, $[\theta[A]]_F$, is Borel.

Gao in [**23**] produces orbit equivalence relations E_{H_1}, E_{H_2} with $E_{H_1} \leq_B E_{H_2}$ but *not* $E_{H_1} \leq_{FB} E_{H_2}$. The example is non-trivial.

DEFINITION 6.39. Let H be a Polish group, Y_0, Y_1 Polish H-spaces. A function $\pi : Y_0 \to Y_1$ is an *H-embedding* if it is one to one and for all $y \in Y_0, h \in H$

$$\pi(g \cdot y) = g \cdot \pi(y).$$

The existence of a Borel H-embedding as above implies $E_H^{Y_0} \leq_B E_H^{Y_1}$.

THEOREM 6.40. *Let G be a GE group and X a Polish G-space. Then the following are equivalent:*

(I) E_G^X admits classification by countable models;

(II) there is a countable language $\mathcal{L}$ with $E_G^X \leq_{FB} \cong|_{\mathrm{Mod}(\mathcal{L})}$;

(III) $x \mapsto \varphi_x$ provides a complete invariant for the orbit equivalence relation;

(IV) there is no turbulent Polish G-space Y and continuous G-embedding from Y to X.

PROOF. (II) implies (I) at once from the definitions, and (III) implies (II) by 6.30. (I) implies (IV) since 3.19 states that Y as in (IV) cannot admit classification by countable models. We are still to see (IV) implying (III) – and this is where the assumption on G is played.

Supposing that $x \mapsto \varphi_x$ is not a complete invariant, find orbit inequivalent $x, y \in X$ such that $\varphi_x = \varphi_y$. By 6.35 we only need show every orbit in $X(\varphi_x)$ is meager. But this follows, since if $z \in X(\varphi_x)$ is G_δ then it is already dense by 6.34, so

$$xE_G zE_G y$$

by G being GE. □

EXERCISE 6.41. Let $G = c_0$ and $X = \mathbb{R}^{\mathbb{N}}$ with the action by translation. Show that for all x

$$\gamma(x) = 0.$$

EXERCISE 6.42. Let $G = S_\infty$ and $V_n = \{g \in S_\infty : \forall i < n(g(i) = i)\}$ as at 6.1. Let X be a Polish S_∞-space.

Provide the following partial reconciliation of the notation from §6.1 and §6.2: If $x, y \in U \subset X$ with $V_n \cdot x, V_n \cdot y \subset U$ then for all $\alpha < \omega_1$

$$\varphi_\alpha(x, V_n) = \varphi_\alpha(y, V_n)$$

implies

$$\varphi_\alpha(x, U, V_n) = \varphi_\alpha(y, U, V_n).$$

6.3. A counterexample

Although it is open whether (I), (II), and (IV) of 6.40 imply one another for arbitrary Polish groups, in general they are *not* equivalent to (III).

DEFINITION 6.43. Let $M_\infty(\mathbb{R})$ be the group of measure preserving transformations of $(\mathbb{R}, \lambda)$, with transformations agreeing a.e. identified, λ being Lebesgue measure.

$M_\infty(\mathbb{R})$ is Polish. For instance it can be viewed as a closed subgroup of the unitary group on $L^2(\mathbb{R}, \lambda)$. More directly if we fix $\{O_i : i \in \mathbb{N}\}$ a basis of $\mathbb{R}$ with each $\lambda(O_i) \leq 1$ then as in 2.11 a complete metric is obtained by letting $d_{\mathbb{R},\lambda}(\pi_0, \pi_1)$ equal

$$\sum 2^{-i}[\lambda(\pi_0(O_i)\Delta\pi_1(O_i)) + \lambda(\pi_0^{-1}(O_i)\Delta\pi_1^{-1}(O_i))].$$

DEFINITION 6.44. Let $X(\lambda, 1)$ equal the set of

$$(f_n)_{n\in\mathbb{N}} = \vec{f} \in (L^2(\mathbb{R}, \lambda))^{\mathbb{N}},$$

each $f_n : \mathbb{R} \to \mathbb{R}$ square integrable, such that:

(1) for each $n \in \mathbb{N}$,

$$\int_{\mathbb{R}} |f_n| \lambda(x) = 1;$$

(2) for $n \neq m \in \mathbb{N}$

$$\int_{\mathbb{R}} |f_n \cdot f_m| \lambda(x) = 0;$$

(3) for each n and a.e. $x \in \mathbb{R}$

$$f_n(x) \in \{0,1\}.$$

$L^2(\mathbb{R},\lambda)$ is a Polish space with the metric

$$d(f,f') = (\int (f-f')^2 \lambda(x))^{1/2}$$

and the understanding that functions agreeing a.e. are identified. $X(\lambda,1)$ is a closed subspace of $(L^2(\mathbb{R},\lambda))^{\mathbb{N}}$, and hence Polish. There is a natural action of $M_\infty(\mathbb{R})$ on $L^2(\mathbb{R},\lambda)$: for $\pi \in M_\infty(\mathbb{R})$, $f \in L^2(\mathbb{R},\lambda)$, $x \in \mathbb{R}$

$$(\pi \cdot f)(x) = f(\pi^{-1}(x)).$$

This action is continuous and by taking its infinite product we obtain that $(L^2(\mathbb{R},\lambda))^{\mathbb{N}}$ is a Polish $M_\infty(\mathbb{R})$-space with the action

$$(\pi \cdot \vec{f})_n(x) = f_n(\pi^{-1}(x)).$$

$X(\lambda,1)$ is a closed invariant subspace of $(L^2(\mathbb{R},\lambda))^{\mathbb{N}}$, and hence a Polish $M_\infty(\mathbb{R})$ space.

Lemma 6.45. *The orbit equivalence induced by the action of $M_\infty(\mathbb{R})$ on $X(\lambda,1)$ is smooth: $E^{X(\lambda,1)}_{M_\infty(\mathbb{R})} \leq_B \mathrm{id}(\mathbb{R})$.*

Proof. A complete invariant for any $\vec{f} \in X(\lambda,1)$ is given by the measure of $\{x \in \mathbb{R} : \forall n(f_n(x) = 0)\}$. Thus

$$\theta : X(\lambda,1) \to \mathbb{R}$$
$$\vec{f} \mapsto 2^{-\lambda(\{x\in\mathbb{R}:\forall n(f_n(x)=0)\})}$$

provides the needed reduction. □

Lemma 6.46. *For all $\vec{f} \in X(\lambda,1)$ and*

$$U \subset X(\lambda,1)$$

open containing $\vec{f}$,

$$V \subset M_\infty(\mathbb{R})$$

a neighbourhood of the identity, there is open $\hat{U} \subset U$ containing $\vec{f}$ such that for all $\vec{h} \in \hat{U}$ the set

$$\mathcal{O}(\vec{h},\hat{U},V)$$

is dense in $\hat{U}$.

Proof. Fix $\vec{f}, U, V$ as indicated, and choose $\delta > 0$ small enough that for some $N \in \mathbb{N}$

$$\hat{U} =_{df} \{\vec{g} \in X(\lambda,1) : \forall n < N(d(g_n,f_n) < \delta)\} \subset U,$$
$$B_\delta(1) \subset V,$$

where $B_\delta(1)$ is the open $d_{\mathbb{R},\lambda}$ ball of radius δ about the identity of the group. Supposing $\vec{g},\vec{h} \in \hat{U}$ and $N \leq M \in \mathbb{N}$ it suffices to find $\pi \cdot \vec{h} \in \mathcal{O}(\vec{f},\hat{U},V)$ such that $(\pi \cdot \vec{h})_n = g_n$ for all $n \leq M$.

For $\vec{k} \in X(\lambda,1)$ and $n \in \mathbb{N}$ let

$$A(n,\vec{k}) = \{x \in \mathbb{R} : k_n(x) = 1\}.$$

Choose $\pi_M \in M_\infty(\mathbb{R})$ such that

$$\pi_M^2 = \mathrm{id},$$

$$\pi_M[A(M,\vec{h}) \setminus A(M,\vec{g})] = A(M,\vec{g}) \setminus A(M,\vec{h}),$$

and for all $x \notin A(M,\vec{h})\Delta(M,\vec{g})$

$$\pi_M(x) = x.$$

Thus in particular we have for all $k \in \mathbb{N}$ and $x \in A(k,\vec{h}) \cap A(k,\vec{g})$

$$\pi_M(x) = x.$$

This can be repeated for $\vec{h}^1 = \pi_M \cdot f$, and we may successively obtain

$$\vec{h}^2, \vec{h}^3, ..., \vec{h}^{M+1}$$

and

$$\pi_{M-1}, \pi_{M-2}, ..., \pi_0$$

such that at each $n \leq M$

$$\vec{h}^{n+1} = \pi_{M-n} \cdot \vec{h}^n,$$

$$\pi_{M-n}^2 = \mathrm{id},$$

$$\pi_{M-n}[A(M-n,\vec{h}^n) \setminus A(M-n,\vec{g})] = A(M-n,\vec{g}) \setminus A(M-n,\vec{h}^n),$$

and for all $x \notin A(M-n,\vec{h}^n)\Delta(M-n,\vec{g})$

$$\pi_{M-n}(x) = x.$$

Thus at each $n \leq M$ and $k \in \mathbb{N}$ we have

$$A(k,\vec{h}^n) \cap A(k,\vec{g}) \supset A(k,\vec{g}) \cap A(k,\vec{h}),$$

and so each $\vec{h}^n$ is in $\hat{U}$.

For each n we may partition $A(M-n,\vec{h}^n) \setminus A(M-n,\vec{g})$ into sets

$$B_{n,0}, B_{n,1}, ..., B_{n,k(n)}$$

with each $\lambda(B_{n,i}) < \delta^2/2$, and then we may define $\pi_{i,M-n}$ by

$$\pi_{i,M-n}|_{B_{n,i}} = \pi_{M-n}|_{B_{n,i}},$$

$$\pi_{i,M-n}|_{\pi[B_{n,i}]} = \pi_{M-n}|_{\pi_i[B_{n,i}]},$$

$$\pi_{i,M-n}|_{\mathbb{R}\setminus(B_{n,i}\cup\pi[B_{n,i}])} = \mathrm{id}|_{\mathbb{R}\setminus(B_{n,i}\cup\pi[B_{n,i}])}.$$

Each such

$$\pi_{i,M-n} \in B_\delta(1) \subset V.$$

Again for each k and n we have

$$\pi_{i,M-n}|_{A(k,\vec{h}^n)\cap A(k,\vec{g})} = \mathrm{id}|_{A(k,\vec{h}^n)\cap A(k,\vec{g})}.$$

Thus if let $\vec{h}^{n,0} = \vec{h}^n$ and each $\vec{h}^{n,i+1} = \pi_{i,M-n} \cdot \vec{h}^{n,i}$ then we obtain each $\vec{h}^{n,i} \in \mathcal{O}(\vec{h},\hat{U},V)$ and we finish. □

This suffices by the following completely general observation.

LEMMA 6.47. *Let G be a Polish group and X a Polish G-space in which every orbit is dense. Suppose as usual we fix some basis $(U_l)_{l\in\mathbb{N}}$ for X and a neighborhood basis $(V_n)_{n\in\mathbb{N}}$ around the identity in G. Suppose that for all $n, m \in \mathbb{N}$ and $x \in U_n$ there is some $\hat{U} \subset U_n$ open containing x with every $y \in \hat{U}$ having $\mathcal{O}(y, \hat{U}, V_m)$ dense in $\hat{U}$.*

Then for all $n \in \mathbb{N}$, $x, y \in U_n$, and $\alpha < \omega_1$

$$\varphi_\alpha(x, U_n, V_m) = \varphi_\alpha(y, U_n, V_m).$$

PROOF. For each $m \in \mathbb{N}$ choose a basis $\{W_n^m : n \in \mathbb{N}\}$ for X such that for each $y \in W_n^m$ the local orbit $\mathcal{O}(y, W_n^m, V_m)$ is dense in W_n^m. By 6.29 it suffices to show that for all $\alpha < \omega_1$, $n, m \in \mathbb{N}$, $x, y \in W_n^m$

$$\varphi_\alpha(x, W_n^m, V_m) = \varphi_\alpha(y, W_n^m, V_m).$$

At once by the definition of the basis we have that for all $n, m \in \mathbb{N}$, $x, y \in W_n^m$

$$\varphi_0(x, W_n^m, V_m) = \varphi_0(y, W_n^m, V_m).$$

Now the general case follows by transfinite induction. □

Thus by 6.27, 6.47, 6.46, for all $\vec{f}, \vec{h} \in X(\lambda, 1)$

$$\varphi_{\vec{f}} = \varphi_{\vec{h}}.$$

6.4. A different direction

Instead of measuring the extent to which the complete invariants for S_∞ work for arbitrary Polish groups, we can also try to craft complete invariants for the general context. Here are just a few steps in this direction.

Ultimately we would want to show that the $\approx_\gamma$ equivalence classes defined at 6.53 correspond to something comprehensible, such as in the case of S_∞ when they can be thought of as hereditarily countable sets. At the very minimum one would hope to show that the $\approx_\gamma$ equivalence relation is Borel reducible to an orbit equivalence relation arising from the group in question.

The approach here is abstract. It completes the task at 6.55, and obtains from a bound in the complexity of the individual orbits in X a bound in the Borel complexity of E_G^X, but it gives no further insight.

DEFINITION 6.48. Let G be a Polish group and X a Polish G-space. For $V_0, V_1 \subset G$ open and non-empty, $x_0, x_1 \in X$, write

$$(x_0, V_0) \leq_1 (x_1, V_1)$$

if

$$\overline{V_0 \cdot x_0} \subset \overline{V_1 \cdot x_1}.$$

Write

$$(x_0, V_0) \leq_{\alpha+1} (x_1, V_1)$$

if for all $W_0 \subset V_0$ open non-empty there is $W_1 \subset V_1$ non-empty and open with

$$(x_1, W_1) \leq_\alpha (x_0, W_0).$$

For λ a limit

$$(x_0, V_0) \leq_\lambda (x_1, V_1)$$

expresses that at every $\alpha < \lambda$

$$(x_0, V_0) \leq_\alpha (x_1, V_1).$$

LEMMA 6.49. *For $W_0 \subset V_0$, $W_1 \supset V_1$, all open and non-empty,*

$$(x_0, V_0) \leq_\alpha (x_1, V_1)$$

implies

$$(x_0, W_0) \leq_\alpha (x_1, W_1).$$

PROOF. Immediate from the definitions. $\square$

From now on let $\mathcal{B}$ be a countable basis for G and let $\mathcal{N}$ be the non-empty elements of $\mathcal{B}$. Assume $G \in \mathcal{B}$.

LEMMA 6.50.

$$(x_0, V_0) \leq_{\alpha+1} (x_1, V_1)$$

if and only if for all $W_0 \subset V_0$ in $\mathcal{N}$ there is $W_1 \subset V_0$ in $\mathcal{N}$ with

$$(x_1, W_1) \leq_\alpha (x_0, W_0).$$

PROOF. From 6.49. $\square$

COROLLARY 6.51. *Let $(\hat{V}_n)_{n\in\mathbb{N}}$ enumerate $\mathcal{N}$. Then for all $\alpha < \omega_1$*

$$\{(x_0, x_1, n, m) : (x_1, \hat{V}_n) \leq_\alpha (x_0, \hat{V}_m)\}$$

is a $\underset{\sim}{\Pi}^0_{\alpha+k}$ set for some $k \in \mathbb{N}$.

LEMMA 6.52. *(a) Each $\leq_\alpha$ is transitive.*

(b) $\alpha < \beta$ implies $\leq_\alpha \supset \leq_\beta$ (in the sense that $(x_1, W_1) \leq_\beta (x_0, W_0)$ implies $(x_1, W_1) \leq_\alpha (x_0, W_0)$).

PROOF. (a) is almost immediate from the definitions. (b) routinely following by transfinite induction once we observe:

Claim: $\leq_1 \supset \leq_2$.

Proof of claim: Suppose that $(x_1, W_1) \leq_2 (x_0, W_0)$. Choose $U_0 \subset X$ open with $W_1 \cdot x_1 \cap U_0 \neq \emptyset$. Then we may choose some $x_1' \in W_1 \cdot x_1 \cap U_0$ and U_1 an open neighbourhood of x_1' with $\overline{U_1} \subset U_0$. We then further choose some non-empty basic open $W_1' \subset W_1$ with $W_1' \cdot x_1 \subset U_1$. By assumption we find $W_0' \subset W_0$ with $(W_0', x_0) \leq_1 (W_1', x_1)$, and hence $W_0' \cdot x_0 \subset \overline{U_1} \subset U_0$. (Claim$\square$)

$\square$

DEFINITION 6.53. Set $x_0 \approx_\gamma x_1$ if for all $i \in \{0, 1\}$, $\beta < \gamma$, $W_i \in \mathcal{N}$, there exists $\hat{W}_i \subset W_i$ with $\hat{W}_i \in \mathcal{N}$ and W_{1-i} with

$$(x_i, \hat{W}_i) \leq_\beta (x_{1-i}, W_{1-i}) \leq_\beta (x_i, W_i).$$

Note that $x_0 E^X_G x_1$ implies $x_0 \approx_\gamma x_1$.

LEMMA 6.54. *Let $B \subset X$ be $\underset{\sim}{\Pi}^0_\alpha$, α some countable ordinal. Suppose $(x_0, V_0) \leq_\alpha (x_1, V_1)$ and $x_1 \in B^{*V_1}$.*

Then $x_0 \in B^{*V_0}$.

PROOF. This is trivial at the base case of $\alpha = 1$, since it amounts to calculating $\overline{V_0 \cdot x_0}$ and $\overline{V_1 \cdot x_1}$. Now suppose it is true at all $\beta < \alpha$ and that

$$B = \bigcap_{i\in\mathbb{N}} B_i$$

with each $B_i \in \underset{\sim}{\Sigma}^0_{\beta(i)}$ some $\beta(i) < \alpha$. Then assuming

$$(x_0, V_0) \leq_\alpha (x_1, V_1)$$

and $i \in \mathbb{N}$ and $W_0 \subset V_0$ with $W_0 \in \mathcal{N}$. We just need to show that $x_0 \in (B_i)^{\Delta W_0}$.

So choose $W_1 \subset V_1$ with $W_1 \in \mathcal{N}$ and

$$(x_0, W_0) \geq_{\beta(i)} (x_1, W_1).$$

Now if x_0 is *not* in $(B_i)^{\Delta W_0}$, then for $C = X \setminus B_i$ we have

$$x_0 \in C^{*W_0}$$

and so by assumption on $\beta(i) < \alpha$ we have

$$x_1 \in C^{*W_1},$$

with a contradiction to $x_1 \in (B_i)^{\Delta W_1}$. $\square$

COROLLARY 6.55. *If every orbit is* $\underset{\sim}{\Pi}^0_\alpha$ *then* E^X_G *is* $\underset{\sim}{\Pi}^0_{\alpha+\omega}$.

PROOF. $x_0 E^X_G$ implies $x_0 \approx_{\alpha+\omega} x_1$. Conversely, $x_0 \approx_{\alpha+\omega} x_1$ implies

$$x_0 \in ([x_1]_G)^{*G}$$

by 6.54 and the assumption on the Borel complexity of $[x_1]_G$, and hence it implies $x_0 E^X_G x_1$. Thus $x_0 E^X_G$ if and only if $x_0 \approx_{\alpha+\omega} x_1$, and so the corollary follows 6.51. $\square$

In particular, if the orbits are bounded in Borel complexity, then the resulting orbit equivalence relation is Borel, a result obtained previously by Sami in [**74**] using very different methods.

CHAPTER 7

GE groups

The GE groups where defined in §6.2 to be the Polish groups such that any Polish G-space with a dense G_δ orbit can have only a single orbit. They were seen to submit to an especially nice analysis of their turbulent versus classifiable actions.

The central results from this chapter in §7.2 and §7.4 are to the effect that there are many GE groups. They include abelian, even nilpotent, invariantly metrizable, and countable products of locally compact. §7.4 further proves the curious fact that the orbit equivalence relations arising from countable products of locally compact groups all admit classification by countable models. In particular locally compact Polish groups never give rise to turbulent orbit equivalence relations.

§7.3 develops some technology on changing the topologies in a Polish G-space, an idea previewed by the proofs of 6.11 and 6.32. This technique will be used in chapter 9, where a method of transfinite changes in topologies is used to give an analysis of turbulence. The technique is also used in an aside on the Becker dichotomy theorem at §7.5, to the effect that any Polish group admitting a complete left invariant metric group has all its non-smooth orbit equivalence relations $\sqsubseteq_c$ above E_0.

7.1. More on Polish groups; Glimm-Effros; G_δ orbits

The results from this section are all well known and mostly classical.

Definition 7.1. Let G be a group. A metric d is said to be *left invariant* (or *right invariant*) if for all $g, h_0, h_1 \in G$

$$d(h_0, h_1) = d(gh_0, gh_1)$$

(respectively $d(h_0, h_1) = d(h_0 g, h_1 g)$). A metric is said to be simply *invariant* if it is both left and right invariant.

Theorem 7.2. *(Birkhoff; Kakutani) Let G be a Polish group. Then G admits a left invariant metric and it admits a right invariant metric.*

Proof. It suffices to find a left invariant metric d, since we then obtain a right invariant metric by $d_r(g, h) = d(g^{-1}, h^{-1})$.

Choose $(V_n)_{n\in\mathbb{N}}$ a neighborhood basis of the identity,

$$V_{n+1}^4 \subset V_n,$$
$$V_n^{-1} = V_n.$$

For $g_1, g_2 \in G$ let

$$\delta(g_1, g_2) = \inf\{2^{-k} : g_1^{-1} g_2 \in V_k\},$$

and let

$$d(g_1, g_2) = \inf\{\sum_{i=0}^{i=k} \delta(h_i, h_{i+1}) : h_0 = g_1, h_{k+1} = g_2\}.$$

Claim: $\forall g_1, g_2 \in G(d(g_1, g_2) \geq \delta(g_1, g_2)/2)$.

Proof of Claim: Suppose instead that there is a counterexample,

$$\delta(g_1, g_2) = 2^{-n},$$

with $(h_j)_{j \leq k+1}$, $h_0 = g_1$, $h_{k+1} = g_2$, and

$$\sum_{j=0}^{j=k} \delta(h_j, h_{j+1}) < 2^{-n-1}.$$

Suppose that k has been chosen as small as possible among counterexamples and over all possible n.

Subclaim: We can find some $0 \leq i(1,0) < i(2,0) \leq k+1$ so that

$$\delta(h_{i(1,0)}, h_{i(2,0)}) = 2^{-n-1}.$$

Proof of subclaim: Since by $(V_{n+2})^2 \subset V_{n+1}$ for *any* $j \leq k+1$ either

$$\delta(h_0, h_j) \geq 2^{-n-1}$$

or

$$\delta(h_j, h_{k+1}) \geq 2^{-n-1}.$$

So we can either take $i(1,0) = 0, i(2,0) = j$, or else $i(1,0) = j, i(2,0) = k+1$. (Subclaim$\square$)

Then for any l with $i(1,0) < l < i(2,0)$ exactly one of $\delta(h_{i(1,0)}, h_l), \delta(h_l, h_{i(2,0)})$ is equal to 2^{-n-1} – at least one else we obtain $\delta(h_{i(1,0)}, h_{i(2,0)}) \leq 2^{-n-1}$ by $(V_{n+2})^4 \subset V_{n+1}$, and at most one since *for all*

$$m_1 < m_2 \leq m_3 < m_4 \leq k+1$$

the minimality of k implies

$$d(h_{m_1}, h_{m_2}) \geq \delta(h_{m_1}, h_{m_2})/2$$

and

$$d(h_{m_3}, h_{m_4}) \geq \delta(h_{m_3}, h_{m_4})/2,$$

and hence by assumption of

$$\sum_{j=0}^{j=k} d(h_j, h_{j+1}) < 2^{-n-1}$$

either

$$\delta(h_{m_1}, h_{m_2}) < 2^{-n-1}$$

or

$$\delta(h_{m_3}, h_{m_4}) < 2^{-n-1}.$$

Repeating this we may successively find $i(1,k)$ and $i(2,k)$ at each k so that $\{i(1,k), i(1,k)+1, ..., i(2,k)\}$ strictly includes $\{i(1,k+1), i(1,k+1)+1, ..., i(2,k+1)\}$ and

$$\delta(h_0, h_{i(1,k)}) < 2^{-n-1},$$

$$\delta(h_{i(1,k)}, h_{i(2,k)}) = 2^{-n-1},$$

$$\delta(h_{i(2,k)}, h_{k+1}) < 2^{-n-1}.$$

At last we arrive at some n with $i(2,k) = i(1,k)+1$, with a contradiction. (Claim $\square$)

By $V_n = V_n^{-1}$ for all $g_1, g_2 \in G$

$$d(g_1, g_2) = d(g_2, g_1).$$

The structure of the definition of d enforces the triangle inequality. By the claim $d(g_1, g_2) = 0$ implies $g_1 = g_2$. So d is a metric.

From the claim and the definition d is compatible with the topology on G. □

THEOREM 7.3. *(folklore) Let G be a Polish group with $\{U_n : n \in \mathbb{N}\}$ a neighborhood basis of the identity consisting of conjugation invariant sets – so that for all $g \in G$ and $n \in \mathbb{N}$*

$$gU_ng^{-1} = U_n.$$

Then G admits an invariant metric.

PROOF. First by replacing U_n by $U_n \cap U_n^{-1}$ we may assume that each $U_n = U_n^{-1}$. Then we may choose $\{V_n : n \in \mathbb{N}\} \subset \{U_n : n \in \mathbb{N}\}$ with V_{n+1} chosen sufficiently small that $(V_{n+1})^4 \subset V_n$.

Now proceeding through the construction of 7.2 we obtain for all $g_1, g_2, h \in G$ that

$$\delta(g_1, g_2) = \inf\{2^{-k} : g_1^{-1}g_2 \in V_k\} = \inf\{2^{-k} : g_1^{-1}g_2 \in hV_kh^{-1}\},$$

since $hV_kh^{-1} = V_k$, and hence in turn equals $\inf\{2^{-k} : h^{-1}g_1^{-1}g_2h \in V_k\} = \delta(g_1h, g_2h)$, so the metric is not just left invariant but also right invariant. □

The conclusion of 7.3 implies the assumption, since if d is an invariant metric, then for each n we may let $U_n = \{g \in G : d(g, 1_G) < 2^{-n}\}$.

LEMMA 7.4. *Any compatible invariant metric on a Polish group is complete.*

PROOF. Let G be the Polish group, d the invariant metric, X the Cauchy completion of (G, d). G is a G_δ subset of X, since if $\hat{d}$ is a complete metric on G, $\{g_i : i \in \mathbb{N}\}$ dense in G, and for each n, i we choose open $\mathcal{O}_{i,n} \subset X$ with

$$\mathcal{O}_{i,n} \cap G = \{g \in G : \hat{d}(g, g_i) < 2^{-n}\}$$

then $G = \{x \in X : \forall n \exists i (x \in \mathcal{O}_{i,n}\}$.

X is a Polish space, since the separability of G imposes separability on X. In fact it may be naturally viewed as a Polish G-space.

For $g_i \to x$ a Cauchy sequence, $x \in X$, then we can define $g \cdot x$ to be the Cauchy limit of $(g \cdot g_i)_{i \in \mathbb{N}}$; this is well defined by left invariance of d. We obtain that the action is continuous – for if $g_i \to x$ and $h_i \to y$ as $i \to \infty$, then for all $g, h \in G$

$$\begin{aligned} d(g \cdot x, h \cdot y) = \lim_{i\to\infty} d(g \cdot g_i, h \cdot h_i) &\leq \lim_{i\to\infty} d(g_i, g^{-1}h \cdot h_i) \\ &\leq \lim_{i\to\infty} d(g_i, h_i) + d(h_i, g^{-1}h \cdot h_i) \end{aligned}$$

which by right invariance of the metric equals

$$\lim_{i\to\infty} d(g_i, h_i) + d(h, g).$$

If we now manage to show that X has only one orbit we will license the conclusion $X = G$.

So fix $x \in X$ the Cauchy limit of $(h_i)_{i \in \mathbb{N}}$. Note that $[x]_G$ is dense, since for $g \in G$ with $d(x, g) < \epsilon$ one has for all $h \in G$ that $d(hg^{-1}x, h) < \epsilon$ by left invariance of d. Note that every $\hat{x} \in [x]_G$ will be the Cauchy limit of $(\hat{g}h_i)_{i \in \mathbb{N}}$ for some $\hat{g} \in G$. Note that if $\hat{x}_0, \hat{x}_1$ are the Cauchy limits of $(\hat{g}_0h_i)_{i \in \mathbb{N}}$, $(\hat{g}_1h_i)_{i \in \mathbb{N}}$ respectively, then $d(\hat{x}_0, \hat{x}_1) = d(g_0, g_1)$. Define

$$\pi : [x]_G \to G$$

$$g \cdot x \mapsto g.$$

By right invariance π is a well defined and a homeomorphism – and indeed not just a homeomorphism but an isometry.

Thus for any $x_1, x_2 \in X$ the orbits $[x_1]_G$ and $[x_2]_G$ are dense G_δ's and so have a point in common. $\square$

THEOREM 7.5. *(Hausdorff) Let X be Polish and Y metrizable. If $\pi : X \twoheadrightarrow Y$ is open, continuous, and onto, then Y is Polish.*

PROOF. Let $\hat{Y}$ be the Cauchy completion of Y. Since Y separable, $\hat{Y}$ is Polish, and so by 2.3(i) it suffices to show Y is a G_δ in $\hat{Y}$.

Define open $N_s \subset X$ for $s \in \mathbb{N}^{<\mathbb{N}}$ by induction on the length of s. $N_\emptyset = X$. $\{N_s : s \in \mathbb{N}^1\}$ will cover X with each N_s having diameter less than 1. Given $\{N_s : s \in \mathbb{N}^{\leq k}\}$ we then choose $\{N_s : s \in \mathbb{N}^{k+1}\}$ so that at each $t \in \mathbb{N}^k$

$$N_t = \bigcup\{N_s \in \mathbb{N}^{k+1} : s \supset t\}$$

and the diameter of N_t is less than 2^{-k-1} with respect to the complete metric in X. We also insist that

$$\overline{N_t} \subset N_s$$

for s strictly included in t.

Then choose open $W_s \subset \hat{Y}$ open for $s \in \mathbb{N}^{<\mathbb{N}}$ so that $\pi[N_s] \cap Y = W_s \cap Y$. Using that open sets are F_σ we can obtain open V_s so that:

(i) $\bigcup\{V_s : s \in \mathbb{N}^1\} = \bigcup\{W_s : s \in \mathbb{N}^1\}$;

(ii) for all $s \in \mathbb{N}^{<\mathbb{N}}$

$$V_s \subset W_s;$$

(iii) for all $k \in \mathbb{N}$, $s \in \mathbb{N}^k$

$$\bigcup\{V_t : t \in \mathbb{N}^{k+1}, t \supset s\} = V_s \cap \bigcup\{W_t : t \in \mathbb{N}^{k+1}, t \supset s\};$$

(iv) for all $y \in \hat{Y}$, $k \in \mathbb{N}$, and $s \in \mathbb{N}^k$, there are only finitely many $t \supset s$ in $\mathbb{N}^{k+1}$ with $y \in V_t$, and there are only finitely many $t \in \mathbb{N}^1$ with $y \in V_t$.

(iv) is the only difficulty in all this, but if each $W_t = \bigcup C_{n,t}$ with each $C_{n,t}$ closed, $C_{n,t} \subset C_{n+1,t}$, then we can demand for $t_0, t_1 \in \mathbb{N}^{k+1}$, $s \in \mathbb{N}^k$, $s \subset t_0, t_1$, if

$$t_0(k) < t_1(k)$$

then

$$W_{t_1} \cap C_{n,t_0} = \emptyset$$

for $n < t_1(k)$. Consequently for each $y \in \hat{Y}$ and $k \in \mathbb{N}$ there are only finitely many $s \in \mathbb{N}^k$ with $y \in V_s$.

All this done, it is clear that

$$Y \subset \bigcap_{k \in \mathbb{N}} \bigcup_{s \in \mathbb{N}^k} V_s.$$

So it is enough that:

Claim: $Y \supset \bigcap_{k \in \mathbb{N}} \bigcup_{s \in \mathbb{N}^k} V_s$.

Proof of Claim (this is König's infinity lemma). Fix $y \in \bigcap_{k \in \mathbb{N}} \bigcup_{s \in \mathbb{N}^k} V_s$ and let

$$T_y = \{s \in \mathbb{N}^{<\mathbb{N}} : \text{ there exists infinitely many } t \supset s \text{ with } y \in V_t\}.$$

T_y is a *tree*, in the sense that

$$s \subset t \in T_y$$

implies

$$s \in T_y.$$

$y \in V_s$ whenever $s \in T_y$ by (iii). Since there can be only finitely many $s \in \mathbb{N}^1$ for which $y \in V_s$ the set T_y is non-empty, and by (iv) for each $s_0 \in T_y$ there must exist $s_1 \in T_y$ such that $s_1 \supset s_0$ and $s_1 \neq s_0$ Thus we find $f \in \mathbb{N}^{\mathbb{N}}$ such that for each $k \in \mathbb{N}$

$$f|_k \in T_y$$

$$\therefore y \in V_{f|_k}.$$

We may also find

$$x \in \bigcap_{k \in \mathbb{N}} N_{f|_k}$$

by the completeness of the metric. Since $\{N_{f|_k} : k \in \mathbb{N}\}$ is a neighbourhood basis of x and the map is continuous it follows that y is in every open set containing $\pi(x)$ and thus $y = \pi(x)$. (Claim□) □

COROLLARY 7.6. *If G be a Polish group and H is a closed subgroup then G/H in the quotient topology is Polish.*

PROOF. If d is a right invariant metric on G then

$$d(g_0H, g_1H) = \inf\{d(h_0, h_1) : h_0 \in g_0H, h_1 \in g_1H\}$$

provides a compatible metric on G/H, and so the quotient is Polish. □

In 7.6 we have that G/H is a Polish G-space under left translation.

The importance of GE groups is indicated by the next few results, the first of which is presented in [**4**] as an abstraction of [**25**] and [**14**]. Recall E_0 as the equivalence relation of eventual agreement on $2^{\mathbb{N}}$, the Cantor space of infinite binary sequences.

THEOREM 7.7. *(Becker, Kechris) Let G be a Polish group and X a properly generically ergodic Polish G-space – that is to say, every orbit dense and meager. Then*

$$E_0 \sqsubseteq_c E_G^X.$$

PROOF. By 2.46 and the fact that E_G^X as a $\underset{\sim}{\Sigma}^1_1$ subset of the product has the Baire property, the orbit equivalence relation must be meager in $X \times X$; so find $\{\mathcal{O}_i : i \in \mathbb{N}\}$ open dense sets in $X \times X$ witnessing this:

$$E_G^X \cap \bigcap_{i \in \mathbb{N}} \mathcal{O}_i = \emptyset.$$

Assume that $\mathcal{O}_n \subset \mathcal{O}_m$ whenever $n \geq m$. For $s \in 2^{<\mathbb{N}}$ build non-empty open sets $V_s \subset X$ and group elements $\{g_{s,t} : s, t \in 2^k\} \subset G$ for each $k \in \mathbb{N}$, so that:

(i) $d(V_s) < 2^{-k}$, for $s \in 2^k$, where d is the complete metric on X; $s \supset t$ and $s \neq t$ implies

$$\overline{V_t} \subset V_s;$$

(ii) for $s, t \in 2^k$ some k,

$$g_{s,t} \cdot V_s = V_t;$$

(iii) for $s, t \in 2^k$ with $s(n) \neq t(n)$,

$$V_s \times V_t \subset \mathcal{O}_n,$$

$$V_s \cap V_t = \emptyset;$$

(iv) if for $s \in 2^k$ and $j \in \{0,1\}$ we define $s^\frown j \in 2^{k+1}$ by the requirement that $s \subset s^\frown j$, $s^\frown j(k) = j$, then for all $k \in \mathbb{N}$ and $s, t \in 2^k$

$$g_{s^\frown 0, t^\frown 0} = g_{s^\frown 1, t^\frown 1} = g_{s,t}.$$

Granting that this may be done, and running on to the finish, note that we could then define $\theta : 2^{\mathbb{N}} \to X$ by letting $\theta(y)$ be the unique point in

$$\bigcap \{V_s : s \in 2^{<\mathbb{N}}, y \supset s\}.$$

This is well defined and continuous by (i); one to one by (iii). If $y_0, y_1 \in 2^{\mathbb{N}}$ with $y_0 E_0 y_1$, then we come to some $N \in \mathbb{N}$ such that for all $k \geq N$ in $\mathbb{N}$

$$y_0(k) = y_1(k)$$

and hence for all $k \geq N$

$$g_{y_0|_k, y_1|_k} = g_{y_0|_N, y_1|_N}$$

by (iv), and hence

$$g_{y_0|_N, y_1|_N} \cdot \theta(y_0) = \theta(y_1)$$

by (ii) and continuity of the action. Alternatively if there are infinitely many k such that $y_0(k) \neq y_1(k)$ then are infinitely many k with

$$(\theta(y_0), \theta(y_1)) \in \mathcal{O}_k$$

by (iii), and hence

$$(\theta(y_0), \theta(y_1)) \in \bigcap_{k \in \mathbb{N}} \mathcal{O}_k$$

by the choice of $(\mathcal{O}_k)_{k \in \mathbb{N}}$ as a decreasing sequence, and so $\theta(y_0)$ and $\theta(y_1)$ are orbit inequivalent.

To perform this construction begin with $W^0_{\langle 0 \rangle}, W^0_{\langle 1 \rangle}$ open nonempty subsets of X with diameter less than $1/2$ and

$$W^0_{\langle 0 \rangle} \times W^0_{\langle 1 \rangle} \subset \mathcal{O}_0.$$

Refining find $W^1_{\langle 0 \rangle} \subset W^0_{\langle 0 \rangle}$ and $W^1_{\langle 1 \rangle} \subset W^0_{\langle 1 \rangle}$ with

$$W^1_{\langle 0 \rangle} \times W^1_{\langle 1 \rangle} \subset \mathcal{O}_0.$$

By every orbit dense we can find $x \in W^1_{\langle 0 \rangle}$, $g \in G$ with $g \cdot x \in W^1_{\langle 1 \rangle}$. By continuity of the action let $V_{\langle 0 \rangle} \subset W^1_{\langle 0 \rangle}$ be open containing x with $g \cdot V_{\langle 0 \rangle} \subset W^1_{\langle 1 \rangle}$. Let $g_{\langle 0 \rangle, \langle 1 \rangle} = g$ and $V_{\langle 1 \rangle} = g \cdot V_{\langle 0 \rangle}$. ($g_{\langle 0 \rangle, \langle 0 \rangle}$ and $g_{\langle 1 \rangle, \langle 1 \rangle}$ are taken to be the identity; $g_{\langle 1 \rangle, \langle 0 \rangle} = g^{-1}_{\langle 0 \rangle, \langle 1 \rangle}$.)

At the next step note that we may first choose disjoint $W^0_{\langle 0,1 \rangle}, W^0_{\langle 0,0 \rangle} \subset V_{\langle 0 \rangle}$ that are non-empty and open having diameter less than 2^{-2}

$$W^0_{\langle 0,0 \rangle} \times W^0_{\langle 0,1 \rangle} \subset \mathcal{O}_1;$$

this uses $\mathcal{O}_1$ open dense. Using the same consideration find non-empty $W^1_{\langle 0,0 \rangle} \subset W^0_{\langle 0,0 \rangle}$ $W^1_{\langle 0,1 \rangle} \subset W^0_{\langle 0,1 \rangle}$ with

$$g_{\langle 0 \rangle, \langle 1 \rangle} \cdot W^1_{\langle 0,0 \rangle} \times g_{\langle 0 \rangle, \langle 1 \rangle} \cdot W^1_{\langle 01 \rangle} \subset \mathcal{O}_1;$$

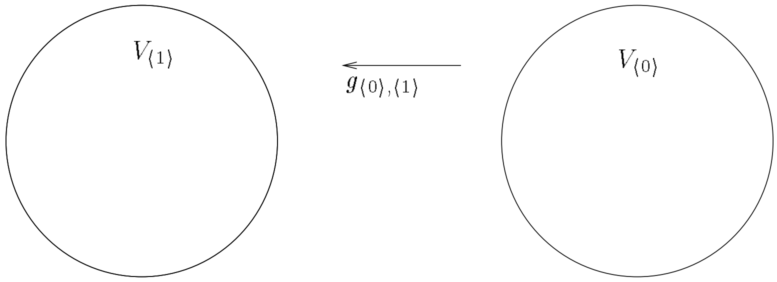

FIGURE 7.1. Production of $V_{\langle 0\rangle}$ and $V_{\langle 1\rangle}$

this can be done, since $\mathcal{O}_1$ is open dense. Going again find $W^2_{\langle 0,0\rangle} \subset W^1_{\langle 0,0\rangle}$ and $W^2_{\langle 0,1\rangle} \subset W^1_{\langle 0,1\rangle}$ such that

$$W^2_{\langle 0,1\rangle} \times W^2_{\langle 0,0\rangle} \subset \mathcal{O}_1;$$

then $W^3_{\langle 0,0\rangle} \subset W^2_{\langle 0,0\rangle}$ and $W^3_{\langle 0,1\rangle} \subset W^2_{\langle 0,1\rangle}$ such that

$$g_{\langle 0\rangle,\langle 1\rangle} \cdot W^3_{\langle 0,1\rangle} \times g_{\langle 0\rangle,\langle 1\rangle} \cdot W^3_{\langle 00\rangle} \subset \mathcal{O}_1;$$

and again for $W^4_{\langle 0,0\rangle} \subset W^3_{\langle 0,0\rangle}$ and $W^4_{\langle 0,1\rangle} \subset W^3_{\langle 0,1\rangle}$ so

$$W^4_{\langle 0,0\rangle} \times g_{\langle 0\rangle,\langle 1\rangle} \cdot W^4_{\langle 0,1\rangle} \subset \mathcal{O}_1;$$

and again $W^5_{\langle 0,0\rangle} \subset W^4_{\langle 0,0\rangle}$ and $W^5_{\langle 0,1\rangle} \subset W^4_{\langle 0,1\rangle}$ so

$$g_{\langle 0\rangle,\langle 1\rangle} \cdot W^5_{\langle 0,1\rangle} \times W^5_{\langle 0,0\rangle} \subset \mathcal{O}_1;$$

again for $W^6_{\langle 0,0\rangle} \subset W^5_{\langle 0,0\rangle}$ and $W^6_{\langle 0,1\rangle} \subset W^5_{\langle 0,1\rangle}$ so

$$W^6_{\langle 0,1\rangle} \times g_{\langle 0\rangle,\langle 1\rangle} \cdot W^6_{\langle 0,0\rangle} \subset \mathcal{O}_1;$$

and again $W^7_{\langle 0,0\rangle} \subset W^6_{\langle 0,0\rangle}$ and $W^7_{\langle 0,1\rangle} \subset W^6_{\langle 0,1\rangle}$ so

$$g_{\langle 0\rangle,\langle 1\rangle} \cdot W^7_{\langle 0,0\rangle} \times W^7_{\langle 0,1\rangle} \subset \mathcal{O}_1;$$

Next let

$$W_{\langle 0,0\rangle} = W^7_{\langle 0,0\rangle},$$
$$W_{\langle 0,1\rangle} = W^7_{\langle 0,1\rangle},$$
$$W_{\langle 1,0\rangle} = g_{\langle 0\rangle,\langle 1\rangle} \cdot W^7_{\langle 0,0\rangle},$$
$$W_{\langle 1,1\rangle} = g_{\langle 0\rangle,\langle 1\rangle} \cdot W^7_{\langle 0,1\rangle}.$$

We may assume that the diameters of $W_{\langle 1,1\rangle}$ and $W_{\langle 1,0\rangle}$ are also less than 2^{-2}, since at worst this only requires us to shrink $W^7_{\langle 0,0\rangle}$ and $W^7_{\langle 0,1\rangle}$ one more time.

Using the dense orbits we find $V_{\langle 0,0\rangle} \subset W_{\langle 0,0\rangle}$ and $g_{\langle 0,0\rangle,\langle 0,1\rangle} \in G$ with

$$g_{\langle 0,0\rangle,\langle 0,1\rangle} \cdot V_{\langle 0,0\rangle} \subset W_{\langle 0,1\rangle}.$$

Let

$$V_{\langle 0,1\rangle} = g_{\langle 0,0\rangle,\langle 0,1\rangle} \cdot V_{\langle 0,0\rangle}.$$

After this the definitions

$$V_{\langle 1,0\rangle} = g_{\langle 0\rangle,\langle 1\rangle} \cdot V_{\langle 0,0\rangle}$$

and

$$V_{\langle 1,1\rangle} = g_{\langle 0\rangle,\langle 1\rangle} \cdot V_{\langle 0,1\rangle}$$

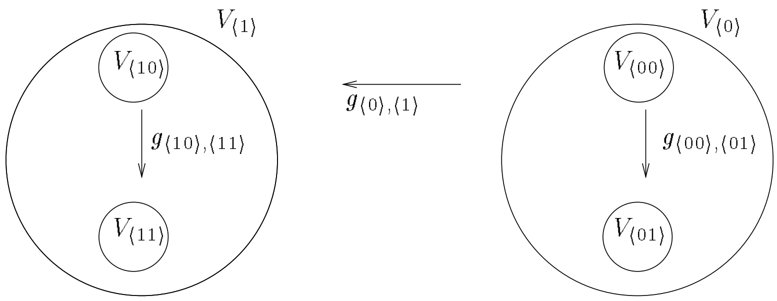

FIGURE 7.2. Subdividing $V_{\langle 0\rangle}$ and $V_{\langle 1\rangle}$

are both necessary and good. The group elements at the second level are defined by the requirement that $g_{\langle 0\rangle,\langle 1\rangle}$ copies the arrangement of $(V_{\langle 0,0\rangle}, V_{\langle 0,1\rangle})$ over to $(V_{\langle 1,0\rangle}, V_{\langle 1,1\rangle})$ and that in general $g_{s,t} = (g_{t,s})^{-1}$:

$$g_{\langle 0,1\rangle,\langle 0,0\rangle} = g^{-1}_{\langle 0,0\rangle,\langle 0,1\rangle};$$
$$g_{\langle 0,1\rangle,\langle 1,1\rangle} = g_{\langle 0,0\rangle,\langle 1,0\rangle} = g_{\langle 0\rangle,\langle 1\rangle}$$

and

$$g_{\langle 1,1\rangle,\langle 0,1\rangle} = g_{\langle 1,0\rangle,\langle 0,0\rangle} = g_{\langle 1\rangle,\langle 0\rangle} = g^{-1}_{\langle 0\rangle,\langle 1\rangle};$$
$$g_{\langle 0,0\rangle,\langle 1,1\rangle} = g_{\langle 0\rangle,\langle 1\rangle} g_{\langle 0,0\rangle,\langle 0,1\rangle};$$
$$g_{\langle 1,0\rangle,\langle 1,1\rangle} = g_{\langle 0\rangle,\langle 1\rangle} g_{\langle 0,0\rangle,\langle 0,1\rangle} g^{-1}_{\langle 0\rangle,\langle 1\rangle}.$$

This completes the construction for $s \in 2^{<\mathbb{N}}$ of length of at most two. Continuing in this fashion we finish. □

In virtue of 3.4 theorem 7.7 may be equivalently reformulated as

COROLLARY 7.8. *Let G be a Polish group and X a Polish G-space. (i) If every orbit meager and at least one orbit dense then*

$$E_0 \sqsubseteq_c E^X_G.$$

(ii) If there is some orbit $[x]_G$ so that for no G_δ orbit $[y]_G$ do we have

$$\overline{[x]_G} = \overline{[y]_G}$$

then

$$E_0 \sqsubseteq_c E^X_G.$$

PROOF. (i) follows directly from 7.7 and 3.5 and that any non-meager subset of the space X_0 described there must be non-meager in X.

For (ii): Suppose we may find $x \in X$ for which there is no y with $[x]_G$ a G_δ and $\overline{[x]_G} = \overline{[y]_G}$. Let

$$X(x) = \{z \in X : \overline{[x]_G} = \overline{[z]_G}\}.$$

$X(x)$ is a G_δ since if $\{U_l : l \in \mathbb{N}\}$ forms a basis of X and $\{g_i : i \in \mathbb{N}\}$ is dense in G then

$$\overline{[x]_G} = \overline{[z]_G}$$

if and only if

$$\forall i, l, \exists j (g_i \cdot x \in U_l \Rightarrow g_j \cdot z \in U_l)$$

and

$$\forall i, l, \exists j (g_i \cdot z \in U_l \Rightarrow g_j \cdot x \in U_l).$$

Since the class of Polish spaces is closed under taking G_δ subsets by 2.3, we have that $X(x)$ is a Polish G-space. Since no orbit in $X(x)$ can be G_δ, every orbit is meager by 6.37. Thus by 7.7

$$E_0 \sqsubseteq_c E_G^{X(x)} \sqsubseteq_c E_G^X.$$

$\square$

THEOREM 7.9. *Let G be a GE group and X a Polish G-space. Then exactly one of the following holds:*

1. $E_G^X \leq_B \mathrm{id}(\mathbb{R})$;
2. $E_0 \sqsubseteq_c E_G^X$.

PROOF. Claim: At most one of 1. and 2.

Proof of claim: It suffices to observe that E_0 is not Borel reducible to $\mathrm{id}(\mathbb{R})$. To prove this note that E_0 is induced by the natural action of $(\mathbb{Z}_2)^{<\mathbb{N}}$ (the subgroup of $(\mathbb{Z}_2)^{\mathbb{N}}$ consisting of $\vec{g}$ that has $\vec{g}(i) = 0$ for all sufficiently large i) on $2^{\mathbb{N}}$ under identification of $\mathbb{Z}_2$ and $\{0, 1\}$ – namely, for $\vec{g} \in (\mathbb{Z}_2)^{<\mathbb{N}}$ and $\vec{x} \in 2^{\mathbb{N}}$ we let $(\vec{g}\cdot\vec{x})(i) = \vec{g}(i) + \vec{x}(i) \bmod 2$. Since every orbit under this action is dense and countable, and so certainly meager, we can appeal to 3.3. (Claim $\square$)

To show that at least one holds, first suppose that for all $x, y \in X$ the closure of the two orbits are equal if and only if the orbits themselves are equal. Fix $\{g_i : i \in \mathbb{N}\}$ a dense subset of G and $\{U_l : l \in \mathbb{N}\}$ a basis. To each point $x \in X$ assign the real $\theta(x)$ in the unit interval whose lth coordinate in its decimal expansion is 5 if there exists some i for which $g_i \cdot x \in U_l$ and 6 otherwise: so if $A_l = G \cdot U_l$ then

$$\theta : X \to [0, 1]$$
$$\theta(x) = 0 \cdot \chi_{A_0}(x)\chi_{A_1}(x)\chi_{A_2}(x)...$$

witnesses the truth of (I).

So now suppose there are $x, y \in X$ with

$$\overline{[x]_G} = \overline{[y]_G}$$

but

$$[x]_G \neq [y]_G.$$

Then by the definition of a GE group there can be no G_δ orbit with the same closure as $[x]_G$ and we have $E_0 \sqsubseteq_c E_G^X$ by 7.9. $\square$

This gives a non historical proof of

THEOREM 7.10. *(Effros) Let G be a locally compact group and X a Polish G-space. Then exactly one of the following holds:*

1. $E_G^X \leq_B \mathrm{id}(\mathbb{R})$;
2. $E_0 \sqsubseteq_c E_G^X$.

PROOF. It suffices to show that locally compact entails GE. But if X_0 is a Polish G-space with every orbit dense and $[x]_G$ is G_δ it must also be that

$$[x]_G \in F_\sigma,$$

since G is locally compact and $g \mapsto g \cdot x$ is continuous; then given any $y \in X$ which is E_G-inequivalent to x we obtain that both $[x]_G$ and its complement are dense G_δ's in X_0 with a contradiction to the Baire category theorem. $\square$

LEMMA 7.11. *(Effros) Suppose that G is a Polish group and X is a Polish G-space. Then the following are equivalent:*

(I) every orbit is G_δ;

(II) for all $x, y \in X$,

$$[x]_G = [y]_G \Leftrightarrow \overline{[x]_G} = \overline{[y]_G};$$

(III) E_G^X is G_δ.

Moreover each of these implies that E_G^X is smooth.

PROOF. (I)$\Rightarrow$(II): Since disjoint G_δ sets cannot have the same closure by the Baire category theorem.

(II)$\Rightarrow$(III): Since for $\{U_l : l \in \mathbb{N}\}$ a basis of X and $\{g_i : i \in \mathbb{N}\}$ a dense subgroup of G we have $\overline{[x]_G} = \overline{[y]_G}$ if and only if

$$\forall i, l, \exists j (g_i \cdot x \in U_l \Rightarrow g_j \cdot y \in U_l)$$

and

$$\forall i, l, \exists j (g_i \cdot y \in U_l \Rightarrow g_j \cdot x \in U_l),$$

both of which are G_δ.

(III)$\Rightarrow$(I). Fix $x \in X$ and suppose

$$E_G^X = \bigcap_{n \in \mathbb{N}} \mathcal{O}_n,$$

each $\mathcal{O}_n$ open in $X \times X$. Then for $\mathcal{O}_n^x = \{y : (x, y) \in \mathcal{O}_n\}$, each $\mathcal{O}_n^x$ is open and

$$[x]_G = \bigcap_{n \in \mathbb{N}} \mathcal{O}_n^x.$$

Note that (II) implies $E_G^X \leq_B \mathrm{id}(\mathbb{R})$, since by the argument of 7.9 we can then assign to each point a code for the closure of its orbit. $\square$

THEOREM 7.12. *(Effros) Suppose that G is a Polish group, X is a Polish G-space, and $x \in X$. Then the following are equivalent:*

(I) the map

$$G \to [x]_G$$
$$g \mapsto g \cdot x$$

is open;

(II) $[x]_G \in G_\delta$;

(III) $[x]_G$ is non-meager in $\overline{[x]_G}$.

PROOF. (I)$\Rightarrow$(II): For then if $G_x =_{df} \{g \in G : g \cdot x = x\}$ we have that G/G_x and $[x]_G$ are homeomorphic, and then 7.6 implies both are Polish spaces, and thus $[x]_G \in G_\delta$.

(II)$\Rightarrow$(I): (II) implies that $[x]_G$ is itself a Polish G-space. Define

$$f : [x]_G \to G/G_x$$

by letting $f(x')$ be the unique $g'G_x$ in G/G_x with $g' \cdot x = x'$. The space G/G_x is Polish by 7.6. It is enough to show that f is continuous.

For any $V \subset G$ open the set $\{x' \in [x]_G : \exists g' \in V(g' \cdot x = x')\}$ is $\underset{\sim}{\Sigma}_1^1$, and so f is Baire measurable by 2.52. Choose $C \subset [x]_G$ comeager on which f is continuous by 2.49. By 2.47 and the fact that G acts transitively on the Polish space $[x]_G$ we have that for all $x' \in [x]_G$

$$\forall^* g \in G (g \cdot x' \in C).$$

Thus for any $x_0, x_1, x_2, ..., x_\infty \in [x]_G$ with

$$x_i \to x_\infty$$

we may find $g \in G$ so that $g \cdot x_\infty \in C$ and each $g \cdot x_i \in C$. Thus if $g_\infty \cdot x = x_\infty$ and at each i we choose g_i with $g_i \cdot x = x_i$ then it follows from the definition of C that

$$gg_iG_x \to gg_\infty G_x$$

and therefore

$$g_iG_x \to g_\infty G_x,$$

which amounts to asserting that

$$f(x_i) \to f(x_\infty)$$

just as required.

(III)⇒(II): This is 6.37.

(II)⇒(III): Dense G_δ's are by definition comeager. □

As a word towards notation, the term *GE group* derives from the following definition of [**42**]:

DEFINITION 7.13. A Polish group G has the *strong Glimm-Effros property* if whenever X is a Polish G-space exactly one of the following hold:

1. $E_0 \sqsubseteq_c E_G^X$;
2. E_G^X is G_δ.

By now it is not much of a surprise that:

LEMMA 7.14. *Any GE group has the strong Glimm-Effros property.*

PROOF. Let X be a Polish G-space. If there are two distinct orbits with the same closure then we obtain $E_0 \sqsubseteq_c E_G^X$ as in the proof of 7.9. Otherwise we obtain that the closure of an orbit classifies the orbit completely and appeal to 7.11. □

It is unknown whether GE groups are the only groups with the strong Glimm-Effros property.

EXERCISE 7.15. Any right invariant metric on a locally compact Polish group is necessarily complete.

EXERCISE 7.16. If one compatible right invariant metric on a Polish group is complete then they all are.

EXERCISE 7.17. If a *compact* Polish group acts continuously on a Polish space then the orbit equivalence relation is smooth.

Solecki has recently proved a converse: If a Polish group G has E_G^X smooth for every Polish G-space X then G is compact. Kechris shows in [**49**] that every locally compact Polish group has all its orbit equivalence relations $\leq_B$ some Borel equivalence relation with countable equivalence classes; he goes on to conjecture *its* converse:

> If a Polish group G is such that whenever it acts continuously on a Polish space X we have $E_G^X \leq_B E$ some Borel equivalence relation with all classes countable, then G is locally compact.

This remains stubbornly open.

EXERCISE 7.18. (See [**4**], 2.3.5(b).) Let H be a closed subgroup of a Polish group $\hat{H}$. Let X be a Polish H-space. With the following we may lift the action of H on X to an action $\hat{H}$ on $\hat{X}$ so that X is a closed subspace of $\hat{X}$ and each orbit $[x]_{\hat{H}}$ meets exactly one orbit in X.

Let $\hat{X}$ be the set of all $[x,g]$ with $x \in X$ and $g \in \hat{H}$ where we set $[x,g] = [x',g']$ if there is some $h \in H$ with $h \cdot x = x'$ and $hg = g'$; give $\hat{X}$ the quotient topology provided by the surjection

$$X \times \hat{H} \twoheadrightarrow \hat{X}$$

$$(x,g) \mapsto [x,g].$$

(i) Show that $\hat{X}$ is Hausdorff.

(ii) Show that for any open $U \subset \hat{X}$ and $z \in U$ we may find open U_0 containing z with $\overline{U_0} \subset U$.

(iii) Use the separability of $\hat{X}$ and Urysohn's metrizability criterion at §8.3 [**73**] to conclude that $\hat{X}$ is metrizable; hence by 7.5 it is Polish.

(iv) In the action

$$g \cdot [x,g'] = [x, g'g^{-1}]$$

$\hat{X}$ is a Polish $\hat{H}$-space.

(v) X may be identified with $\{[x,1] : x \in X\}$, so that E_H^X equals the restriction of $E_{\hat{H}}^{\hat{X}}$ to X.

[**53**] shows this lift to preserve turbulence, so if X is a turbulent Polish H-space then $\hat{X}$ is a turbulent $\hat{H}$-space.

EXERCISE 7.19. (Gao [**22**]) If H is Polish and G is a closed normal subgroup with both G and H/G admitting a complete left invariant metric, then H admits a complete left invariant metric, and thus all solvable Polish groups have complete left invariant metrics.

EXERCISE 7.20. A *homogeneous* compact metric space is one on which the homeomorphism group acts transitively. Use 7.12 to show that if (K,d) is a homogeneous compact metric space then for all $\epsilon > 0$ there is $\delta > 0$ such that for all $x, y \in X$ with $d(x,y) < \delta$ there is some $\pi \in \mathrm{Hom}(K)$ such that

$$\pi(x) = y$$

and for all $z \in K$

$$d(\pi(z), z) < \epsilon.$$

This abrupt consequence of Effros' theorem turned out to be profoundly useful in [**28**].

EXERCISE 7.21. (Compare §6.3 of [**4**] and the proofs of 6.11 and 6.17 above.)

(i) Suppose $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$. Show that the following three things are equivalent:

(a) $[\mathcal{M}]_{S_\infty} \in G_\delta$ (with respect to the usual topology on $\mathrm{Mod}(\mathcal{L})$).

(b) For all $n_0, n_1, ..., n_k \in \mathbb{N}^{<\mathbb{N}}$ there is a quantifier free formula φ such that

$$\mathcal{M} \models \exists \vec{z} \varphi(\vec{z}, n_0, n_1, ..., n_k)$$

and for *all* formulas $\psi(x_0, ..., x_k)$ in the language we have either

$$\mathcal{M} \models \forall x_0 ... \forall x_k (\exists \vec{z} \varphi(\vec{z}, \vec{x}) \Rightarrow \psi(\vec{x}))$$

or

$$\mathcal{M} \models \forall x_0 ... \forall x_k (\exists \vec{z} \varphi(\vec{z}, \vec{x}) \Rightarrow \neg\psi(\vec{x})).$$

(c) For all $n_0, n_1, ..., n_k \in \mathbb{N}^{<\mathbb{N}}$ there is a quantifier free formula φ such that

$$\mathcal{M} \models \exists \vec{z} \varphi(\vec{z}, n_0, n_1, ..., n_k)$$

and for all quantifier free formulas $\psi(x_0, ..., x_k, y_0, ..., y_l)$ in the language we have either

$$\mathcal{M} \models \forall x_0 ... \forall x_k (\exists \vec{z} \varphi(\vec{z}, \vec{x}) \Rightarrow \exists y_0 ... \exists y_l \psi(\vec{x}, \vec{y}))$$

or

$$\mathcal{M} \models \forall x_0 ... \forall x_k (\exists \vec{z} \varphi(\vec{z}, \vec{x}) \Rightarrow \forall y_0 ... \forall y_l \neg\psi(\vec{x}, \vec{y})).$$

(ii) (Miller; Suzuki) Let τ_{FO} be the Polish topology from 2.43. Show that the following are equivalent:

(a) $[\mathcal{M}]_{S_\infty}$ is G_δ with respect to the τ_{FO}.

(b) $\mathcal{M}$ is an *atomic model*; in other words, for all $n_0, n_1, ..., n_k \in \mathbb{N}^{<\mathbb{N}}$ there is some formula φ such that

$$\mathcal{M} \models \varphi(n_0, n_1, ..., n_k)$$

and for all $\psi(x_0, ..., x_k)$ in the language we have either

$$\mathcal{M} \models \forall x_0 ... \forall x_k (\varphi(\vec{x}) \Rightarrow \psi(\vec{x}))$$

or

$$\mathcal{M} \models \forall x_0 ... \forall x_k (\varphi(\vec{x}) \Rightarrow \neg\psi(\vec{x})).$$

(Hint for (i): For (c)⇒(a), show that $\gamma^*(\mathcal{M}) = 0$ and appeal to 6.20 and 7.12. For (a)⇒(b), we can let $V = \{g \in S_\infty : g(n_i) = n_i \text{ all } i \leq k\}$; quoting the definition of the topology and 7.12 we have some quantifier free φ as above)

Clearly there is some kind of combinatorial difference in (i)(b) and (ii)(b), since for the topology generated by quantifier free formulas one does not have the set $\{\mathcal{N} : \exists \vec{z} \varphi(\vec{z}, n_0, ..., n_k)\}$ as a clopen set even for φ quantifier free. An important step in the proof of Becker's theorem at §7.5 is to pass to a topology which approximates this property of τ_{FO}; we do not quite obtain for any basic open $U \subset X$ and $V \subset G$ that $U^{\Delta V}$ is clopen, but rather the approximation indicated at 7.32 and 7.48(III).

Exercise 7.22. (The Vitali equivalence relation produced by rational translation is equireducible with E_0. See also [**13**].)

(i) $E_0 \sqsubseteq_c \mathbb{R}/\mathbb{Q}$ – that is to say, there is a continuous injection reducing E_0 to the orbit equivalence relation resulting from $\mathbb{Q}$ acting by translation on $\mathbb{R}$.

(ii) Let G be the group $\mathbb{Q}/\mathbb{Z}$ obtained by taking the algebraic quotient of $\mathbb{Q}$ by the subgroup $\mathbb{Z}$; let

$$\pi_1 : \mathbb{Q} \twoheadrightarrow G$$

be the quotient map. Let $\mathbb{T}$ be the space $\mathbb{R}/\mathbb{Z}$ obtained by taking the topological quotient of $\mathbb{R}$ by the closed equivalence relation $\{(x, y) \in \mathbb{R}^2 : x - y \in \mathbb{Z}\}$; let

$$\pi_2 : \mathbb{R} \twoheadrightarrow \mathbb{T}$$

be the canonical surjection.

We can then let G act of $\mathbb{T}$ in the obvious manner:

$$(\pi_1(q)) \cdot (\pi_2(x)) = \pi_2(q + x).$$

One then has

$$\mathbb{R}/\mathbb{Q} \leq_B \mathbb{T}/G.$$

(iii) G may be written as

$$G = \bigcup_{n \in \mathbb{N}} G_n$$

where each G_n is a finite subgroup of G and $G_n \subset G_{n+1}$.

(iv) Choose $\{I_s : s \in \mathbb{N}^{<\mathbb{N}}\}$ half-open intervals in $\mathbb{T}$ such that $I_\emptyset = \mathbb{T}$, each I_s is the disjoint union of $(I_{s^\frown i})_{i \in \mathbb{N}}$, $d(I_s)$ approaches 0 as the length of s approaches ∞, for some choice of a compatible metric d, and for each $s \in \mathbb{N}^k$ and $i \in \mathbb{N}$

$$G_k \cdot I_{s^\frown i} \cap I_s = I_{s^\frown i}.$$

(v) Let A be the set $\mathbb{N}^{<\mathbb{N}}$. For each $x \in \mathbb{T}$ define $\theta(x) \in 2^A$ by

$$(\theta(x))(s) = 1$$

if and only if the length of s equals $n+1$ some n and $G_n \cdot x \cap I_s \neq \emptyset$ where $n+1$. For $x, y \in \mathbb{T}$ we have that $\theta(x)$ and $\theta(y)$ agree off of a finite set if and only if for some n

$$G_n \cdot x = G_n \cdot y.$$

(vi) Thus by (v) and (ii)

$$\mathbb{R}/\mathbb{Q} \leq_B E_0.$$

7.2. Invariantly metrizable and nilpotent are GE

Both these results appear in [**42**]. The first to be proved, and even then initially in the special case of G abelian, was 7.23 and at the time it was very surprising.

The point is that the we should not be so shocked that smoothness for locally compact groups implies every orbit is G_δ. Local compactness of the group is sufficient to imply every orbit F_σ, and so the further reduction to the class $G_\delta \cap F_\sigma$ is a small one.

However the situation with abelian Polish groups was already known from Solecki's work at [**76**] to be completely different. Abelian Polish groups are capable of giving rise to orbit equivalence relations that are not even Borel. Thus part of the content of the next theorem is to say that when an orbit equivalence relation induced by an abelian Polish group action is smooth we necessarily have an enormous reduction in the potential Borel complexity. This was the first result of its kind.

THEOREM 7.23. *(Solecki) Any Polish group admitting an invariant metric is GE.*

PROOF. Let d be an invariant metric on a Polish group G, which in turn will be complete by 7.4, and let X a Polish G-space with $[x]_G$ a dense G_δ. Suppose $y \in X$ has

$$\overline{[y]_G} = \overline{[x]_G}.$$

Choose $(g_i)_{i \in \mathbb{N}}$ and $(h_i)_{i \in \mathbb{N}}$ two sequences in G with

$$g_i \cdot x \rightarrow y$$

$$h_i \cdot y \rightarrow x.$$

It suffices to show that there are $c_0, c_1, ..., c_\infty \in G_x$ and $g_\infty \in G$ so that

$$g_i c_i \rightarrow g_\infty c_\infty.$$

For each $n \in \mathbb{N}$ let
$$\hat{V}_n = \{g \in G : d(g,1) < 2^{-n}\}$$
and let $V_n \subset \hat{V}_n$ be an open conjugation invariant neighbourhood of the identity such that $V_n^{-1} = V_n$ and $(V_n)^2 \subset \hat{V}_n$. The map $g \mapsto g \cdot x$ is open as a function from G to $[x]_G$ by 7.12, so we can find open sets U_n containing x with $V_n \cdot x = U_n \cap [x]_G$. By speeding up the sequences (g_i) and (h_i) we may assume that for all n and for all $i, j \geq n$
$$h_i g_j \cdot x \in V_n.$$
Thus in particular
$$h_j g_j \in V_j \cdot G_x$$
and so we may find $c'_j, c''_{j+1} \in G_x$ such that
$$h_j g_j \in V_j c'_j$$
$$h_j g_{j+1} \in V_j c''_{j+1},$$
$$\therefore (g_j)^{-1} g_{j+1} \in (c'_j)^{-1}(V_j)^{-1} V_j c''_{j+1},$$
and since $V_j = (V_j)^{-1} = c'_j(V_j)^{-1}(c'_j)^{-1}$
$$(g_j)^{-1} g_{j+1} \in (V_j)^2 (c'_j)^{-1} c''_{j+1}.$$
Let $\hat{c}_j = ((c'_j)^{-1} c''_{j+1})^{-1}$; this is in G_x, since the stabilizer of any point forms a subgroup of G. Rephrasing the above we have
$$(g_j)^{-1} g_{j+1} \hat{c}_j \in (V_j)^2 \subset \hat{V}_j$$
and thus by definition of the $\hat{V}_j$
$$d((g_j)^{-1} g_{j+1} \hat{c}_j, 1) < 2^{-j},$$
so by left invariance by d
$$d(g_{j+1}\hat{c}_j, g_j) < 2^{-j}.$$
Thus if we define $c_j = \hat{c}_{j-1}\hat{c}_{j-2}...\hat{c}_0$ then for all j
$$d(g_{j+1}c_{j+1}, g_j c_j) < 2^{-j}$$
by the right invariance of d, and so for all $l > j$
$$d(g_l c_l, g_j c_j) < \sum_{i \geq j} 2^{-i} = 2^{-j+1}$$
as required. □

Corollary 7.24. *(A) Let G be an abelian Polish group and X a Polish G-space. Then exactly one of the following holds:*

1. $E_G^X \leq_B \mathrm{id}(\mathbb{R})$;
2. $E_0 \sqsubseteq_c E_G^X$.

(B) More generally let G be an invariantly metrizable Polish group and X a Polish G-space. Then exactly one of the following holds:

1. $E_G^X \leq_B \mathrm{id}(\mathbb{R})$;
2. $E_0 \sqsubseteq_c E_G^X$.

Proof. (A) follows from (B) in light of 7.3. (B) follows from 7.23 by 7.11 and the definition of GE group. □

Theorem 7.25. *Any nilpotent Polish group is GE.*

PROOF. Supposing G to be a nilpotent Polish group amounts to asserting there is a finite sequence $G = Z_k \rhd Z_{k-1} \rhd ....Z_0 = \{1\}$ with each Z_i a normal subgroup of G such that Z_{i+1}/Z_i is the *centralizer* of G/Z_i: so that

$$Z_{i+1}/Z_i = \{\bar{g} \in G/Z_i : \forall \bar{h} \in G/Z_i(\bar{g}\bar{h} = \bar{h}\bar{g})\}.$$

Let X be a Polish G space, $x \in X$. For each i let $Z_i^x = \overline{\langle Z_i, G_x \rangle}$ be the closure of the subgroup generated by Z_i and the stabilizer of x.

Claim(I): Each Z_i^x/G_x is closed in G/G_x.

Proof of Claim: It is a general fact that if G is a topological group and $H_0 < H_1 < G$ are closed subgroups then the inclusion of H_0 in H_1 implies H_1/H_0 is closed in G/H_0. (Claim(I)$\Box$)

Claim(II): Each Z_i^x is normal in Z_{i+1}^x.

Proof of Claim: Since the operation of conjugating one element by another is jointly continuous it suffices to show that for all $g \in \langle Z_{i+1}, G_x \rangle$ and $h \in \langle Z_i, G_x \rangle$

$$ghg^{-1} \in \langle Z_i, G_x \rangle.$$

So fix such $g = z_0 k_0$ and $h = z_1 k_1$, with $z_0 \in Z_{i+1}$, and $z_1 \in Z_i$, and $k_0, k_1 \in G_x$. Then

$$ghg^{-1} = z_0 k_0 z_1 k_1 (k_0)^{-1}(z_0)^{-1} = z_0(z_0)^{-1}\hat{z}_1 k_0 k_1 (k_0)^{-1},$$

for some $\hat{z}_1 \in Z_i$ by the definition of Z_{i+1},

$$= \hat{z}_1 \hat{k}$$

for $\hat{k} = k_0 k_1 (k_0)^{-1} \in G_x$. (Claim(II)$\Box$)

Claim(III): Each Z_{i+1}^x/Z_i^x is abelian.

Proof of Claim: It suffices to show that there is a dense abelian subgroup. Letting

$$Pr_{i+1} : Z_{i+1}^x \to Z_{i+1}^x/Z_i^x$$
$$z \mapsto zZ_i^x$$

be the projection map. $\langle Z_{i+1}, G_x \rangle$ is dense in Z_{i+1}^x and hence its image is dense in Z_{i+1}^x/Z_i^x under Pr_{i+1}.

Fix $g_1, g_2 \in Z_{i+1}$ and $k_1, k_2 \in G_x$ with $Pr_{i+1}(g_1 k_1) = \bar{g}_1$ and $Pr_{i+1}(g_2 k_2) = \bar{g}_2$ both in Z_{i+1}^x/Z_i^x. Then by the definition of Z_{i+1}

$$\bar{g}_2 \bar{g}_1 = Pr_{i+1}(g_2 k_2 g_1 k_1) = Pr_{i+1}(g_1 g_2 k_1 k_2 k \epsilon),$$

for some $k = k_2^{-2} k_1^{-1} k_2 k_1 \in G_x$ and $\epsilon \in Z_i$

$$= Pr_{i+1}(g_1 k_1 g_2 k_2 k \epsilon') = \bar{g}_1 \bar{g}_2$$

some $\epsilon' \in Z_i$. (Claim(III)$\Box$)

Now suppose the orbit $[x]_G$ is G_δ and that $y \in X$ has

$$\overline{[y]_G} = \overline{[x]_G}.$$

Assume that for some $l \in \{0, 1, ..., k\}$ there are $g_i \in Z_l^x$ and $h_i \in Z_{l+1}^x$ with

$$g_i \cdot x \to y$$

and

$$h_i \cdot y \to x,$$

and work towards showing that we can choose $\hat{g}_i \in Z_{l-1}^{x'}$ and $\hat{h}_i \in Z_l^{x'}$ with

$$\hat{g}_i \cdot x' \to y$$

and

$$\hat{h}_i \cdot y \to x'$$

for some $x' \in [x]_G$. (In the case that $l = k$ take the h_i's to be in Z^x_k and note that one of the steps below is superfluous.) This will prove that x and y are orbit equivalent, since by repeating the same argument over and over we at last obtain $l = 0$.

Claim(IV): There exists $\{z_i : i \in \mathbb{N}\} \subset Z_l$ for which

$$z_i \cdot x \to y.$$

Proof of Claim: First choose $\epsilon_i \to 1$ in G and $z_i \in Z_l$ and $k_i \in G_x$ with $g_i = \epsilon_i z_i k_i$ by assumption of $g_i \in Z^x_l$. We may assume that there are open sets $\{O_i : i \in \mathbb{N}\}$ around y and V_i at the identity of G thus:

$$\{y\} = \bigcap_{i \in \mathbb{N}} O_i;$$

$$V_i \cdot O_i \subset O_{i-1};$$

$$g_i \cdot x \in O_i,$$

$$\epsilon_i \in (V_i)^{-1}.$$

Then

$$z_i \cdot x = z_i k_i \cdot x = \epsilon_i^{-1} \epsilon_i z_i k_i \cdot x = \epsilon_i^{-1} g_i \cdot x \in O_{i-1}.$$

(Claim(IV)$\square$)

Claim(V): There exist $\{\hat{z}_i : i \in \mathbb{N}\} \subset Z^x_l$ for which

$$\hat{z}_i \cdot y \to x.$$

Proof of Claim: Fixing the z_i from claim(IV) and recalling $h_i \cdot y \to x$ we may assume that

$$h_i z_i \cdot x \to x.$$

Then by 7.12 and claim (I)

$$[x]_{Z^x_{l+1}} =_{df} \{g \cdot x : g \in Z^x_{l+1}\}$$

is G_δ; so one more application of 7.12 gives that

$$Z^x_{l+1}/G_x \to [x]_{Z^x_{l+1}}$$

$$gG_x \mapsto g \cdot x$$

is open. Hence there are $\epsilon_i \to 1$ in Z^x_{l+1} such that

$$\epsilon_i h_i z_i x = x.$$

Hence there are $k_i \in G_x$ such that

$$\epsilon_i h_i z_i = k_i.$$

Since $Z_l \triangleleft G$ is a normal subgroup of G there are $\bar{z}_i \in Z_l$ such that

$$\bar{z}_i \epsilon_i h_i = k_i$$

$$\therefore h_i = (\epsilon_i)^{-1}(\bar{z}_i)^{-1} k_i.$$

Since $(\epsilon_i)^{-1} \to 1$ the argument from claim (IV) shows that setting $\hat{z}_i = (\bar{z}_i)^{-1} k_i$ succeeds. (Claim(V)$\square$)

Summing up: We now have $[x]_{Z^x_l} \in G_\delta$ and $\overline{[x]_{Z^x_l}} = \overline{[y]_{Z^x_l}}$, with witnesses $\{z_i : i \in \mathbb{N}\} \subset Z_l$ and $\{\hat{z}_i : i \in \mathbb{N}\} \subset Z^x_l$ for which

$$z_i \cdot x \to y,$$

$$\hat{z}_i \cdot y \to x.$$

As in the proof of 7.23 the openness of the map $g \mapsto g \cdot x$ entails the existence of open neighborhoods V_i of 1 in G so that

$$\hat{z}_i z_j \in V_i G_x$$

for all $j \geq i$,

$$V_{i+1} \subset V_i,$$

and

$$\{1\} = \bigcap_{i \in \mathbb{N}} V_i.$$

Thus if d^* is an invariant two sided metric on Z_l^x / Z_{l-1}^x (which exists for any abelian group by 7.3 and is complete by 7.4) and $Pr_l : Z_l^x \to Z_l^x / Z_{l-1}^x$ is the projection then we may assume that for all i and $j \geq i$

$$d^*(Pr_l(\hat{z}_i z_j), 1) < 2^{-i-1}$$

and thus for all $j, k \geq i$

$$d^*(Pr_l(\hat{z}_i) Pr_l(z_j), Pr_l(\hat{z}_i) Pr_l(z_k)) < 2^{-i}$$

$$\therefore d^*(Pr_l(z_j), Pr_l(z_k)) < 2^{-i}$$

by right invariance. Hence we have a Cauchy sequence in this necessarily complete metric, and may therefore find $z_\infty \in Z_l^x$ with

$$Pr_l(z_i) \to Pr_l(z_\infty).$$

Thus there exist $z_i^* \in Z_{l-1}^x$ with

$$z_i z_i^* \to z_\infty.$$

Thus choose $\delta_i \in Z_l^x$ with $\delta_i \to 1$ and at each i

$$\delta_i z_i z_i^* = z_\infty.$$

Since Z_{l-1}^x is normal in Z_l^x we can find $\bar{z}_i \in Z_{l-1}^x$ with

$$\bar{z}_i \delta_i z_i = z_\infty.$$

Let $x' = z_\infty \cdot x$. Since $z_i \cdot x \to y$ and $\delta_i \to 1$ we obtain

$$\delta_i z_i x \to y$$

and then

$$y \in \overline{[x']_{Z_{l-1}^x}}$$

since

$$(\bar{z}_i)^{-1} \cdot x' = (\bar{z}_i)^{-1} z_\infty x = (\bar{z}_i)^{-1} \bar{z}_i \delta_i z_i \cdot x = \delta_i z_i \cdot x.$$

Claim(VI): $G_x \subset Z_{l-1}^{x'} (=_{df} \overline{\langle Z_{l-1}, G_{x'} \rangle})$.

Proof of Claim: Fix $h \in G_x$. Then by $z_\infty \in Z_l^x$ there are $z_i' \in Z_l$ and $k_i \in G_x$ for which

$$(z_i')^{-1} (k_i)^{-1} h k_i z_i' \to (z_\infty)^{-1} h z_\infty$$

$$\therefore z_\infty (z_i')^{-1} z_\infty^{-1} z_\infty k_i^{-1} z_\infty^{-1} z_\infty h z_\infty^{-1} z_\infty k_i z_\infty^{-1} z_\infty z_i' z_\infty^{-1} \to h$$

and so since $G_{x'} = z_\infty G_x z_\infty^{-1}$, for some $k_i^*, h^* \in G_{x'}$

$$z_\infty (z_i')^{-1} z_\infty^{-1} k_i^* h^* (k_i^*)^{-1} z_\infty z_i' z_\infty^{-1} \to h;$$

and therefore by normality of Z_l there are $\tilde{z}_i \in Z_l$ so that

$$\tilde{z}_i^{-1} k_i^* h^* (k_i^*)^{-1} \tilde{z}_i \to h;$$

so by the definition of Z_l/Z_{l-1} as the center of G/Z_{l-1} there are $\dot{z}_i \in Z_{l-1}$ with

$$\dot{z}_i k_i^* h^* (k_i^*)^{-1} \to h.$$

(Claim(VI)□)

So: $Z_{l-1}^x \subset Z_{l-1}^{x'}$ and $Z_l^x \subset Z_l^{x'}$.

We achieved

$$y \in \overline{[x']_{Z_{l-1}^x}} \subset \overline{[x']_{Z_{l-1}^{x'}}}$$

just prior to claim(VI), and

$$x' \in \overline{[y]_{Z_l^x}} \subset \overline{[y]_{Z_l^{x'}}}$$

was claim(V). □

COROLLARY 7.26. *Let G be a nilpotent Polish group and X a Polish G-space. Then exactly one of the following holds:*

1. $E_G^X \leq_B \mathrm{id}(\mathbb{R})$;
2. $E_0 \sqsubseteq_c E_G^X$.

PROOF. As at 7.24(B). □

EXERCISE 7.27. (See [**42**].) Let $\mathbb{Z}^{\mathbb{Z}}$ be given the product topology and group structure of pointwise multiplication. Define

$$\psi : \mathbb{Z} \to \mathrm{Hom}(\mathbb{Z}^{\mathbb{Z}})$$

by

$$(\psi(l))(g_k) = g_{k-l}.$$

Let $\mathbb{Z} \ltimes \mathbb{Z}^{\mathbb{Z}}$ be the semi-direct product of $\mathbb{Z}$ and $\mathbb{Z}^{\mathbb{Z}}$ along ψ, so that

$$(n_0, \vec{g}^0)(n_1, \vec{g}^1) = (n_0 + n_1, ((\psi(n_1))(\vec{g}^0))\vec{g}^1).$$

Let $\mathbb{Z} \cup \{\infty\}$ be the one point compactification of $\mathbb{Z}$, and let $\mathbb{Z} \ltimes \mathbb{Z}^{\mathbb{Z}}$ act on $(\mathbb{Z} \cup \{\infty\})^{\mathbb{Z}}$ by

$$((l, \vec{g})(\vec{x}))_k = g_{k+l} + x_{k+l},$$

where $n + \infty$ is always ∞. Then $(\mathbb{Z} \cup \{\infty\})^{\mathbb{Z}}$ is a Polish $\mathbb{Z} \ltimes \mathbb{Z}^{\mathbb{Z}}$-space.

(i) $(..., 0, 0, 0, ...)$ and $(.....0, 0, \infty, 0, 0, ...)$ both have dense orbits.

(ii) The orbit of $(....0, 0, 0, ...)$ is G_δ.

(iii) $\mathbb{Z} \ltimes \mathbb{Z}^{\mathbb{Z}}$ is a solvable non-GE Polish group.

The group $\mathbb{Z} \ltimes \mathbb{Z}^{\mathbb{Z}}$ is (realizable as) a closed subgroup of S_∞, and thus by §6.1 and 7.18 all its orbit equivalence relations are classifiable by countable models. It seems that nothing specific can be said about the forms of the Glimm-Effros dichotomy satisfied by closed subgroups of S_∞.

7.3. Dynamic changes in topologies

For the purpose of §7.5 we work towards a theorem from [**4**] stating that given a Polish G-space X and a Borel set B we can expand the topology on X so that it is still a Polish G-space but $B^{\Delta V}$ now becomes open. The proof below is a little different and provides an exact evaluation of the damage inflicted on the original topology.

Definition 7.28. Let G be a Polish group and d a right invariant metric on G. We let $\mathcal{L}(G,d)$ be the space of all

$$f: G \to \mathbb{R}^{\geq 0}(=_{df} \{r \in \mathbb{R} : r \geq 0\})$$

such that for all $g_0, g_1 \in G$

$$|f(g_0) - f(g_1)| \leq d(g_0, g_1).$$

We equip this with a topology whose subbasic open sets are of the form

$$\{f \in \mathcal{L}(G,d) : f(\hat{g}) < q_1\}$$

and

$$\{f \in \mathcal{L}(G,d) : q_0 < f(\hat{g})\}$$

where $\hat{g} \in G$ and $q_0, q_1 \in \mathbb{Q}^+(=_{df} \{q \in \mathbb{Q} : q > 0\})$. For $g_0 \in G$ and $f_0 \in \mathcal{L}(G,d)$ define $g_0 \cdot f_0$ by the requirement that for all $g \in G$

$$g_0 \cdot f_0(g) = f_0(gg_0);$$

this is well defined since $g_0 \cdot f_0 \in \mathcal{L}(G,d)$, by the right invariance of the metric; it *is* an action since for all $g_0, g_1, g \in G$ and $f \in \mathcal{L}(G,d)$

$$g_1 \cdot (g_0 \cdot f(g))) = g_0 \cdot f(gg_1) = f(gg_1g_0).$$

Lemma 7.29. *$\mathcal{L}(G,d)$ is a Polish G-space.*

Proof. (i) It is a Polish space: Fix $\{g_i : i \in \mathbb{N}\}$ a countable dense subset of G. For $f_0, f_1 \in \mathcal{L}(G,d)$ let

$$d_{\mathcal{L}}(f_0, f_1) = \sum_{i \in \mathbb{N}} 2^{-i}|f_0(g_i) - f_1(g_i)|/(1 + |f_0(g_i) - f_1(g_i)|).$$

This metric generates the original topology, since for all $\hat{g} \in G$ and $f \in \mathcal{L}(G,d)$ and rationals with $q_0 < f(\hat{g}) < q_1$ we can find some $1 > \epsilon > 0$ and $i \in \mathbb{N}$ such that

$$d(g_i, \hat{g}) < \epsilon$$

$$q_0 + 2\epsilon < f(\hat{g}) < q_1 - 2\epsilon.$$

Thus

$$f \in \{\hat{f} \in \mathcal{L}(G,d) : d_{\mathcal{L}}(f, \hat{f}) < \epsilon 2^{-i-1}\} \subset \{\hat{f} \in \mathcal{L}(G,d) : q_0 < f(\hat{g}) < q_1\}.$$

(ii) The action is continuous: It suffices to check this for subbasic open sets. Fix $g, h \in G$ and U including some subbasic open neighbourhood

$$\{\hat{f} \in \mathcal{L}(G,d) : |\hat{f}(h) - g \cdot f(h)| < 2\epsilon\}$$

of some $g \cdot f \in \mathcal{L}(G,d)$. Fix open

$$U_0 = \{\hat{f} \in \mathcal{L}(G,d) : |\hat{f}(hg) - f(hg)| < \epsilon\}$$

and V be an open neighbourhood of g such that

$$\forall \hat{g} \in V(d(h\hat{g}, hg) < \epsilon).$$

Then for all $\hat{g} \in V$ and $\hat{f} \in U_0$ we have

$$|\hat{g} \cdot \hat{f}(h) - g \cdot f(h)| = |\hat{f}(h\hat{g}) - f(hg)|$$

$$\leq |\hat{f}(hg) - \hat{f}(h\hat{g})| + |\hat{f}(hg) - f(hg)| < \epsilon + \epsilon.$$

□

DEFINITION 7.30. Let X be a Polish G-space, d a right invariant metric on G. For $\mathcal{O} \subset X$ open define and $x \in X$ define $f_x^{\mathcal{O}} : G \to [0,1]$ by

$$f_x^{\mathcal{O}}(\hat{g}) = \inf\{d(g,\hat{g})/(1+d(g,\hat{g})) : g \cdot x \in \mathcal{O}\},$$

where this $f_x^{\mathcal{O}}(\hat{g})$ is understood to be 1 when $[x]_G \cap \mathcal{O} = \emptyset$.

LEMMA 7.31. *The function*

$$\pi_{\mathcal{O}} : X \to \mathcal{L}(G,d)$$
$$x \mapsto f_x^{\mathcal{O}}$$

is G-map for all $\mathcal{O} \subset X$ and $x \in X$.

PROOF. Fix $x \in X$ and $\mathcal{O} \subset X$ and $\hat{g}_0, \hat{g}_1 \in G$. Then

$$\hat{g}_0 \cdot f_x^{\mathcal{O}}(\hat{g}_1) = \inf\{d(\hat{g}_1\hat{g}_0, g) : g \cdot x \in \mathcal{O}\} = \inf\{d(\hat{g}_1, g\hat{g}_0^{-1}) : g \cdot x \in \mathcal{O}\}$$

by the right invariance of the metric; this in turn equals

$$= \inf\{d(\hat{g}_1, g) : g \cdot (\hat{g}_0 \cdot x) \in \mathcal{O}\} =_{df} f_{\hat{g}_0 \cdot x}(\hat{g}_1).$$

□

The next definition and lemma formulates the precise extent the topology must be expanded to make the function $\pi_{\mathcal{O}} : x \mapsto f_x^{\mathcal{O}}$ continuous. The main point is that we need only add a collection of F_σ sets – each of the form $B^{\Delta V} =_{df} \{x \in X : \{g \in V : g \cdot x \in B\}$ is non-meager $\}$ for various closed sets in $B \subset X$. While that main, brutal point is all we quote at §7.5, later in chapter 9 we will need a precise determination of the collection of F_σ sets and the more subtle point that a presentation of a basis for the topology may be *effectively produced*: From bases for the topologies on X and G and a dense subgroup of G that all happen to be countable we can definably produce a countable basis for the new topology; the subtle point is that this can be done without first finding a countable enumeration of the basis for X. I will go ahead and present the notation $\mathcal{B}^\sharp$ and $\mathcal{B}^+$ that enables us to keep exact account of the new bases, but with the understanding that we will not draw on such a detailed analysis until 9.

DEFINITION 7.32. Let $\mathcal{B}$ be a basis of X, $\{g_i : i \in \mathbb{N}\} = G_0$ a countable dense subgroup of G, and $\mathcal{B}_0$ be a basis for G including all sets of the form $\{g \in G : d(g, g_i) < q\}$ for $i \in \mathbb{N}$ and $q \in \mathbb{Q}$. Let C be a closed set in X. Then set $\mathcal{B}^\sharp(\mathcal{B}, \mathcal{B}_0, C)$ to consist of all sets of the form

$$(U \cap C^{*V_0} \cap C^{*V_1} \cap ... \cap C^{*V_k})^{\Delta W},$$

where $V_0, V_1, ..., V_k, W$ in $\mathcal{B}_0$.

For $\mathcal{C}$ a collection of closed sets in X, let $\mathcal{B}^+(\mathcal{B}, \mathcal{B}_0, \mathcal{C})$ be the smallest collection of sets including $\mathcal{B}$ and $\{(X \setminus C) : C \in \mathcal{C}\}$, closed under finite intersections and unions, and such that for all $U \in \mathcal{B}^+(\mathcal{B}, \mathcal{B}_0, \mathcal{C})$

$$\mathcal{B}^\sharp(\mathcal{B}^+(\mathcal{B}, \mathcal{B}_0, \mathcal{C}), \mathcal{B}_0, U \setminus X) \subset \mathcal{B}^+(\mathcal{B}, \mathcal{B}_0, \mathcal{C}).$$

Note that this last inclusion is nothing other than the requirement that for all $U_0 \in \mathcal{B}^+(\mathcal{B}, \mathcal{B}_0, \mathcal{C})$ and $V_0, V_1, ..., V_k, W$ in $\mathcal{B}_0$ we have

$$(U_0 \cap C^{*V_0} \cap C^{*V_1} \cap ... \cap C^{*V_k})^{\Delta W} \in \mathcal{B}^+(\mathcal{B}, \mathcal{B}_0, \mathcal{C})$$

for $C = X \setminus U$.

LEMMA 7.33. *Let $\mathcal{B}$ and $\mathcal{B}_0$ be be bases of X and G as in 7.32. Let $\mathcal{O} \subset X$ be open and $C = X \setminus \mathcal{O}$ be the closed set that forms its complement. Then for every subbasic $\hat{U} \subset \mathcal{L}(G,d)$ the pullback $\pi_{\mathcal{O}}^{-1}[\hat{U}]$ is a countable union of elements in $\mathcal{B}^\sharp(\mathcal{B}, \mathcal{B}_0, C)$.*

PROOF. The pullbacks of subbasic open sets of the form

$$\{f \in \mathcal{L}(G,d) : f(g) < q\}$$

are already open in X for $q > 0$ and $g \in G$. On the other hand for subbasic sets of the form

$$\{f_x^{\mathcal{O}}(g) > q\}$$

it suffices as in the proof of 7.29(i) to consider the case that $q \in \mathbb{Q}$ and $g = g_i$ for some $i \in \mathbb{N}$. But then if we have some fixed $x_0 \in X$ with

$$f_{x_0}^{\mathcal{O}}(g_i) > q$$

then there will be some rational $\bar{q} > q$ such that

$$f_{x_0}^{\mathcal{O}}(g_i) \geq \bar{q}$$

which amounts to saying that

$$\forall g \in G(d(g, g_i) < \bar{q} \Rightarrow g \cdot x_0 \in C);$$

by right invariance of d this amounts to asserting that for $V = \{\hat{g} \in G : d(\hat{g}, 1) < \bar{q}\}$

$$\forall g \in Vg_i(gg_i \cdot x_0 \in C);$$

equivalently:

$$x_0 \in C^{*Vg_i}.$$

In short we have seen that for any x_0 in the set

$$\pi^{-1}[\{f_{x_0}^{\mathcal{O}}(g_i) > q\}]$$

we may find a set $C^{*Vg_i} \in \mathcal{B}^\sharp(\mathcal{B}, \mathcal{B}_0, C)$ with

$$x_0 \in C^{*Vg_i} \subset \pi^{-1}[\{f_{x_0}^{\mathcal{O}}(g_i) > q\}].$$

□

Lemma 6.9 implies that all the sets in $\mathcal{B}^\sharp(\mathcal{B}, \mathcal{B}_0, C)$ are certainly no worse than F_σ. Thus the open sets in the topology generated by $\mathcal{B}^\sharp(\mathcal{B}, \mathcal{B}_0, C)$ are all F_σ with respect to the original topology on X.

DEFINITION 7.34. Let X, G, d be as above, $\mathcal{O} \subset X$ open. Let $\hat{X}(\mathcal{O}) = \{(x, f) \in X \times \mathcal{L}(G,d) : f = f_x^{\mathcal{O}}\}$.

LEMMA 7.35. *Let X, G, d be as above, $\mathcal{O} \subset X$ open. Then $\hat{X}(\mathcal{O})$ is a G_δ subset of $X \times \mathcal{L}(G,d)$ (in the product topology).*

PROOF. Fix $\{g_i : i \in \mathbb{N}\}$ a dense subgroup of G. For each $i \in \mathbb{N}$ and $q \in \mathbb{Q}$ the set

$$\{(x, f) \in X \times \mathcal{L}(G,d) : f(g_i) > q > f_{\mathcal{O}}^x(g_i)\}$$

is open by definition and the set

$$\{(x, f) \in X \times \mathcal{L}(G,d) : f(g_i) < q < f_{\mathcal{O}}^x(g_i)\}$$

is F_σ by 7.33. □

LEMMA 7.36. *Let X be a set with τ a Polish topology on X. Suppose that G is a Polish group acting on X, (X, τ) is a Polish G-space, and $C \subset X$ is closed with respect to τ. Then there is a new topology $\hat{\tau}$ so that:*

1. $\tau \subset \hat{\tau}$;
2. $(X, \hat{\tau})$ *is a Polish G-space;*
3. *for each $W \subset G$ open*
$$C^{\Delta W} \in \hat{\tau};$$
4. *every open set in $(X, \hat{\tau})$ is F_σ in (X, τ);*

moreover if $\mathcal{B}$ and $\mathcal{B}_0$ are bases as in 7.32 we may also obtain:

1. $\hat{\tau}$ *has $\mathcal{B}^\sharp(\mathcal{B}, \mathcal{B}_0, C)$ as a basis.*

PROOF. Fix $\{g_i : i \in \mathbb{N}\} = G_0$ a dense subgroup of G and $\mathcal{B}$ a basis of (X, τ). Let $\mathcal{O} = X \setminus C$ and define $(\mathrm{id}, \pi_{\mathcal{O}}) : X \to \hat{X}(\mathcal{O})$ by
$$x \mapsto (x, f^x_{\mathcal{O}}).$$
Let $\hat{\tau}$ be the topology generated by the pullback of open sets in $\hat{X}(\mathcal{O})$.

It is then immediate from the definition of the product topology on $X \times \mathcal{L}(G, d)$ that $\hat{\tau} \supset \tau$. 2 follows from 7.31 and 7.35 and the general fact from 2.3 that G_δ subspaces of Polish spaces are Polish. As for 3, note that for $x \in X$ and $W \subset G$ open
$$x \in C^{\Delta W} \Leftrightarrow \exists g \in W(f^x_{\mathcal{O}}(g) > 0).$$
4 follows from 7.33, since all the sets in $\mathcal{B}^\sharp(\mathcal{B}, \mathcal{B}_0, C)$ are F_σ in X.

For the further statement at 5 we need to do a little more in the defining of $\hat{\tau}$. Let $\vec{\mathcal{O}} = (\mathcal{O}_i)_{i \in \mathbb{N}}$ enumerate all the open sets arising as complements of sets of the form $C^{*V_0} \cap C^{*V_1} \cap ... \cap C^{*V_k}$ where each $V_i \in \mathcal{B}_0$ and then by analogy with above define
$$(\mathrm{id}, \pi_{\mathcal{O}_0}, \pi_{\mathcal{O}_1}, ...) : X \to \hat{X}(\vec{\mathcal{O}})$$
$$x \mapsto (x, f^x_{\mathcal{O}_0}, f^x_{\mathcal{O}_1}, ...)$$
and take the pullback of the topology along $(\mathrm{id}, \pi_{\mathcal{O}_0}, \pi_{\mathcal{O}_1}, ...)$. We obtain that this is a Polish topology on X as in the proof of 2 above. By the argument for 3 above we have enforced that $\mathcal{B}^\sharp(\mathcal{B}, \mathcal{B}_0, C)$ is included in this new topology. Finally to see that the new topology has $\mathcal{B}^\sharp(\mathcal{B}, \mathcal{B}_0, C)$ as a basis note that
$$(C^{*V_0} \cap C^{*V_1} \cap ... \cap C^{*V_k})^{*W} = C^{*V_0 W} \cap C^{*V_1 W} \cap ... \cap C^{*V_k W},$$
and so we can apply 7.33. □

COROLLARY 7.37. *For G, X, $\mathcal{B}$, $\mathcal{B}_0$ as above, $\mathcal{C}$ a countable collection of closed sets, the space X in the topology generated by $\mathcal{B}^+(\mathcal{B}, \mathcal{B}_0, \mathcal{C})$ remains a Polish G-space.*

PROOF. By applying 7.36 countably many times we find sequence of Polish topologies $\tau_0 = \tau \subset \tau_1 \subset ...\tau_i...$ with countable bases $\mathcal{B}^*_0 = \mathcal{B} \subset \mathcal{B}^*_1 \subset ...\mathcal{B}^*_i...$, each closed under finite intersections and unions, each included in $\mathcal{B}^+(\mathcal{B}, \mathcal{B}_0, \mathcal{C})$, such that for each $i \in \mathbb{N}$ and $U \in \mathcal{B}^*_i$, there is some $j > i$ with
$$\mathcal{B}^*_{j+1} \supset \mathcal{B}^\sharp(\mathcal{B}^*_j, \mathcal{B}_0, (X \setminus U))$$
and for each $C \in \mathcal{C}$ there is some k with
$$\mathcal{B}^*_{k+1} \supset \mathcal{B}^*_k \cup \{U \setminus C : U \in \mathcal{B}^*_k\}.$$

By 2.3 we can finish with $\hat{\tau}$ the Polish topology generated by

$$\hat{\mathcal{B}} = \bigcup_{i\in\mathbb{N}} \mathcal{B}_i^*.$$

We do then have that for all $U_0, U \in \mathcal{B}_i^*$, and $V_0, V_1, ..., V_k, W \in \mathcal{B}_0$, if we let $C = X \setminus U$ then there is some $j > i$ with

$$(U_0 \cap C^{*V_0} \cap C^{*V_1} \cap ... \cap C^{*V_k})^{\Delta W} \in \mathcal{B}_j^*.$$

Hence $\mathcal{B}^+(\mathcal{B}, \mathcal{B}_0, \mathcal{C}) \subset \hat{\mathcal{B}}$. We maintained the reverse inclusion at every step of the construction, and thus we have $\mathcal{B}^+(\mathcal{B}, \mathcal{B}_0, \mathcal{C}) = \hat{\mathcal{B}}$, as required. □

THEOREM 7.38. *(Becker, Kechris) Let (X, τ) be a Polish G-space and $B \subset X$ Borel. Then there is a topology $\hat{\tau}$ such that:*

1. $\tau \subset \hat{\tau}$;
2. $(X, \hat{\tau})$ *is a Polish G-space;*
3. *for each $W \subset G$ open*
$$B^{\Delta W} \in \hat{\tau};$$
4. *every open set in $\hat{\tau}$ is Borel in τ.*

PROOF. Let $\mathcal{D}$ be the collection of B for which there is some $\hat{\tau}$ as in the conclusion of the theorem. It includes all open sets and it is closed under countable unions by 2.3(iii). As for complementation note that

$$(X \setminus B)^{\Delta V}$$

is equal to the union over basic open W_0 and W_1 with $W_0 W_1 \subset V$ of the sets

$$((X \setminus B)^{*W_0})^{\Delta W_1}.$$

Thus if we can find a topology $(\tau)^\sharp$ in which each

$$B^{\Delta W_0}$$

becomes open then we may find a topology $(\tau)^{\sharp\sharp}$ in which the Δ Vaught transform of the complement

$$((X \setminus B)^{*W_0})^{\Delta W_1}$$

become open by 7.36. □

EXERCISE 7.39. ([**32**]) For $B \in \underset{\sim}{\Sigma}^0_\alpha(X, \tau)$ at we may tighten part 4 of 7.38
4*. every open set in $\hat{\tau}$ is $\underset{\sim}{\Sigma}^0_\alpha(X, \tau)$.

7.4. Products of locally compact groups

THEOREM 7.40. *Let $G = \prod_{i\in\mathbb{N}} G_i$ in the product topology and group structure with each G_i a locally compact Polish group. Then G is a GE group.*

PROOF. Let X be a Polish G-space and $x, y \in X$ with

$$[x]_G \in G_\delta$$

$$\overline{[x]_G} = \overline{[y]_G} = X.$$

NOTATION 7.41. Let

$$H_l = \{g \in G : \forall n < l (g(n) = 1)\}$$

be the closed normal subgroup of G consisting of all elements that take the identity on coordinates below l. For $x' \in [x]_G$ let

$$H_l^{x'} = \overline{\langle H_l, G_{x'} \rangle}$$

(where as usual $G_{x'}$ indicates $\{g \in G : g \cdot x' = x'\}$ and the overline indicates closure). For

$$g \in \prod_{i \leq k} G_i$$

and

$$h \in \prod_{i > k} G_i$$

let $g^\frown h \in G$ be defined by $(g^\frown h)(n) = g(n)$ for $n \leq k$ but $(g^\frown h)(n) = h(n)$ for $n > k$.

The form of the proof is to inductively obtain sequences of

$$\hat{\delta}_l \in \prod_{i < l} G_i$$

$$h_l^* \in G$$

$$x_l = h_l^* \cdot x_0 = h_l^* \cdot x$$

such that for d^k some fixed right invariant metric on G_k and $l > k$

$$d^k(\hat{\delta}_l(k), 1) < 2^{-l}$$

and such that

$$h_{l+1}^*(h_l^*)^{-1}|_{l+1} = \hat{\delta}_l;$$

this will also be done so

$$y \in \overline{[x_l]_{H_l^{x_l}}},$$

$$d(x_l, y) < 2^{-l}.$$

If we achieve all this the theorem is proved: Since any right invariant metric on a locally compact group is complete so it follows that there exists h_∞ with

$$h_l^*(i) \to h_\infty(i)$$

at each i, and thus

$$h_l^* \to h_\infty$$

and thus

$$h_\infty \cdot x = y.$$

So let us just concentrate on showing we can complete the construction of the

$$x_0, x_1, ..., \hat{\delta}_0, \hat{\delta}_1, ..., h_0^*, h_1^*, ...$$

Assume inductively that we have $x_l \in [x]_G$ and

$$y \in \overline{[x_l]_{H_l^{x_l}}}.$$

Claim(I): $x_l \in \overline{[y]_{H_l^{x_l}}}$.

Proof of claim: Choose $h_i \in G$

$$h_i \cdot y \to x_l$$

and $\hat{g}_i \in H_l^{x_l}$ with

$$\hat{g}_i \cdot x_l \to y.$$

Note that we may choose $g_i \in H_l$ and

$$\epsilon_i \to 1_G,$$

and

$$k_i \in G_{x_l}$$

so that

$$\hat{g}_i = \epsilon_i g_i k_i.$$

Subclaim: $g_i \cdot x_l \to y$.

Proof of subclaim: Then for any open O containing y we may choose open $O_0 \subset O$ containing y and V containing 1_G so that

$$V \cdot O_0 \subset O.$$

Then for for i large enough that $\epsilon_i^{-1} \in V$ and $\hat{g}_i \cdot x_l \in O_0$

$$g_i \cdot x_l = \epsilon_i^{-1} \epsilon_i g_i k_i \cdot x_l = \epsilon_i^{-1} \hat{g}_i \cdot x_l \in O.$$

(Subclaim$\square$)

We may assume

$$h_i g_i \cdot x_l \to x_l.$$

By 7.12 the map

$$G/G_{x_l} \to G \cdot x_l$$
$$gG_{x_l} \mapsto g \cdot x_l$$

is a homeomorphism, so there are $\hat{k}_i \in G_{x_l}$ and

$$\delta_i \to 1_G$$

such that

$$\delta_i h_i g_i = \hat{k}_i.$$

$$\therefore \delta_i h_i = \hat{k}_i (g_i)^{-1} = \bar{g}_i \hat{k}_i$$

some $\bar{g}_i \in H_l$ by normality of H_l.

$$\therefore \delta_i^{-1} \bar{g}_i \hat{k}_i \cdot y = h_i \cdot y \to x_l$$

$$\therefore \bar{g}_i \hat{k}_i \cdot y \to x_l,$$

by the argument of the subclaim. (Claim(I)$\square$)

Let $X_l = \overline{[x_l]_{H_l^{x_l}}}$, the closure of the orbit x_l of under $H_l^{x_l}$. From claim(I), $[y_l]_{H_l^{x_l}}$ is dense in X_l.

Since $H_l^{x_l}/G_{x_l}$ is closed in G/G_{x_l} it follows by 7.12 that $[x_l]_{H_l^{x_l}}$ is relatively closed in $[x_l]_G$, and hence it is a G_δ set; so the map

$$H_l^{x_l}/G_{x_l}$$
$$hG_{x_l} \mapsto h \cdot x_l$$

is a homeomorphism by 7.12 again. Thus letting

$$V \subset \prod_{k \leq l} G_k$$

be an open set with compact closure and letting

$$V_0 = \{h \in H_l^{x_l} : h|_{l+1} \in V\}$$

then

$$\overline{V_0 \cdot x_l} \cap [y]_{H_l^{x_l}} \neq \emptyset.$$

Fix $y_1 \in \overline{V_0 \cdot x_l} \cap [y]_{H_l^{x_l}}$.

Claim(II): There exists $x' \in [x_l]_{H_l^{x_l}}$ with $y_1 \in \overline{H_{l+1} \cdot x'}$.

Proof of claim: By the assumption on y_1 choose $\bar{g}_i = h_i^\frown g_i \in V_0$, so that $h_i \in V$ and $g_i \in \prod_{k>l} G_k$,

$$\bar{g}_i \cdot x_l \to y_1.$$

By the compactness of $\overline{V}$ we may assume that there is some $h \in \bar{V}$ such that

$$h_i \to h.$$

Let $\vec{1} \in \prod_{k>l} G_k$ be the group element that is constantly the identity at every coordinate and let 1^l be the identity element in $\prod_{k \leq l} G_k$. Each $h_i^\frown \vec{1}$ is in $H_l^{x_l}$ since $h_i g_i \in H_{l+1} \subset H_l^{x_l}$. Since $H_l^{x_l}$ is closed

$$h^\frown \vec{1} \in H_l^{x_l}.$$

Since $h_i \to h$ there are

$$\delta_i \to 1_G$$

such that

$$\delta_i(h^\frown \vec{1}) h_i^\frown \vec{1}.$$

Hence

$$\bar{g}_i = \delta_i(h^\frown \vec{1})((1^l)^\frown g_i) = \delta_i((1^l)^\frown g_i)(h^\frown \vec{1})$$

and therefore as in the subclaim from claim(I),

$$((1^l)^\frown g_i)(h^\frown \vec{1}) \cdot x_l \to y_1,$$

and so $(h^\frown \vec{1}) \cdot x_l = x'$ is as required. (Claim(II)$\Box$)

By the assumption on y_1 choose $g \in H_l^{x_l}$ with $g \cdot y = y_1$. Then for $x_{l+1} = g^{-1} \cdot x' \in [x_l]_{H_l^{x_l}}$ we have

$$\begin{aligned} y = g^{-1} y_1 \in g^{-1} \overline{H_{l+1} \cdot x'} &= \overline{g^{-1} H_{l+1} \cdot x'} \\ &= \overline{g^{-1} H_{l+1} g g^{-1} \cdot x'} = \overline{H_{l+1} g^{-1} \cdot x'}, \end{aligned}$$

since H_{l+1} is normal,

$$= \overline{H_{l+1} x_{l+1}},$$

by definition of x_{l+1}.

So we have $x_{l+1} \in [x_l]_{H_l^{x_l}}$ with $y \in \overline{H_{l+1} \cdot x_{l+1}}$ and so certainly

$$y \in \overline{H_{l+1}^{x_{l+1}} \cdot x_{l+1}}.$$

By possibly replacing x_{l+1} by some other point in $H_{l+1} \cdot x_{l+1}$ we may assume that

$$d(x_{l+1}, y) < 2^{-l-1}.$$

By $x_{l+1} \in [x_l]_{H_l^{x_l}}$ fix $\bar{\delta}_i \in \prod_{k<l} G_k$ and $\hat{h}_i \in \prod_{k \geq l} G_k$ with

$$\bar{\delta}_i \to 1$$

$$x_{l+1} = (\bar{\delta}_i^\frown \hat{h}_i) \cdot x_l.$$

Then for i large enough that $\bar{\delta}_i$ close enough to the identity that for all $k \leq l$

$$d^k(\bar{\delta}_i, 1_{G_k}) < 2^{-l-1}$$

we can take

$$\hat{\delta}_l = \overline{\delta_i}$$

$$h^*_{l+1} = (\bar{\delta}_i^\frown \hat{h}_i)h^*_l$$

and continue with the construction. $\square$

THEOREM 7.42. *Let $G = \prod_{i\in\mathbb{N}} G_i$ be a product of locally compact Polish groups. Then G has no turbulent actions.*

PROOF. Let X be a Polish G-space. Assuming every orbit is meager we will attempt to head off part (iii) in the definition of turbulence.

For each n let d^n be a complete right invariant metric on

$$G^n =_{df} \prod_{i\leq n} G_n.$$

As in 7.28 let $\mathcal{L}(G^n, d^n)$ be the Polish G^n-space of function $f : G^n \to \mathbb{R}^{\geq 0}$ with $|f(h) - f(h')|$ always bounded by $d^n(h, h')$. Let

$$X_n = \prod_{\mathbb{N}} \mathcal{L}(G^n, d^n)$$

be the product of $\mathcal{L}(G^n, d^n)$ with itself $\aleph_0$ many times, viewed as a Polish G^n-space in the product topology and G^n-action.

Fixing a countable basis $\{U_m : m \in \mathbb{N}\}$ for X define for each $x \in X$ and $n, m \in \mathbb{N}$ the function

$$f^x_{\langle n,m\rangle} : G^n \to \mathbb{R}$$

$$g \mapsto \inf\{d^n(\hat{g}, g) : \exists \vec{h} \in G(\hat{g} \subset \vec{h}, \hat{g}\cdot x \in U_m)\}.$$

We then define

$$\phi_n : X \mapsto X_n$$

$$x \mapsto (f^x_{\langle n,m\rangle})_{m\in\mathbb{N}}.$$

Claim(I): For all $g \in G^n$ and $\vec{h} \in G$ with $g \subset \vec{h}$ and $x \in X$

$$\phi_n(\vec{h}\cdot x) = g \cdot \phi_n(x).$$

Proof of claim: As in the proof of 7.31. (Claim(I)$\square$)

By Borel or even $\underset{\sim}{\Sigma}^1_1$ sets having Baire property we may find a comeager $C \subset X$ on which each ϕ_n is continuous. Choose $x \in C$ so that $\forall^* g \in G(g\cdot x \in C)$. We will show that x fails (iii) from the definition of turbulence.

Since $[x]_G$ is meager, we may find closed nowhere dense $B_i \subset X$ with

$$[x]_G \subset \bigcup_{i\in\mathbb{N}} B_i.$$

Since the map from G to X given by $g \mapsto g\cdot x$ is continuous we can find some $h \in G$ and basic open neighbourhood $V \subset G$ of 1 so that

$$hV\cdot x \subset B_i,$$

$$\therefore V\cdot x \subset h^{-1}B_i.$$

Note that $h^{-1}B_i$ is still closed nowhere dense.

Since V is basic open it is supported on only finitely many coordinates of G. For notational simplicity assume that there is some open $W \subset G_0$ such that

$$V = \{\vec{g} \in G : g(0) \in W\}.$$

Refining we can find an open neighbourhood $W_0 \subset W$ of the identity in G_0 such that

$$(W_0)^{-1} = W_0$$
$$(W_0)^5 \subset W$$

and then let

$$V_0 = \{\vec{g} \in G : g(0) \in W_0\}.$$

Claim(II): There is some open $O \subset X_0$ containing $\varphi_0(x)$ and such that for all $g \in (W_0)^4$

$$g \cdot \phi_0(x) \in O$$

implies

$$g \cdot \phi_0(x) \in W_0 \cdot \phi_0(x).$$

Proof of claim: (An equivalent point is made in the course of [**49**].) Otherwise let O_n be open sets around $\phi_0(x)$ with $O_{n+1} \subset O_n$ and

$$\{\phi_0(x)\} = \bigcap_{n \in \mathbb{N}} O_n.$$

Then choose $g_i \in (W_0)^4$

$$g_i \cdot \phi_0(x) \in O_i$$
$$\therefore g_i \cdot \phi_0(x) \rightarrow \phi_0(x),$$

but

$$g_i \cdot \phi_0(x) \notin W_0 \cdot \phi_0(x).$$

By compactness of $\overline{W}$ there is some $g_\infty \in G_0$ with

$$g_i \rightarrow g_\infty$$
$$\therefore g_\infty \cdot \phi_0(x) = \phi_0(x).$$

So procure

$$\delta_i \rightarrow 1$$

with

$$\delta_i g_i = g_\infty.$$

Then at i large enough that $(\delta_i)^{-1} \in W_0$ we have

$$g_i \cdot \phi_0(x) = (\delta_i)^{-1} g_\infty \cdot \phi_0(x) \in W_0 \cdot \phi_0(x).$$

(Claim(II)□)

For this O and our comeager C on which all our functions are continuous, choose open $O' \subset X$ so that $\phi_0^{-1}[O] \cap C = O' \cap C$.

Claim(III): For all $z \in \mathcal{O}(x, O', V_0)$

$$\phi_0(z) \in (W_0)^2 \cdot \phi_0(x).$$

Proof of claim. Let $g_i \in V_0$ and $x_i \in O'$ be the witnesses to this, so that $x_0 = x$, $x_{i+1} = g_i \cdot x_i$, and $z = x_k$. We need to show by induction on i that

$$\phi_0(x_i) \in (W_0)^2 \cdot \phi_0(x).$$

Given its truth at i, we may discharge the assumption $\forall^* g \in G(g \cdot x \in C)$ to find $g \in V_0$ such that

$$g \cdot x_{i+1} = g g_i \cdot x_i \in O' \cap C$$
$$\therefore \phi_0(g \cdot x_{i+1}) \in (W_0)^4 \cdot \phi_0(x) \cap O$$
$$\therefore \phi_0(g \cdot x_{i+1}) \in W_0 \cdot \phi_0(x)$$
$$\therefore \phi_0(x_{i+1}) \in (W_0)^2 \cdot \phi_0(x)$$

by claims(I) and (II). (Claim(III)$\Box$)

Let

$$B = \{z \in X : \forall g \in V_0(g \cdot z \in h^{-1}B_i\}.$$

This contains x and is closed. By 2.47 we have a comeager set of y for which

$$\forall^* g \in G(g \cdot y \in (X \setminus h^{-1}B_i));$$

for such y

$$\forall g \in G(g \cdot y \notin B).$$

It suffices to show that for some U' and V' the closure of the local U'-V'−orbit of x does not include any point in $[y]_G$.

Now let y' be in $[y]_G$, and assume for a contradiction that $y' \in \overline{\mathcal{O}(x, O', V_0)}$. Since y' is not in the closed set B we may choose basic open U_m containing y' with U_m and B disjoint. Then for any $z \in \mathcal{O}(x, O', V_0)$ with $z \in U_m$ it follows that

$$f^z_{\langle 0,m\rangle}(1_{G_0}) = 0$$

$$\therefore \exists g \in (V_0)^2(f^{g\cdot x}_{\langle 0,m\rangle}(1_{G_0}) = 0)$$

by claim (III). Thus by definition of this function we may find

$$\epsilon_i \to 1_{G_0}$$

$$h_i \in \prod_{n\geq 1} G_n$$

$$(\epsilon_i^\frown h_i)g \cdot x \in U_m,$$

and so for great enough i

$$\epsilon_i \in W_0$$

and thus

$$(\epsilon_i^\frown h_i)g \in (V_0)^3,$$

with a contradiction to $(V_0)^3 \cdot x \subset B$ and $B \cap U_m = \emptyset$. $\Box$

Corollary 7.43. *Let $G = \prod G_n$ be a countable product of locally compact groups and X a Polish G-space. Then there is a countable language $\mathcal{L}$ and Polish subspace $Y \subset \mathrm{Mod}(\mathcal{L})$ and Borel*

$$\theta : X \to \mathrm{Mod}(\mathcal{L})$$

$$\rho : B \to X$$

such that

1. *θ witnesses $E^X_G \leq_B E^{\mathrm{Mod}(\mathcal{L})}_{S_\infty}$;*
2. *ρ witnesses $E^Y_{S_\infty} \leq_B E^X_G$;*
3. *θ and ρ are inverses up to orbit equivalence, in that for all $x \in X$*

$$\rho \circ \theta(x) E^X_G x.$$

Proof. By 7.42, 7.40, and 6.40. $\Box$

EXERCISE 7.44. Let G be a locally compact Polish group and X a Polish G-space. Show that for all $x \in X$

$$\gamma(x) \leq 1,$$

and thus since locally compact groups are GE

$$E_G^X \leq_B F_2.$$

(But [**49**] has a sharper result: we can find a countable discrete group H and a Polish H-space Y with $E_G^X \leq_B E_H^Y$.)

EXERCISE 7.45. Given a countable partially ordered set $(I, \leq)$, locally compact Polish groups $(G_i)_{i \in I}$ and continuous onto homomorphic maps $\phi_{ij} : G_i \to G_j$ for $i \geq j$ with the commutativity condition that $i \geq j \geq k$ implies $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$, we define the direct limit of $(G_i, \phi_{i,j})_{i \leq j \in I}$ to be the "least" G for which there are continuous onto homomorphisms $\phi_i : G \to G_i$ so that for all $i \geq j$ we have $\phi_{ij} \circ \phi_i = \phi_j$ – least in the sense that given any other H with $\psi_i : H \twoheadrightarrow G_i$ and each $\phi_{ij} \circ \psi_i = \psi_j$ we have $\pi : H \twoheadrightarrow G$ with each $\psi_i = \phi_i \circ \pi$. (Compare §2 [**72**].)

Then G is a closed subgroup of

$$\prod_{i \in I} G_i,$$

and thus 7.43 and 7.18 all its orbit equivalence relations admit classification by countable models.

7.5. CLI groups have the weak Glimm-Effros property

DEFINITION 7.46. A Polish group G is said to have the *weak Glimm-Effros property* if whenever X is a Polish G-space one of the following must hold:

(I) $E_G^X \leq_B \mathrm{id}(\mathbb{R})$;
(II) $E_0 \sqsubseteq_c E_G^X$.

DEFINITION 7.47. A Polish group is said to be *CLI* if it admits a complete left invariant metric.

We could similarly define a group to be *RLI* if it admits a complete right invariant complete metric, but this is an equivalent notion. Given d_l a left invariant complete metric we obtain a right invariant complete metric with

$$d_r(g, h) = d_l(g^{-1}, h^{-1}).$$

An essential step in the proof of Howard Becker's theorem at 7.49 below is to prepare the topology of the Polish G-space. This is expressed at 7.48. Ideally we might like to have a basis for the topology that is closed under all Boolean combinations and Vaught transforms, but this is beyond hope. 7.48(3) instead negotiates an approximation: we will at least have the Δ Vaught transforms of all Boolean combinations of basic open sets.

LEMMA 7.48. *Let G be a Polish group, (X, τ) a Polish G-space, and $\mathcal{B}$ a countable basis for (X, τ) and $\mathcal{B}_0$ a countable basis for G. Then there is a topology $\hat{\tau} \supset \tau$ with basis $\hat{\mathcal{B}}$ such that:*

1. *$\tau \subset \hat{\tau}$ and $\mathcal{B} \subset \hat{\mathcal{B}}$;*
2. *$(X, \hat{\tau})$ is a Polish G-space;*
3. *for all $W_0, W_1 \subset G$ basic open and $U, U' \in \hat{\mathcal{B}}$*

$$((X \setminus U)^{*W_0} \cap U')^{\Delta W_1} \in \hat{\mathcal{B}};$$

4. *every open set in $\hat{\tau}$ is Borel in τ.*

PROOF. $\mathcal{B}^+(\mathcal{B}, \mathcal{B}_0, \emptyset)$ from 7.32 suffices in light of 7.37. □

LEMMA 7.49. *Let G be a Polish group and X a Polish G-space. Let $[x]_G$ be a G_δ orbit. Then for any open neighborhood W of G and $x_0 \in [x]_G$ we may find some open $U' \subset X$ such that:*

1. *$x_0 \in U'$;*
2. *for any other $x_1 \in U' \cap [x]_G$ and any other open U with*
$$[x]_G \cap U \cap U' \neq \emptyset$$
there exists $g \in W$ with
$$g \cdot x_1 \in U.$$

PROOF. Choose $V \subset W$ an open neighborhood of the identity with
$$V^2 \subset W,$$
$$V^{-1} = V.$$
Then by 7.12 we have the function
$$G \to [x]_G$$
$$g \mapsto g \cdot x_0$$
to be open. Thus in particular we have that $V \cdot x_0$ is a relatively open set in $[x]_G$, and thus we may find open $U' \subset X$ with $U' \cap [x]_G = V \cdot x_0$.

Now for any other $x_1 \in U' \cap [x]_G$ we may find some $g_1 \in V$ with
$$g_1 \cdot x_0 = x_1.$$
And if U is an open set with non-empty intersection with
$$U' \cap [x]_G = V \cdot x_0$$
then we may find some $g_2 \in V$ with
$$g_2 \cdot x_0 \in U.$$

Then
$$g_2 g_1^{-1} \cdot x_1 \in U',$$
as required. □

LEMMA 7.50. *Let G be a Polish group and (X, τ) a Polish G-space. Let $\mathcal{B}_0$ be basis for G and $\mathcal{B}$ a basis for (X, τ). Suppose that for all $W_0, W_1 \in \mathcal{B}_0$ and $U, U' \in \cap B$ we have that the set*
$$((X \setminus U)^{*W_0} \cap U')^{\Delta W_1}$$
is τ-open.

Suppose $[x]_G$ is a dense G_δ orbit.

Then for any V an open neighborhood of the identity in G there is open U' containing x such that for all open $U \subset X$ and $y \in U'$
$$U \cap U' \neq \emptyset \Rightarrow (y \in U^{\Delta V}).$$

PROOF. Choose $W \in \mathcal{B}_0$ a basic open neighborhood of the identity with $W^2 \subset V$. Choose U' as in 7.49 for W, and after possibly going to an open subset we may assume $U' \in \mathcal{B}$. Suppose $U \subset X$ is basic open with $U \cap U' \neq \emptyset$; we wish to show that for any $y \in U'$ we have some $g \in V$ with $g \cdot y \in U$.

But if not, then we must have for this y

$$\forall g \in W \forall h \in W (h \cdot (g \cdot y) \in (X \setminus U)),$$

$$\therefore \forall g \in W (g \cdot y \in (X \setminus U)^{*W}).$$

On the other hand, clearly

$$\exists g \in W (g \cdot y \in U').$$

Thus we have

$$y \in ((X \setminus U)^{*W} \cap U')^{\Delta W}.$$

Since this is an open set in our topology on X and since $[x]_G$ is dense, we may find some

$$x_1 \in ((X \setminus U)^{*W} \cap U')^{\Delta W} \cap [x]_G.$$

Then in particular we may choose $g_1 \in W$ with

$$g_1 \cdot x_1 \in (X \setminus U)^{*W} \cap U'.$$

Since U is open this implies that for all $g_2 \in W$

$$g_2 \cdot (g_1 \cdot x) \notin U,$$

which crashes into a contradiction with the assumption on U' and the facts that

$$g_1 \cdot x \in [x]_G$$

and

$$U \cap U' \neq \emptyset.$$

$\square$

COROLLARY 7.51. *Let G be a Polish group with basis $\mathcal{B}_0$ and compatible metric d and (X, τ) a Polish G-space with basis $\mathcal{B}$. Suppose $\mathcal{B}$ has the closure property indicated at 7.48(3): for all $W_0, W_1 \subset G$ basic open and $U, U' \in \hat{\mathcal{B}}$*

$$((X \setminus U)^{*W_0} \cap U')^{\Delta W_1} \in \hat{\mathcal{B}}.$$

Suppose $[x]_G$ is G_δ.

Then there is a sequence of open sets

$$U_{n+1} \subset U_n$$

with

$$\{x\} = \bigcap_{n \in \mathbb{N}} U_n,$$

such that for all n and for all $y \in U_n$ with

$$\overline{[y]_H} = \overline{[x]_H}$$

and for all $U \subset X$ open with $[x]_G \cap U_n \cap U \neq \emptyset$ there exists $g \in G$ such that

$$d(1, g) < 2^{-n}$$

$$g \cdot y \in U.$$

PROOF. Apply 7.50 to

$$\{g \in G : d(1,g) < 2^{-n}\}$$

inside the Polish space

$$\{y \in X : \overline{[y]_G} \subset \overline{[x]_G}\}.$$

□

THEOREM 7.52. *(Becker) Every CLI group has the weak Glimm-Effros property.*

PROOF. Let G be a CLI group and X a Polish G-space. Let d_r be a complete right invariant complete metric on G. Let $\mathcal{B}_0$ be a countable basis for G.

Starting with lemma 7.48 we may assume that X has a countable basis $\mathcal{B}$ such that for all $U, U' \in \mathcal{B}$ and $W_0, W_1 \in \mathcal{B}_0$ the set $((X \setminus U)^{*W_0} \cap U')^{\Delta W_1} \in \mathcal{B}$. Assume that $E_0 \sqsubseteq_c E_G^X$ does not hold. By 7.11 it suffices to show that for all $y \in X$ we have

$$[y]_G \in G_\delta.$$

By the assumption that $E_0 \not\sqsubseteq_c E_G^X$ and 7.8 we can find for all $y \in X$ some $x \in X$ with

$$\overline{[x]_G} = \overline{[y]_G},$$
$$[x]_G \in G_\delta.$$

Let us assume then that $[x]_G \in G_\delta$ and $\overline{[x]_G} = \overline{[y]_G}$ and attempt to show $x E_G^X y$.

Following 7.51 we may find a sequence of $U_n \in \mathcal{B}$ so that

$$\bigcap U_n = \{x\}$$

at each n and for each $y' \in [y]_G \cap U_n$ and $U \subset X$ open with $U_n \cap U \neq \emptyset$ there exists $g \in G$ such that

$$d_r(1,g) < 2^{-n}$$
$$g \cdot y' \in U.$$

We may assume that $d(U_n) < 2^{-n}$ for d the metric on X.

With this in hand we now wish to find a sequence of $y_0 = y, y_1, y_2, ... \in [y]_G$ and $g_0, g_1, g_2, ... \in G$ such that:

(i) $g_i \cdot y_i = y_{i+1}$;
(ii) $y_{i+1} \in U_i$;
(iii) $d_r(g_{i+1}, 1) < 2^{-i}$.

If this can be fulfilled then we are done, since for $h_i = g_i g_{i-1} ... g_0$ we have $(h_i)_{i \in \mathbb{N}}$ is d_r-Cauchy by right invariance and (iii), and so has a limit h_∞. Yet by (ii) we have

$$h_i \cdot y = y_{i+1} \to h_\infty \cdot y = x.$$

So suppose we have found the y_j and g_j for $j \leq i$. Then since $y_i \in U_i$ and $U_i \cap U_{i+1} \cap [x]_G \neq \emptyset$ we can use assumption on the U_i to find some $g_{i+1} \in G$ with

$$g_{i+1} \cdot y_i \in U_{i+1}$$
$$d_r(1, g_{i+1}) < 2^{-i}.$$

□

COROLLARY 7.53. *Every solvable Polish group has the weak Glimm-Effros property.*

PROOF. By 7.52 and 7.19. □

CHAPTER 8

The dark side

Here we take time with the non-Polishable groups and other pathologies. 8.2 gives a proof of the Kechris-Louveau theorem that there is a Borel equivalence relation not Borel reducible to the orbit equivalence relation of a continuous Polish group action. We also recall that there are non-Polishable Borel groups, that there are Borel groups which cannot be realized as Borel subgroups of Polish groups, and that there are even natural examples of abstract groups of cardinality the continuum with no Polish topology compatible with their group structure.

DEFINITION 8.1. For $\vec{x} = (x_0, x_1, x_2, ...), \vec{y} = (y_0, ...) \in \mathbb{R}^{\mathbb{N}}$ set $\vec{x} E_1 \vec{y}$ if and only if there is some $N \in \mathbb{N}$ so that

$$\forall m > N(x_m = y_m).$$

THEOREM 8.2. *(Kechris, Louveau) If G is a Polish group and X is a Polish G-space then*

$$E_1 \not\leq_B E_G^X.$$

PROOF. Suppose

$$\theta : \mathbb{R}^{\mathbb{N}} \to X$$

is Borel such that

$$\forall \vec{x}, \vec{y} \in \mathbb{R}^{\mathbb{N}}(\vec{x} E_1 \vec{y} \Rightarrow \theta(\vec{x}) E_G^X \theta(\vec{y})).$$

For each n let H_n be $\mathbb{R}^n$ in the usual topological and group structure. At each n we have $\mathbb{R}^{\mathbb{N}}$ as a Polish H_n-space in the natural sense: for $\vec{x} \in \mathbb{R}^{\mathbb{N}}$ and $h \in H_n$ define

$$(h \cdot \vec{x})(l)$$

to be $h(l) + x_l$ for $l < n$ and just x_l for $l \geq n$. Moreover

$$E_1 = \bigcup_{n \in \mathbb{N}} E_{H_n}^{\mathbb{R}^{\mathbb{N}}}.$$

Fix d_G a complete metric on G.

Following 2.49 and 3.17 we can find a comeager $C \subset \mathbb{R}^{\mathbb{N}}$ so:

(i) $\theta|_C : C \to X$ is continuous;

(ii) for all W an open neighbourhood of G, $n \in \mathbb{N}$, and $\vec{x} \in C$, there is some open neighbourhood V of the identity in H_n such that

$$\forall^* h \in V(\exists g \in W(\theta(h \cdot \vec{x}) = g \cdot \theta(\vec{x})));$$

(iii) and for notational convenience assume that there are open dense $\mathcal{O}_i \subset \mathbb{R}^{\mathbb{N}}$ such that

$$C = \bigcap_{i \in \mathbb{N}} \mathcal{O}_i.$$

For each n the set

$$C_n =_{df} \{\vec{x} \in \mathbb{R}^{\mathbb{N}} : \forall^* h \in H_n(h \cdot x \in C))\}$$

is comeager by 2.47. There is a detail here: if

$$\vec{y} \in \bigcap_{m \geq n} C_m$$

then

$$\forall h \in H_n (h \cdot \vec{y} \in \bigcap_{m \geq n} C_m).$$

We may now find sequences of $\vec{x} = \vec{x}_0, \vec{x}_1, ...$ in $\mathbb{R}^{\mathbb{N}}$, open $U_0, U_1, ...$ in $\mathbb{R}^{\mathbb{N}}$, and $h_0, h_1, ...$ with each $h_n \in H_n$, and $g_0, g_1, ...$ in G so that at each i

(iv) $\vec{x}_{i+1} = h_i \cdot \vec{x}_i \in U_{i+1} \cap C \cap \bigcap \{C_j : j > i\}$;
(v) $h_i(l) > 0$ for each $l < i$;
(vi) $h_i(l) < 2^{-i}$ for each $l < i$;
(vii) $\overline{U}_{i+1} \subset U_i \subset \mathcal{O}_i$;
(viii) $g_i \cdot \theta(x) = \theta(x_i)$;
(ix) $d_G(g_{i+1}, g_i) < 2^{-i}$.

Once we do all this we are quickly done. By (ix) there exists some g_∞ with

$$g_i \to g_\infty$$

and from (vi) there will be some $\vec{x}_\infty \in \mathbb{R}^{\mathbb{N}}$ with

$$\vec{x}_i \to \vec{x}_\infty.$$

By (v), $\vec{x}_\infty$ is not E_1-equivalent to $\vec{x}_0 = \vec{x}$, whilst (vii) gives

$$\vec{x}_\infty \in C$$

and so by (i)

$$\theta(\vec{x}_i) = g_i \cdot (\theta(\vec{x})) \to \theta(\vec{x}_\infty) = g_\infty \cdot (\theta(\vec{x})),$$

so

$$\theta(\vec{x}) E_G \theta(\vec{x}_\infty).$$

Now to prove such sequences exists. Suppose we have g_j, U_j, $\vec{x}_j$ for $j \leq i$ and h_j for $j < i$. Start by fixing an open neighbourhood W of 1_G so that for all $g \in W$

$$d_G(g_i, gg_i) < 2^{-i}.$$

Then by assumption of $\vec{x}_i \in C$ we may find open V containing the identity of 1_{H_i} so that

$$\forall^* h \in V(\exists g \in W(g \cdot \theta(\vec{x}_i) = \theta(h \cdot \vec{x}_i))).$$

Now we can find $V' \subset V$ open and non-empty so that for all $h \in V'$ and $l \leq i$

$$h(l) > 0$$

$$h(l) < 2^{-i-1}$$

$$h \cdot \vec{x}_i \in U_i.$$

By assumption

$$\vec{x}_i \in \bigcap_{j \geq i} C_j$$

we have that

$$\forall^* h \in V'(h \cdot \vec{x}_i \in C).$$

Choosing $h_i \in V'$ meeting all the relevant comeager sets

$$\vec{x}_{i+1} = h_i \cdot \vec{x}_i \in C$$

and there exists $g \in W$ with

$$g \cdot \theta(\vec{x}_i) = \theta(\vec{x}_{i+1}).$$

Taking $g_{i+1} = gg_i$ and U_{i+1} a sufficiently small open set around $\vec{x}_{i+1}$ continues the construction. $\square$

In particular this gives another proof of:

THEOREM 8.3. *(folklore) There are non-Polishable Borel groups.*

PROOF. Let c_{00} be the subgroup of $\mathbb{R}^{\mathbb{N}}$ consisting of sequences with finite support, so that $\vec{x} \in c_{00}$ if there is some N so that

$$\forall n > N(x_n = 0).$$

If τ was a Polish topology on this group, then the inclusion

$$(c_{00}, \tau) \hookrightarrow \mathbb{R}^{\mathbb{N}}$$

would be continuous by 2.56, and so $\mathbb{R}^{\mathbb{N}}$ would be a Polish c_{00}-space with orbit equivalence relation E_1, violating 8.2. $\square$

THEOREM 8.4. *There is a Borel group that cannot be realized as a Borel subgroup of a Polish group.*

PROOF. We begin with

$$A =_{df} \{(y, x) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}\times\mathbb{N}} : \exists n(y = x(n, \cdot))\}.$$

Let τ_0 and τ_1 be the usual product topologies on $2^{\mathbb{N}}$ and $2^{\mathbb{N}\times\mathbb{N}}$ respectively.

Claim(I): There are no Polish topologies $\hat{\tau}_0$ and $\hat{\tau}_1$ on $2^{\mathbb{N}}$ and $2^{\mathbb{N}\times\mathbb{N}}$ compatible with the usual Borel structure in which A becomes closed in the product.

Proof of claim. Otherwise let $\hat{\tau}_0$ and $\hat{\tau}_1$ be the topologies and consider the Borel injections:

$$i_0 : (2^{\mathbb{N}}, \tau_0) \to (2^{\mathbb{N}}, \hat{\tau}_0),$$
$$i_1 : (2^{\mathbb{N}\times\mathbb{N}}, \tau_1) \to (2^{\mathbb{N}\times\mathbb{N}}, \hat{\tau}_1),$$

and go to comeager $C_0 \subset 2^{\mathbb{N}}$ and $C_1 \subset 2^{\mathbb{N}\times\mathbb{N}}$ on which these maps are continuous. Viewing $2^{\mathbb{N}\times\mathbb{N}}$ as an S_∞-space as at 2.22, where $(g \cdot x)(n, m) = x(g^{-1}(n), m)$, we can use Kuratowski-Ulam to find some $x \in C_1$ so that

$$\forall^* g \in S_\infty(g \cdot x \in C_1)$$

and $[x]_{S_\infty}$ is dense in $2^{\mathbb{N}\times\mathbb{N}}$. But then for $y = x(0, \cdot)$ we have

$$\{y\} \times ([x]_{S_\infty}) \cap C_1 \subset A$$

is dense in $\{y\} \times C_1$. Since A is relatively closed this would assert that for all $\hat{x} \in C_1$

$$\exists n((\hat{x}(n, \cdot) = x(0, \cdot)),$$

which flies in the face of

$$\{\hat{x} \in 2^{\mathbb{N}\times\mathbb{N}} : (\hat{x}(n, \cdot) = x(0, \cdot)\}$$

being meager for each n. (Claim(I)$\square$)

Define $F(2^{\mathbb{N}})$ and $F(2^{\mathbb{N}\times\mathbb{N}})$ to be the free groups on the generating sets $2^{\mathbb{N}}$ and $2^{\mathbb{N}\times\mathbb{N}}$ with the natural Borel structure. These are standard Borel spaces, since we may identify them with a Borel subsets of

$$(2^{\mathbb{N}} \cup (2^{\mathbb{N}})^{-1})^{<\mathbb{N}} \cup \{1\}$$

and

$$(2^{\mathbb{N}\times\mathbb{N}} \cup (2^{\mathbb{N}\times\mathbb{N}})^{-1})^{<\mathbb{N}} \cup \{1\},$$

where $(\cdot)^{-1}$ is understood as providing a formal inverse and $(2^{\mathbb{N}})^{-1}$ and $(2^{\mathbb{N}\times\mathbb{N}})^{-1}$ are viewed as homeomorphic copies of $2^{\mathbb{N}}$ and $2^{\mathbb{N}\times\mathbb{N}}$. For $y \in 2^{\mathbb{N}}$ and $x \in 2^{\mathbb{N}\times\mathbb{N}}$ let

$$\psi_y(x) = x$$

if $(y, x) \in A$ and

$$\psi_y(x) = x^{-1}$$

otherwise. By the defining property of the free group this extends to homomorphism

$$\psi_y : F(2^{\mathbb{N}\times\mathbb{N}}) \to F(2^{\mathbb{N}\times\mathbb{N}}).$$

Thus we have a map

$$\psi : 2^{\mathbb{N}} \to \mathrm{Hom}(F(2^{\mathbb{N}\times\mathbb{N}}))$$

$$y \mapsto \psi_y.$$

Again by the property of being a free group this extends to a homomorphism

$$\psi : F(2^{\mathbb{N}}) \to \mathrm{Hom}(F(2^{\mathbb{N}\times\mathbb{N}})).$$

Let $H = F(2^{\mathbb{N}\times\mathbb{N}}) \rtimes_\psi F(2^{\mathbb{N}})$ be the semi-direct product along ψ. So H will have underlying set $F(2^{\mathbb{N}\times\mathbb{N}}) \times F(2^{\mathbb{N}})$ and group operation

$$(g_0, h_0)(g_1, h_1) = (g_0\psi_{h_0}(g_1), h_0 h_1).$$

Since the set A is Borel the map

$$(g, h) \to \psi_h(g)$$

is Borel and so H is a Borel group. It suffices then to see:

Claim(II) H is not a Borel subgroup of any Polish group.

Proof of Claim: Suppose otherwise and let (G, τ) be such a Polish group. By continuity of multiplication the set

$$\{(g, \hat{g}) \in G \times G : g\hat{g}g^{-1}\hat{g}^{-1} = 1\}$$

is τ-closed. Then by 2.3 we can find $\hat{\tau} \supset \tau$ a richer Polish topology in which

$$B_0 =_{df} \{1\} \times 2^{\mathbb{N}}$$

$$B_1 =_{df} 2^{\mathbb{N}\times\mathbb{N}} \times \{1\}$$

are both closed.

So let $\hat{\tau}_0$ and $\hat{\tau}_1$ be the Polish topologies on $2^{\mathbb{N}}$ and $2^{\mathbb{N}\times\mathbb{N}}$ under their natural identification with the (closed) subspaces B_0 and B_1 of $(G, \hat{\tau})$. But then A in the product topology for $(2^{\mathbb{N}}, \hat{\tau}_0)$ and $(2^{\mathbb{N}\times\mathbb{N}}, \hat{\tau}_1)$ is closed since it can be identified with the set

$$\{(g, h) \in B_0 \times B_1 : g\hat{g}g^{-1}\hat{g}^{-1} = 1\}$$

contradicting claim(I). (Claim(II)$\square$) $\square$

Exercise 8.5. (See 6.2 of [**4**].) Every $\underset{\sim}{\Sigma}^1_1$ equivalence relation is induced by a Borel action of a Borel group.

EXERCISE 8.6. Let B_∞ be the Baire category analogue of M_∞. More precisely, let B_∞ be the class of Borel bijections on $2^{\mathbb{N}}$ which are homeomorphisms on some comeager subset of $2^{\mathbb{N}}$, with identification of π_1 and π_2 whenever

$$\forall^* x \in 2^{\mathbb{N}}(\pi_1(x) = \pi_2(x)).$$

The next few steps indicate that there is no Polish group topology on B_∞ compatible with its *algebraic* structure: Suppose for a contradiction that (B_∞, τ) is a Polish group. Let $\mathcal{B}(B_\infty, \tau)$ be the Borel structure on B_∞ generated by this Polish topology and $\underset{\sim}{\Sigma}^1_1(B_\infty, \tau)$ the resulting collection of $\underset{\sim}{\Sigma}^1_1$ sets.

(i) For $U \subset 2^{\mathbb{N}}$ open let

$$A(U) = \{\pi \in B_\infty : \pi|_U = \mathrm{id}|_U\}.$$

Each such $A(U) \in \mathcal{B}(B_\infty, \tau)$. (For suitably chosen $\{\pi_i : i \in \mathbb{N}\} \in A(X \setminus \overline{U})$ we have

$$\pi \in A(U) \Leftrightarrow \forall i(\pi_i \circ \pi = \pi \circ \pi_i).)$$

(ii) For $U, V \subset 2^{\mathbb{N}}$ open let

$$B(U, V) = \{\pi \in B_\infty : \pi[U] \subset V\}.$$

If U and V are open then the set $B(U, V)$ is $\underset{\sim}{\Sigma}^1_1(B_\infty, \tau)$. (By translating V it suffices to show that $B(U, U) \in \underset{\sim}{\Sigma}^1_1(B_\infty, \tau)$. But if $(W_n)_{n\in\mathbb{N}}$ is a basis for X, then $\pi_0 \in B(U, U)$ if and only if for all n

$$W_n \setminus \overline{U} = \emptyset$$

or

$$\exists \pi_1 \in A(X \setminus \overline{W_n \cup U})(\pi_1 \circ \pi_0 \in A(U)).)$$

(iii) For $\{V_n : n \in \mathbb{N}\}$ a basis of $2^{\mathbb{N}}$ and $U \neq \emptyset, 2^{\mathbb{N}}$ open there will be some $V_n \neq \emptyset, 2^{\mathbb{N}}$ with $B(U, X \setminus \overline{V_n})$ non-meager.

(iv) Then (iii) implies for *any* open $U, V \neq \emptyset, 2^{\mathbb{N}}$ that $B(U, V)$ is non-meager.

(v) Let

$$C(U, V) = \{\pi \in B_\infty : \pi[U] \subset V, \neq V\}$$

and

$$D(U, V) = \{\pi \in B_\infty : \pi[U] = V\}$$

for open $U, V \neq \emptyset, 2^{\mathbb{N}}$. (ii) implies that all these sets have Baire property. Use the separability of τ to conclude that for no such U and V is $D(U, V)$ open.

(vi) Conclude that for each open $U \neq \emptyset, 2^{\mathbb{N}}$ the set $C(U, U)$ is non-meager.

(vii) Now note that $C(U, U)$ is open for U as above. (First from 2.56(iv) we get that $C(U, U)$ contains some non-empty open set, and then that any two elements of $C(U, U)$ may be transformed one to the other by left and right multiplication from $D(U, U)$.)

(viii) For U as above and W_0 and W_1 open, disjoint, and non-empty, and not meeting U, and for $\pi_0, \pi_1 \in B_\infty$ with

$$\pi_i[U] = U \cup W_i,$$

we have that

$$\pi_i[C(U, U)] =_{df} \{\pi_i \circ \pi : \pi \in C(U, U)\} = C(U, U \cup W_i)$$

are open for $i = 0, 1$, and

$$\therefore D(U, U) = C(U, U \cup W_0) \cap C(U, U \cup W_1)$$

is open, to the grief of (v).

CHAPTER 9

Beyond Borel

In various special cases chapter 6 proved the equivalence of *classifiable by countable structures* and *not reducing a turbulent orbit equivalence relation.* Many of the technical difficulties arising there as a result of the desire to work solely in the Borel category vanish when considering reductions that are $\underset{\sim}{\Delta}^1_2$ or even only in $L(\mathbb{R})$; for instance, the process that extracts the code for a Scott sentence from a countable model is $\underset{\sim}{\Delta}^1_2$, as well as universally Baire measurable in the codes, but not Borel. Allowing the most general class of reductions for which we still have regularity properties such as Baire measurability perhaps matches more closely the vague concept of *effective* or *definable* cardinality. In measuring the effective cardinality or classification difficulty of a collection of objects we should presumably be willing to use any reasonably well behaved class of definable functions; we would probably be relatively unconcerned with the minutiae of whether a specific reduction can be arrived at in a certain complexity class.

The present chapter accordingly considers the theory of reductions much more complicated than Borel, but still having a plausible claim on being *reasonable* or *definable*.

Below §9.1 proves general results for both the generic ergodicity and turbulence. The first of these has similarities with recent work on Ulm-type classifications.

§9.2 considers cardinalities in the universe $L(\mathbb{R})$ of all objects generated from the reals by transfinite iterations of definable subsets. These results would seem to be devoid of "practical application", and are included for the philosophical interest. The preference for countable models is given support by a proof that if we can assign any kind of discrete well orderable structures as a complete invariant to E^X_G then we can do so with countable structures.

This chapter will be unreadable without a substantial background in modern set theory.

9.1. Two theorems by transfinite changes in topologies

Definition 9.1. A function

$$\theta : \mathbb{N}^{\mathbb{N}} \to \mathrm{HC}$$

is *absolutely* $\underset{\sim}{\Delta}^1_2$ if there is some parameter $z \in \mathbb{N}^{\mathbb{N}}$ and formula ψ such that:

(I) for all $x \in \mathbb{N}^{\mathbb{N}}$ and $y \in \mathrm{HC}$

$$\theta(x) = y$$

if and only if there is some countable transitive set $\mathcal{M}$ including x, y, and z such that

$$(\mathcal{M}, \in) \models \psi(x, y, z);$$

(II) this *absolutely* defines a function, in the sense that in all generic extensions it continues to be the case that for all $x \in \mathbb{N}^{\mathbb{N}}$ there exists $y \in \mathrm{HC}$ and countable transitive $\mathcal{M}$ with

$$(\mathcal{M}, \in) \models \psi(x, y, z).$$

We can then extend this to functions between arbitrary Polish spaces in the natural manner. For X_1 and X_2 Polish spaces,

$$\sigma_1 : X_1 \to \mathbb{N}^{\mathbb{N}}$$

$$\sigma_2 : X_2 \to \mathbb{N}^{\mathbb{N}}$$

Borel bijections, we may say that $\theta : X_1 \to X_2$ is *absolutely* $\underset{\sim}{\Delta}^1_2$ if the induced

$$\sigma_2 \circ \theta \circ (\sigma_1)^{-1} : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$$

is absolutely $\underset{\sim}{\Delta}^1_2$ in the sense above. This further definition does not depend on the choice of σ_1 and σ_2.

Similarly for X Polish and

$$\sigma : X \to \mathbb{N}^{\mathbb{N}}$$

a Borel bijection, we may say that $\theta : X \to \mathrm{HC}$ is *absolutely* $\underset{\sim}{\Delta}^1_2$ if the induced

$$\theta \circ (\sigma)^{-1} : \mathbb{N}^{\mathbb{N}} \to \mathrm{HC}$$

is absolutely $\underset{\sim}{\Delta}^1_2$.

This definition adapts for functions whose domain is a natural subclass of the hereditarily countable sets. For instance a function

$$\pi : \omega_1 \to \mathrm{HC}$$

is *absolutely* $\underset{\sim}{\Delta}^1_2$ if there is some parameter $z \in \mathbb{N}^{\mathbb{N}}$ and formula ψ such that:

(I) for all countable ordinals α and $y \in \mathrm{HC}$

$$\theta(\alpha) = y$$

if and only if there is some countable transitive model $\mathcal{M}$ including α, y, and z such that

$$\mathcal{M} \models \psi(\alpha, y, z);$$

(II) through all generic extensions it continues to be the case that for all countable α there exists $y \in \mathrm{HC}$ and countable transitive $\mathcal{M}$ with

$$\mathcal{M} \models \psi(\alpha, y, z).$$

For equivalence relations E and F

$$E \leq_{a\Delta^1_2} F$$

indicates there is some absolutely $\underset{\sim}{\Delta}^1_2$ function θ so that for all x_1, x_2

$$x_1 E x_2 \Leftrightarrow \theta(x_1) F \theta(x_2).$$

Similarly $E \leq_{a\Delta^1_2} \mathrm{id}(2^{<\omega_1})$ indicates that there is an absolutely $\underset{\sim}{\Delta}^1_2$ function assigning bounded subsets of ω_1 as complete invariants and $E \leq_{a\Delta^1_2} \mathrm{id}(\mathrm{HC})$ states that we may in an absolutely $\underset{\sim}{\Delta}^1_2$ fashion assign hereditarily countable sets as complete invariants.

THEOREM 9.2. *(folklore) Any absolutely $\underset{\sim}{\Delta}^1_2$ function is Baire measurable.*

PROOF. By the devices of [**78**].

We need to show that the absolutely $\underset{\sim}{\Delta}^1_2$ function is continuous on a comeager set.

It suffices to prove this for functions between uncountable recursive Polish spaces, so let z and ψ be the witnesses to

$$\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$$

absolutely $\underset{\sim}{\Delta}^1_2$ and choose some countable transitive

$$X \prec V_{\omega+\omega}$$

containing z. Form N as the transitive collapse of X. There will be a comeager set of $C \subset 2^{\mathbb{N}}$ of reals Cohen generic over N. For any $x \in C$

$$N[x] \models \exists y, \mathcal{M}(\mathcal{M} \models \psi(x, y, z))$$

and hence

$$\theta(x) = y$$

since $\mathcal{M}$ must be well founded in V.

Now, as follows from the usual development of forcing, for each basic open $U \subset 2^{\mathbb{N}}$ there will be some open $V \subset 2^{\mathbb{N}}$ so that for all $x \in C$

$$N[x] \models \exists y \in U \exists \mathcal{M}(\mathcal{M} \models \psi(x, y, z))$$

if and only if

$$x \in V,$$

for we can simply let V be the union of the conditions forcing that $\theta(x)$ is in U. $\square$

Since absolutely $\underset{\sim}{\Delta}^1_2$ includes Borel and is closed under composition, every absolutely $\underset{\sim}{\Delta}^1_2$ function is universally Baire measurable.

THEOREM 9.3. *(folklore) If*

$$\pi : \omega_1 \to 2^{\mathbb{N}}$$

is absolutely $\underset{\sim}{\Delta}^1_2$, then there are $\alpha < \beta < \omega_1$ with

$$\pi(\alpha) = \pi(\beta).$$

PROOF. (Compare similar arguments in [**20**].) Fix the parameter z and the formula ψ witnessing π absolutely $\underset{\sim}{\Delta}^1_2$. Thus z and ψ will continue to define a function through all generic extensions, and by Shoenfield absoluteness it suffices in some extension to find $\alpha \neq \beta$ with

$$\pi(\alpha) = \pi(\beta).$$

Let $\mathbb{P}$ =Coll$(\omega, (2^{\aleph_0})^+)$. For $\alpha < (2^{\aleph_0})^+$ let τ_α be the term for $\pi(\alpha)$ as calculated in $V^{\mathbb{P}}$, in the sense that

$$\mathbb{P} \Vdash \exists \, \text{transitive} \, \mathcal{M}, \mathcal{M} \models \psi(\alpha, \tau_\alpha[\dot{G}], z).$$

Claim: For all $n \in \mathbb{N}, p_0, p_1 \in \mathbb{P}$,

$$p_0 \Vdash_{\mathbb{P}} (\tau_\alpha[\dot{G}])(n) = 0$$

excludes

$$p_1 \Vdash_{\mathbb{P}} (\tau_\alpha[\dot{G}])(n) = 1.$$

Proof of claim. Otherwise let $G_l, G_r \subset \mathbb{P}$ be mutually generic filters below p_0, p_1 respectively. Then by the definability assumption on π

$$V[G_l] \models \exists\,\text{transitive}\, \mathcal{M}, \mathcal{M} \models \psi(\alpha, \tau_\alpha[G_l], z),$$

then this goes up to its generic extension so that

$$V[G_l, G_r] \models \exists\,\text{transitive}\, \mathcal{M}, \mathcal{M} \models \psi(\alpha, \tau_\alpha[G_l], z),$$

and so

$$V[G_l, G_r] \models \pi(\alpha) = \tau_\alpha[G_l].$$

Similarly on the $V[G_r]$ side we obtain

$$V[G_l, G_r] \models \pi(\alpha) = \tau_\alpha[G_r],$$

and a contradiction to

$$(\tau_\alpha[G_l])(n) = 0 \neq 1 = (\tau_\alpha[G_r])(n).$$

(Claim□)

Thus we obtain for any generic $G \subset \mathbb{P}$ that

$$(\tau_\alpha[G])(n) = 0$$

if and only if there is *some* $p_0 \in \mathbb{P}$ with

$$p_0 \Vdash_{\mathbb{P}} (\tau_\alpha[\dot{G}])(n) = 0.$$

In particular, this element of Cantor space is already in V. Thus by the pigeonhole principle we may find $\alpha \neq \beta < (2^{\aleph_0})^+$ with $\tau_\alpha[G] = \tau_\beta[G]$. □

A nearly identical proof leads to the following result:

THEOREM 9.4. *(folklore) If*

$$\pi : \omega_1 \longrightarrow \mathcal{P}_{\aleph_0}(2^{\mathbb{N}})$$

is absolutely $\underset{\sim}{\Delta}^1_2$*, then there are* $\alpha < \beta < \omega_1$ *with*

$$\pi(\alpha) = \pi(\beta).$$

PROOF. Essentially 9.3 used the claim established in the course of its proof to see that if τ is a $\mathbb{P}$ term for a real, and in the product forcing we have

$$\mathbb{P} \times \mathbb{P} \Vdash \tau[\dot{G}_l] \times \tau[\dot{G}_r],$$

where $\dot{G}_l \times \dot{G}_r$ names the generic object on the product forcing notion, then there is some a in the ground model with

$$\mathbb{P} \Vdash a = \tau[\dot{G}].$$

However one can easily prove by transfinite induction on the rank of the terms involved that whenever

$$(p_1, p_2) \Vdash_{\mathbb{P}\times\mathbb{P}} \tau_1[\dot{G}_l] \times \tau_2[\dot{G}_r]$$

then there will be some $a \in V$ with $\mathbb{P} \Vdash a = \tau[\dot{G}]$. The point is that if

$$(p_1, p_2) \Vdash_{\mathbb{P}\times\mathbb{P}} \tau_1[\dot{G}_l] \times \tau_2[\dot{G}_r],$$

and σ_1 is a term for an element of τ_1, then there will be some σ_2, a term for an element of τ_2, and $q_1 \leq p_1$, $q_2 \leq p_2$ with

$$(q_1, q_2) \Vdash_{\mathbb{P}\times\mathbb{P}} \sigma_1[\dot{G}_l] \times \sigma_2[\dot{G}_r],$$

and since we may assume that the terms σ_1 and σ_2 have lower rank than τ_1 and τ_2 we can apply the inductive hypothesis. □

THEOREM 9.5. *Let G be a Polish group and X a Polish G-space. Then exactly one of the following holds:*

1. $E_G^X \leq_{a\Delta_2^1} \mathrm{id}(2^{<\omega_1})$;
2. *there is a properly generically ergodic Polish G-space Y and a continuous G-embedding $\pi : Y \to X$.*

PROOF. The incompatibility of 1. and 2. is a known result, and in some form can be found in [**20**] or [**37**]. From the conjunction of the two and 9.2 we obtain a comeager set $C \subset Y$ and a continuous in the codes reduction on C to $\mathrm{id}(2^{<\omega_1})$. Using the boundedness principle for $\underset{\sim}{\Sigma}_1^1$, there will be a single $\delta < \omega_1$ for which the reduction on C provides subsets of γ as complete invariants. Since γ is countable this gives a Baire measurable reduction of the orbit equivalence relation E_G^Y to $\mathrm{id}(\mathbb{R})$, contradicting 3.3.

Instead we concentrate on showing *at least* one to be true. The technique is to define a transfinite sequence of topologies around any orbit in the space. The proof resembles the method of §6.2. The process will be canonical and in some sense absolutely $\underset{\sim}{\Delta}_2^1$. Using 9.3 it will be seen that for any point the transfinite process must terminate. We will then obtain a dichotomy based on whether the topology terminates with the subjected orbit a dense G_δ. If for every orbit it is so, then we will be able to produce a bounded subset of ω_1 as a code; if for some $[x]_G$ not we obtain the set of points with the same closure as $[x]_G$ forming a properly generically ergodic Polish G-space.

As in 7.32 fix a basis $\mathcal{B} = \{U_l : l \in \mathbb{N}\}$ of X, $\{g_i : i \in \mathbb{N}\} = G_0$ a countable dense subgroup of G, $\mathcal{B}_0 = \{\hat{V}_k : k \in \mathbb{N}\}$ a countable basis for G including all sets of the form $\{g \in G : d(g, g_i) < q\}$ for $i \in \mathbb{N}$ and $q \in \mathbb{Q}$, and $\{V_n : n \in \mathbb{N}\}$ a neighborhood basis at 1_G with $V_n = (V_n)^{-1}$ all n.

Focusing on some particular $x \in X$ we assign to each countable ordinal α a basis $\mathcal{B}_\alpha^x$ of a Polish topology τ_α^x on X. We will also produce Polish spaces $(X_\alpha^x, \tau_\alpha^x)$ in which every orbit is dense.

From $\tau_{\alpha+1}^x$ we let $X_{\alpha+1}^x$ be the set of $y \in X_\alpha^x$ whose orbit has the same $\tau_{\alpha+1}^x$ closure as $[x]_G$. At λ at limit we take

$$X_\lambda^x = \bigcap_{\alpha<\lambda} X_\alpha^x.$$

So in particular for $\alpha_1 < \alpha_2$ the space $X_{\alpha_1}^x$ includes $X_{\alpha_2}^x$.

We will maintain inductively for $\alpha_1 < \alpha_2$ that

$$\mathcal{B}_{\alpha_1}^x \subset \mathcal{B}_{\alpha_2}^x$$

and at each α

$$(X, \tau_\alpha^x) \text{ is a Polish } G-\text{space}$$

and mandate the following closure on $\mathcal{B}_\alpha^x$: for all $W_0, W_1 \in \mathcal{B}_0$ and $U, \hat{U} \in \mathcal{B}_\alpha^x$

$$((X \setminus U)^{*W_0} \cap \hat{U})^{\Delta W_1} \in \mathcal{B}_\alpha^x.$$

We may assume that the basis $\mathcal{B}$ has these closure properties by 7.48 and so just let the process begin with

$$\mathcal{B}_0^x = \mathcal{B}.$$

At limit stages we take

$$\mathcal{B}^x_\lambda = \bigcup_{\alpha<\lambda} \mathcal{B}^x_\alpha.$$

Note that at limit λ every $y \in X^x_\lambda$ has orbit with the same τ^x_λ-closure as $[x]_G$.

All the work happens at successor steps and requires a split into cases. For that purpose at each $n \in \mathbb{N}$ let $O_{x,n,\alpha}$ be the union of all open $U \in \mathcal{B}^x_\alpha$ such that for all $z \in X^x_\alpha \cap U$ and $\hat{U} \in \mathcal{B}^x_\alpha$ with $\hat{U} \cap U \neq \emptyset$

$$\exists g_i \in V_n(g_i \cdot z \in \hat{U} \cap U));$$

note there that $O_{x,n,\alpha}$ is defined by a $\underset{\sim}{\Pi}^1_1$ union of basic open sets in $\mathcal{B}^x_\alpha$.

Case(A): For some n the open set $O_{x,n,\alpha}$ is not dense in $(X^x_\alpha, \tau^x_\alpha)$.

Then let

$$\mathcal{B}^x_{\alpha+1} = \mathcal{B}^x_\alpha;$$

from which it follows that for all $\gamma > \alpha$

$$\mathcal{B}^x_\gamma = \mathcal{B}^x_\alpha.$$

Claim(1): There is no non-meager orbit in $(X^x_\alpha, \tau^x_\alpha)$.

Proof of claim: Suppose instead that $[y]_G$ is a non-meager orbit in $(X^x_\alpha, \tau^x_\alpha)$. By 7.12 $[y]_G$ is G_δ. Following 7.50 applied to the G_δ orbit $[y]_G$ we may find around each $y' \in [y]_G$ a basic open set in $U_{y'}$ in $(X^x_\alpha, \tau^x_\alpha)$ so that for all $z \in U_{y'}$ and $\hat{U} \in \mathcal{B}^x_\alpha$ with

$$U_{y'} \cap \hat{U} \neq \emptyset$$

there will exist $g \in V_n$ with

$$g \cdot z \in U_{y'} \cap \hat{U}.$$

Since $[y]_G$ is dense in $(X^x_\alpha, \tau^x_\alpha)$ by the definition of the space, we have a contradiction to case assumption. (Claim(1)$\square$)

So by 7.7 we are in alternative 2. of the theorem with $Y = (X^x_\alpha, \tau^x_\alpha)$.

Case(B): For each n the set $O_{x,n,\alpha}$ is dense in $(X^x_\alpha, \tau^x_\alpha)$ but $[x]_G$ is not a G_δ set in $(X^x_\alpha, \tau^x_\alpha)$.

Claim(2) There is some n with

$$x \in (X \setminus O_{x,n,\alpha})^{\Delta G}.$$

Proof of claim: Since $[x]_G$ is meager we may find a sequence $(C_k)_{k\in\mathbb{N}}$ of τ^x_α closed nowhere dense sets in $(X^x_\alpha, \tau^x_\alpha)$ so that

$$[x]_G \subset \bigcup_{k\in\mathbb{N}} C_k.$$

Thus we may find a basic open V containing 1_G and $x' \in [x]_G$ such that for some k

$$\forall g \in V(g \cdot x' \in C_k).$$

Choose n so that

$$(V_n)^2 \subset V$$

and then it follows from the nowhere density of C_k and the definition of $O_{x,n,\alpha}$ that for all $g \in V_n$

$$g \cdot x' \in X \setminus O_{x,n,\alpha}.$$

(Claim(2)$\square$)

Now take
$$\mathcal{B}^x_{\alpha+1} = \mathcal{B}^+(\mathcal{B}^x_\alpha, \mathcal{B}_0, \{X \setminus O_{x,n,\alpha} : n \in \mathbb{N}\}).$$
The definition of the $\mathcal{B}^+$ operation will allow us to choose canonically from a wellordering of $\mathcal{B}^x_\alpha$ a wellordering of $\mathcal{B}^x_{\alpha+1}$; each $O_{x,n,\alpha}$ can be thought of as being labeled by an ordinal α and a natural number n.

Case(C) For each n the set $O_{x,n,\alpha}$ is dense in $(X^x_\alpha, \tau^x_\alpha)$ and $[x]_G$ *is* a G_δ set in $(X^x_\alpha, \tau^x_\alpha)$.

We then simply set $\mathcal{B}^x_{\alpha+1} = \mathcal{B}^x_\alpha$, from which for all $\gamma > \alpha$
$$\mathcal{B}^x_\gamma = \mathcal{B}^x_\alpha.$$

This completes the description of $\mathcal{B}^x_{\alpha+1}$ from $\mathcal{B}^x_\alpha$.

Claim(3): For each $x \in X$ there is some $\alpha < \omega_1$ with
$$\mathcal{B}^x_{\alpha+1} = \mathcal{B}^x_\alpha.$$

Proof of claim: Suppose otherwise. Then at every $\alpha < \omega_1$ we are in case (B) and $[x]_G$ is meager in $(X^x_\alpha, \tau^x_\alpha)$.

From claim(2) there will be some basic open neighbourhood of the identity, $V_m \in \mathcal{B}_0$, and $x_0 \in [x]_G$ so that
$$x_0 \in (X \setminus O_{x,n,\alpha})^{*V_m}.$$
Then choose $m(0), m(1)$ so that
$$V_{m(1)}(V_{m(1)})^{-1} \subset V_{m(0)}$$
$$V_{m(0)}V_{m(1)} \subset V_m.$$

Subclaim(3a): $x_0 \in ((X \setminus O_{x,n,\alpha})^{*V_{m(0)}})^{*V_{m(1)}}$

Proof of subclaim: Otherwise there is some $g \in V_{m(1)}$ with
$$g \cdot x_0 \in (O_{x,n,\alpha})^{\Delta V_{m(0)}}$$
$$\therefore x_0 \in (O_{x,n,\alpha})^{\Delta V_{m(0)}g} \subset (O_{x,n,\alpha})^{\Delta V_m}.$$
(Subclaim(3a)$\Box$)

Thus in particular $x_0 \in ((X \setminus O_{x,n,\alpha})^{*V_{m(0)}})^{\Delta V_{m(1)}}$.

Subclaim(3b): $((X \setminus O_{x,n,\alpha})^{*V_{m(0)}})^{\Delta V_{m(1)}} \subset (X \setminus O_{x,n,\alpha})^{*V_{m(1)}} \subset X \setminus O_{x,n,\alpha}$.

Proof of subclaim: The second inclusion follows by the continuity of the action with respect to τ^α_x. For the first, note that if $z \in (O_{x,n,\alpha})^{\Delta V_{m(1)}}$ then for *all* $g \in V_{m(1)}$
$$g \cdot z \in (O_{x,n,\alpha})^{\Delta V_{m(1)}g^{-1}} \subset (O_{x,n,\alpha})^{\Delta V_{m(0)}}.$$
(Subclaim(3b)$\Box$)

Thus the set $\mathcal{U}_{x,n,m(0),m(1),\alpha}$ of $z \in (X^x_\alpha, \tau^x_\alpha)$ for which
$$z \in ((X \setminus O_{x,n,\alpha})^{*V_{m(0)}})^{\Delta V_{m(1)}}$$
becomes open in $\tau^x_{\alpha+1}$ and has non-empty intersection with $[x]_G$; note that
$$\mathcal{U}_{x,n,m(0),m(1),\alpha} \cap [x]_G$$
is not an open set in $([x]_G, \tau^x_\alpha)$ by subclaim 3(b) and density of the open $O_{x,n,\alpha}$.

Let $n(x,\alpha), m(0,x,\alpha), m(1,x,\alpha)$ be least such that
$$\mathcal{U}_{x,n(x,\alpha),m(0,x,\alpha),m(1,x,\alpha),\alpha} \cap [x]_G$$

not open in $([x]_G, \tau^x_\alpha)$. Then let

$$\hat{\pi}_x(\alpha) = \{l \in \mathbb{N} : \hat{V}_l \cdot x \subset \mathcal{U}_{x,n(x,\alpha),m(0,x,\alpha),m(1,x,\alpha),\alpha}\}.$$

The set of $g \in G$ with

$$g \cdot x \in \mathcal{U}_{x,n(x,\alpha),m(0,x,\alpha),m(1,x,\alpha),\alpha}$$

is open by $(X^x_\alpha, \tau^x_{\alpha+1})$ being a Polish G-space. The $\hat{\pi}_x$ is absolutely $\underset{\sim}{\Delta}^1_2$ since $\hat{\pi}_x(\alpha)$ is uniformly definable from α over the $\alpha+1$th admissible structure containing x and an appropriate code for G, X, and the action of G on X.

Subclaim(3c): For $\beta < \alpha$ we have $\hat{\pi}_x(\alpha) \neq \hat{\pi}_x(\beta)$.

Proof of subclaim: As remarked above, $\mathcal{U}_{x,n,m(0),m(1),\gamma} \cap [x]_G$ is τ^x_δ-open if and only if $\gamma < \delta$. This gives that

$$\mathcal{U}_{x,n,m(0),m(1),\alpha} \cap [x]_G = \bigcup_{l \in \hat{\pi}_x(\alpha)} (\hat{V}_l \cdot x),$$

$$\mathcal{U}_{x,n,m(0),m(1),\beta} \cap [x]_G = \bigcup_{l \in \hat{\pi}_x(\beta)} (\hat{V}_l \cdot x),$$

and

$$\mathcal{U}_{x,n,m(0),m(1),\alpha} \cap [x]_G \neq \mathcal{U}_{x,n,m(0),m(1),\beta} \cap [x]_G.$$

Thus $\hat{\pi}_x(\alpha) \neq \hat{\pi}_x(\beta)$. (Subclaim(3c)$\Box$)

This then gives a contradiction to 9.3. (Claim(3)$\Box$)

So it terminates. If the process ever finishes in case(A) then we are done and have 2. Instead suppose it to unfailingly finish in (C). We will produce the absolutely $\underset{\sim}{\Delta}^1_2$ reduction. I will only concentrate on formally defining

$$\theta : X \to 2^{<\omega_1}$$

and not verifying that it has the correct level of complexity. This last point is similar to say the proof of $L \cap$HC having a Δ^1_2 in the codes wellordering.

Fix $x \in X$ and let $\alpha(x)$ be the ordinal at which for first time $\mathcal{B}^x_{\alpha(x)} = \mathcal{B}^x_{\alpha(x)+1}$; by case assumption $[x]_G$ is a dense G_δ in $(X^x_{\alpha(x)}, \tau^x_{\alpha(x)})$. For each $\alpha \leq \alpha(x)$ we describe a countable set $A(x,\alpha)$ of countable ordinals, an ordinal $\gamma(x,\alpha) =_{df} \sup A(x,\alpha)$, and an onto function

$$F^x_\alpha : A(x,\alpha) \twoheadrightarrow \mathcal{B}^x_\alpha.$$

The function only depends on the orbit equivalence class of x and for $\alpha < \beta$

$$F^x_\alpha \subset F^x_\beta.$$

At limit stages take unions and at $\alpha = 0$ just begin with $\gamma(x,0) = \omega$ and a function provided by some countable enumeration of $\mathcal{B}$. All the content in the definition takes place at successor steps.

So suppose that $F^x_\alpha : A(x,\alpha) \to \mathcal{B}^x_\alpha$ has been defined and we wish to continue to $F^x_{\alpha+1}$. We define

$$F^x_\alpha = F^x_{\alpha,0} \subset F^x_{\alpha,1} \subset F^x_{\alpha,2} \subset \dots$$

with $A(x,\alpha,i)$ the the domain of $F^x_{\alpha,i}$, $\gamma(x,\alpha,i)$ the supremum of $A(x,\alpha,i)$, and take

$$F^x_{\alpha+1} = \bigcup_{i \in \mathbb{N}} F^x_{\alpha,i}$$

$$A(x,\alpha+1) = \bigcup_{i \in \mathbb{N}} A(x,\alpha,i).$$

Fix some Gödel pairing function

$$s : \omega_1^{<\omega} \to \omega_1$$

which is injective and uniformly Δ_1 inside any admissible set. Given any $n \in \mathbb{N}$ let

$$F^x_{\alpha,1}(\gamma(x,\alpha,0) + s(\langle n \rangle)) = O_{x,n,\alpha}.$$

We continue this at the later stages, making room for the Δ Vaught transforms, intersections and unions. If $U, U' \in \mathcal{B}^x_{\alpha+1}$ with

$$F^x_{\alpha,i}(\delta) = U$$

$$F^x_{\alpha,i}(\delta') = U'$$

let

$$F^x_{\alpha,i+1}(\gamma(x,\alpha,i) + s(\langle 0, \delta, \delta' l(0), ..., l(k), l \rangle)) =$$
$$U \cap (X \setminus U')^{*W_{l(0)}} \cap ... \cap (X \setminus U')^{*W_{k(l)}})^{\Delta W_l}$$
$$F^x_{\alpha,i+1}(\gamma(x,\alpha,i) + s(\langle 1, \delta, \delta' \rangle)) = U \cap U'$$

and

$$F^x_{\alpha,i+1}(\gamma(x,\alpha,i) + s(\langle 2, \delta, \delta' \rangle)) = U \cup U'.$$

We take $A(x, \alpha, i+1)$ to be the union of $A(x, \alpha, i)$ and the new ordinals defined to be in the domain of $F^x_{\alpha,i+1}$ by the above.

Finally we need to describe

$$\theta(x) \in 2^{<\omega_1}.$$

But we can let $\theta(x)$ be the set of $s(\langle \alpha, \delta \rangle)$ such that $\alpha \leq \alpha(x)$ and $\delta \in A(x, \alpha)$ and for

$$U = F^x_\alpha(\delta)$$

in the space $(X^x_\alpha, \tau^x_\alpha)$ we have

$$U \cap [x]_G \neq \emptyset.$$

From the definition of these objects the assignment

$$x \mapsto \theta(x)$$

is an invariant of the orbit. We do need to worry about some other E^X_G-inequivalent $y \in X$ receiving the same invariant, but any such $[y]_G$ cannot also be a dense G_δ in

$$(X^x_{\alpha(x)}, \tau^x_{\alpha(x)});$$

whence it is either not dense in some

$$(X^x_\alpha, \tau^x_\alpha)$$

for $\alpha \leq \alpha(x)$ or else

$$\alpha(x) < \alpha(y).$$

In either case

$$\theta(x) \neq \theta(y).$$

□

Corollary 9.6. *(Becker; Hjorth, Kechris) Let G be a Polish group and X a Polish G-space. Then exactly one of the following holds:*

1. $E^X_G \leq_{a\Delta^1_2} \mathrm{id}(2^{<\omega_1})$;
2. $E_0 \sqsubseteq_c E^X_G$.

Proof. By 7.7 and 9.5. □

The original proof of 9.6 can be found in [**37**], bears no resemblance to 9.5, and fails to give the further information obtained above. On the other hand this earlier argument showed that in 9.6(1.) we can draw the stronger conclusion that there is a *C-measurable* reduction to $\mathrm{id}(2^{<\omega_1})$. [**37**] also mentions that we may not in general replace 1. above by $E_G^X \leq_{a\Delta_2^1} \mathrm{id}(\mathbb{R})$ or even $E_G^X \leq_{L(\mathbb{R})} \mathrm{id}(\mathbb{R})$.

THEOREM 9.7. *Let G be a Polish group and X a Polish G-space. Then exactly one of the following holds:*

1. $E_G^X \leq_{a\Delta_2^1} \mathrm{id}(\mathrm{HC})$;
2. *there is a turbulent Polish G-space Y and a continuous G-embedding $\pi : Y \to X$.*

PROOF. This is near to the proof of 9.5 so we should concentrate on the differences. At most one of the alternatives above can hold by 3.18.

For establishing at least one, we again define a transfinite sequences of topologies τ_x^α with bases $\mathcal{B}_\alpha^x$ so that for $\alpha_1 < \alpha_2$

$$\mathcal{B}_{\alpha_1}^x \subset \mathcal{B}_{\alpha_2}^x.$$

Again it will stop the moment $[x]_G$ becomes G_δ in τ_α^x, and again (I) corresponds to every x thus ending as a G_δ. Again at limits

$$\tau_\lambda^x = \bigcup_{\alpha<\lambda} \tau_\alpha^x,$$

$$X_\lambda^x = \bigcap_{\alpha<\lambda} X_\alpha^x.$$

Again we will maintain inductively that if $W_0, W_1 \subset G$ are basic open sets with $U, U' \in \mathcal{B}_\alpha^x$ then

$$((X \setminus U)^{*W_0} \cap U')^{\Delta W_1} \in \mathcal{B}_\alpha^x.$$

The change is in the manufacture of $\tau_{\alpha+1}^x$ from τ_α^x. It now becomes a multi step process, with an intermediate stage given by the generalized Scott analysis.

From τ_α^x we produce $\tau_{\alpha,0}^x$ as an intermediate topology according to 6.31. Letting $\varphi_x^{\tau_\alpha^x}$ be the generalized Scott sentence calculated in $(X_\alpha^x, \tau_\alpha^x)$ for x, we define the intermediate space

$$X_{\alpha,0}^x =_{df} \{y \in X_\alpha^x : \varphi_x^{\tau_\alpha^x} = \varphi_y^{\tau_\alpha^x}\}.$$

We then take

$$\tau_{\alpha,0}^x =_{df} \tau(\varphi_x^{\tau_\alpha^x})$$

to be the topology with basis

$$\mathcal{B}_{\alpha,0}^x = \mathcal{B}_\alpha^x \cup \{U(\varphi_{\gamma(x)}^{\tau_\alpha^x}(\hat{x}, U, V_n)) : n \in \mathbb{N}, U \in \mathcal{B}_\alpha^x, \hat{x} \in [x]_G \cap U\} \supset \mathcal{B}_\alpha^x,$$

where $\gamma(x)$ is defined as at 6.23 in the space $(X_\alpha^x, \tau_\alpha^x)$, using the basis $\mathcal{B}_\alpha^x$, and

$$U(\varphi_\beta^{\tau_\alpha^x}(\hat{x}, U, V_n)) = \{z \in X_\alpha^x \cap U : \varphi_\beta^{\tau_\alpha^x}(\hat{x}, U, V_n) = \varphi_\beta^{\tau_\alpha^x}(z, U, V_n)\}$$

for any $\beta < \omega_1$.

Case(A): There is no G_δ orbit in $(X_{\alpha,0}^x, \tau_{\alpha,0}^x)$.

Thus by 6.37 and 6.35 we have that the action of G on $(X_{\alpha,0}^x, \tau_{\alpha,0}^x)$ is turbulent. Since the topology $\tau_{\alpha,0}^x$ is richer than the original topology we have that some turbulent orbit equivalence relation allows a continuous G-embedding into X and we finish.

Case(B): There is a G_δ orbit in $(X^x_{\alpha,0}, \tau^x_{\alpha,0})$.

Case(B1): $[x]_G$ is that G_δ orbit.
Then we at once stop the process. We let $\mathcal{B}^x_{\alpha+1} = \mathcal{B}^x_{\alpha,0}$, and for all $\gamma > \alpha$

$$\mathcal{B}^x_\gamma = \mathcal{B}^x_{\alpha+1}.$$

Case(B2): $[x]_G$ is not a G_δ orbit.

Let $y \in X^x_{\alpha,0}$ be the point with a G_δ orbit. For each $n \in \mathbb{N}$ and $y' \in [y]_G$ using 7.12 as 7.49 we may choose some basic open (in the $\tau^x_{\alpha,0}$ topology) set $U_{y',n} \in \mathcal{B}^x_{\alpha,0}$ containing y' such that for all $y_0 \in [y]_G$ and open U with $U \cap U_{y',n}$ non-empty (in $X^x_{\alpha,0}$) we have

$$\exists g \in V_n(g \cdot y_0 \in U).$$

Claim(1): For $U_{y',n}$ as above and U open with

$$U \cap U_{y',n} \neq \emptyset,$$

and for $W_0, W_1 \subset G$ in $\mathcal{B}_0$ with

$$V_n W_1 W_1^{-1} \subset W_0$$

we have that the set

$$((X^x_{\alpha,0} \setminus U)^{*W_0} \cap U_{y',n})^{\Delta W_1}$$

is not relatively open in $[x]_G$.

Proof of claim: Suppose instead $U' \in \mathcal{B}^x_{\alpha,0}$ has non-empty intersection with $[x]_G$ and

$$[x]_G \cap U' \subset ((X^x_{\alpha,0} \setminus U)^{*W_0} \cap U_{y',n})^{\Delta W_1}.$$

Then since $[y]_G$ is dense we may find $y_1 \in [y]_G \cap U' \cap (U_{y',n})^{\Delta W_1}$. Thus in particular we may find some $g_1 \in W_1$ with

$$g_1 \cdot y_1 \in U_{y',n}.$$

By assumption on $U_{y',n}$ we can find $g_2 \in V_n$ with

$$g_2 \cdot (g_1 \cdot y_1) \in U.$$

Thus we have

$$y_1 \in U^{\Delta V_n W_1}.$$

Thus by density of the orbit $[x]_G$ in $X^x_{\alpha,0}$ we have that there is some $x_1 \in [x]_G$ with

$$x_1 \in U' \cap U^{\Delta V_n W_1}.$$

By assumptions on U' we may find $h_1 \in W_1$ with

$$h_1 \cdot x_1 \in (X^x_{\alpha,0} \setminus U)^{*W_0}.$$

But by assumption on x_1 we must also have

$$h_1 \cdot x_1 \in U' \cap U^{\Delta V_n W_1 h_1^{-1}} \subset U^{\Delta V_n W_1 W_1^{-1}},$$

with a contradiction to $V_n W_1 W_1^{-1} \subset W_0$. (□Claim)

Now as in the proof of 9.5 we let $O_{x,n,\alpha}$ be the union of all $U \in \mathcal{B}^x_{\alpha,0}$ which are such that whenever $y_0 \in U \cap [y]_G$ and U' is open with non-empty intersection with U then

$$\exists g \in V_n(g \cdot y_0 \in U').$$

Note that each $O_{x,n,\alpha}$ is dense in $(X^x_{\alpha,0}, \tau^x_{\alpha,0})$.

Case(B2a): For some $W_0, W_1, U_{y',n}$ as in claim(1)

$$[x]_G \cap ((X^x_{\alpha,0} \setminus U)^{*W_0} \cap U_{y',n})^{\Delta W_1} \neq \emptyset.$$

Then we let

$$\mathcal{B}^x_{\alpha+1} = \mathcal{B}^+(\mathcal{B}^x_{\alpha,0}, \mathcal{B}_0, \{X \setminus O_{x,n,\alpha} : n \in \mathbb{N}\})$$

and we let $\tau^x_{\alpha+1}$ be the topology generated by $\mathcal{B}^x_{\alpha+1}$. Thus by claim(1) there is certainly a $\tau^x_{\alpha+1}$-open set O with $O \cap [x]_G$ not relatively open in $\tau^x_{\alpha,0}$, and hence not open in $\tau^x_{\alpha,0}$.

Case(B2b): For every $W_0, W_1, U_{y',n}$ as in claim(1)

$$[x]_G \cap ((X^x_{\alpha,0} \setminus U)^{*W_0} \cap U_{y',n})^{\Delta W_1} = \emptyset.$$

Claim(2): For some n

$$x \in (X \setminus O_{x,n,\alpha})^{\Delta G}.$$

Proof of claim: Since $[x]_G$ is included in a countable union of closed nowhere dense sets, may find $n \in \mathbb{N}$, $x_1 \in [x]_G$, and W_0, W_1 basic open neighborhoods of the identity such that

$$W_1^{-1} = W_1,$$
$$V_n W_1 W_1^{-1} \subset W_0,$$

and $W_0^2 \cap x_1$ is nowhere dense (in $(X^x_{\alpha,0}, \tau^x_{\alpha,0})$).

But then if $g_1 \cdot x_1 \in U_{y',n}$ any $g_1 \in W_1$ and $U_{y',n}$ as in the statement of claim(1), then by case assumption we have that for all open U meeting $U_{y',n}$ there is some $\epsilon \in W_1$ close to the identity with

$$\epsilon g_1 \cdot x \notin (X^x_{\alpha,0})^{*W_0},$$
$$\therefore \exists g_0 \in W_0 (g_0 \epsilon g_1 \cdot x_1 \in U).$$

Thus

$$W_0 \epsilon g_1 \cdot x_1 \subset W_0 W_1^2 \cdot x_1 \subset W_0^2 \cdot x_1$$

is somewhere dense, contradicting assumption on x_1 and W_0. ($\square$Claim)

Claim(3): For no open $W \subset G$ do we have that

$$[x]_G \cap (X \setminus O_{x,n,\alpha})^{*W}$$

includes a relatively open (in $[x]_G$) non-empty set.

Proof of claim: Otherwise let O be such an open set in $\tau^x_{\alpha,0}$ with

$$[x]_G \cap O \subset (X \setminus O_{x,n,\alpha})^{*W}.$$

Then since $[y]_G$ is dense we may find $y_1 \in O$. By the definition of the $O_{x,n,\alpha}$ we clearly have

$$y_1 \in O^{\Delta W}_{x,n,\alpha}.$$

By density of $[x]_G$ we may find

$$x_1 \cap [x]_G \cap O^{\Delta W}_{x,n,\alpha} \cap O,$$

with a contradiction to the assumption on O. ($\square$Claim)

Then we let

$$\mathcal{B}^x_{\alpha+1} = \mathcal{B}^+(\mathcal{B}^x_{\alpha,0}, \mathcal{B}_0, \{X \setminus O_{x,n,\alpha} : n \in \mathbb{N}\}).$$

By claim(3) we have that there is some open set $U \in \mathcal{B}^x_{\alpha+1}$ whose intersection with $[x]_G$ is not relatively open with respect to $\mathcal{B}^x_{\alpha,0}$, and thus certainly not with respect to $\mathcal{B}^x_\alpha$.

This completes the description of the transfinite changes in topologies. We need to see that this process terminates, and that around each orbit there are less than ω_1 many changes of the topology.

Fix $x \in X$. For each $\alpha \in \omega_1$ and $U \in \mathcal{B}^x_\alpha$ we can let

$$W_{x,U} = \{g \in G : g \cdot x \in U\},$$

$$A_{x,U} = \{k \in \mathbb{N} : \forall g \in \hat{V}_k(g \cdot x \in U)\}.$$

Since each $(X^x_\alpha, \tau^x_\alpha)$ is a Polish G-space, we have that each $W_{x,U}$ is open in G. The $A_{x,U}$ is simply a set of natural numbers describing $W_{x,U}$ as a union of basic open sets. Finally for each $\alpha < \omega_1$ let

$$\hat{\pi}_x : \omega_1 \to \mathcal{P}_{\aleph_0}(2^{\mathbb{N}}),$$

$$\hat{\pi}_x : \alpha \mapsto \{A_{x,U} : U \in \mathcal{B}^x_\alpha\}.$$

One can easily verify that this function is absolutely $\underset{\sim}{\Delta}^1_2$ in the codes, $\alpha < \beta$ implies

$$\hat{\pi}_x(\alpha) \subset \hat{\pi}_x(\beta),$$

and that as long as the process has not terminated, that is to say, as long as we are in case(B2) above, claim (3) implies

$$\hat{\pi}_x(\alpha) \neq \hat{\pi}_x(\alpha+1).$$

Thus by 9.4 we have some α at which $\hat{\pi}_x(\alpha)$ equals $\hat{\pi}_x(\alpha+1)$ and we are in case(A) or case(B1). Case(A) for any x leaves us in alternative 2. of the theorem, so let us suppose instead that for every x we eventually come to rest in (B1).

For each x let $\alpha(x)$ be least α such that $\mathcal{B}^x_\alpha = \mathcal{B}^x_{\alpha+1}$. We always have that $[x]_G$ is a G_δ in the topology generated by $\mathcal{B}^x_{\alpha(x)}$, and we wish to use its closure in that topology to represent the orbit by an HC set.

To this purpose we describe countable sets

$$A(x,\alpha), A_0(x,\alpha) \subset \text{HC}$$

for $\beta \leq \alpha(x)$ and onto functions

$$F^x_\alpha : A(x,\alpha) \twoheadrightarrow \mathcal{B}^x_\alpha$$

$$F^x_{\alpha,0} : A_0(x,\alpha) \twoheadrightarrow \mathcal{B}^x_{\alpha,0}$$

for each $\alpha \leq \alpha(x)$.

The construction of $F^x_{\alpha+1}$ from $F^x_{\alpha,0}$ parallels 9.5. We build $F^x_{\alpha+1}$ as the union over $i \in \mathbb{N}$ of $F^x_{\alpha,0,i}$ and $A(\beta+1,x)$ as the union of the $A_0(x,\beta,i)$. We start with $F^x_{\alpha,0,0} = F^x_{\alpha,0}$.

$$F^x_{\alpha,0,1}(\langle \alpha, A_0(x,\alpha), n\rangle) = O_{x,n,\alpha}.$$

Granted $F^x_{\alpha,0,i}$ at $i > 0$ and given $l(0), l(1), ..., l(k), l, n \in \mathbb{N}$ and $a, a' \in A_0(x,\alpha,i)$ with $U, U' \in \mathcal{B}^x_{\alpha,0}$ and

$$F^x_{\alpha,0,i}(a) = U$$

$$F^x_{\alpha,0,i}(a') = U'$$

we let[1]

$$F^x_{\alpha,i+1}(\langle \alpha, A_0(x,\alpha,i), 0, a, a', l(0), ..., l(k), l\rangle) =$$

[1]Here the pairing function is from $\text{HC}^{<\omega}$ to HC – for instance

$$\langle a_1, a_2, ..., a_n\rangle = \{\{a_1, a_2, ..., a_n\}, \{a_2, ..., a_n\}, ..., \{a_n\}\}.$$

$$(U \cap (X \setminus U')^{*W_{l(0)}} \cap ... \cap (X \setminus U')^{*W_{l(k)}})^{\Delta W_l}$$

$$F^x_{\alpha,i+1}(\langle \alpha, A_0(x,\alpha,i), 1, \delta, \delta' \rangle) = U \cap U'$$

$$F^x_{\alpha,i+1}(\langle \alpha, A_0(x,\alpha,i), 2, \delta, \delta' \rangle) = U \cup U'.$$

We can let each $A_0(x,\alpha,i+1)$ be $A(x,\alpha,i)$ along with the hereditarily countable sets above on which $F^x_{\alpha,i+1}$ was just defined.

The new ground is in defining $F^x_{\alpha,0}$ from F^x_α. We construct

$$F^x_{\alpha,0,\beta} : A(x,\alpha,\beta) \twoheadrightarrow \mathcal{B}^x_\alpha \cup \{U(\varphi^{\tau^x_\alpha}_\beta(\hat{x}, U, V_n)) : n \in \mathbb{N}, U \in \mathcal{B}^x_\alpha, \beta' \leq \beta, \hat{x} \in [x]_G\}$$

by induction on $\beta \leq \gamma(x)$, with each $F^x_{\alpha,0,\beta}$ extending the function F^x_α.

At the first level, if $F^x_\alpha(a) = U$ then

$$F^x_{\alpha,0,0}(\langle \alpha, A(x,\alpha), a, n, \{b : \mathcal{O}(\hat{x}, U, V_n) \cap F^x_\alpha(b) \neq \emptyset\}\rangle) = U(\varphi^{\tau^x_\alpha}_0(\hat{x}, U, V_n))$$

for any $\hat{x} \in [x]_G$ and $A(x,\alpha,0)$ is the union of $A(x,\alpha)$ and all the

$$\langle A(x,\alpha), a, n, \{b : \mathcal{O}(\hat{x}, U, V_n) \cap F^x_\alpha(b) \neq \emptyset\}\rangle$$

indicated above. At successor stages of these definitions let

$$F^x_{\alpha,0,\beta+1}(\langle \alpha, A(x,\alpha,\beta), a, n, \{b : \mathcal{O}(\hat{x}, U, V_n) \cap F^x_{\alpha,0,\beta}(b) \neq \emptyset\}\rangle) = U(\varphi^{\tau^x_\alpha}_\beta(\hat{x}, U, V_n))$$

for any $\hat{x} \in [x]_G$; again the set $A(x,\alpha,\beta+1)$ is the union of $A(x,\alpha)$ and all the sets of the form

$$\langle A(x,\alpha,\beta), a, n, \{b : \mathcal{O}(\hat{x}, U, V_n) \cap F^x_{\alpha,\beta}(b) \neq \emptyset\}\rangle.$$

At limits $\lambda \leq \gamma(x)$

$$F^x_{\alpha,0,\lambda} = \bigcup_{\beta<\lambda} F^x_{\alpha,0,\beta}.$$

Finish with

$$F^x_{\alpha,0} = F^x_{\beta,0,\gamma(x)}.$$

At the very final moment obtain

$$\theta(x) = \{a \in A(x,\alpha(x)) : [x]_G \cap F^x_{\alpha(x)} \neq \emptyset\}.$$

□

EXERCISE 9.8. If X is a Polish S_∞-space, then $x \mapsto \varphi^*_x$ (as defined in §6.1) is absolutely $\underset{\sim}{\Delta}^1_2$.

EXERCISE 9.9. If F is F_2-ergodic, then it is E_{S_∞}-ergodic, for any equivalence relation E_{S_∞} induced by the continuous action of S_∞ on a Polish space. By induction on α, if π is a Baire measurable map satisfying $x_1 F x_2 \Rightarrow \pi(x_1) E_{S_\infty} \pi(x_2)$ then $x \mapsto V_\alpha \cap \varphi^*_{\pi(x)}$ is constant on a comeager set – where here V_α denotes the sets whose transitive closures have $\in$-rank less than α. (Essentially this observation is implicit in Harvey Friedman's proof that the cosets of the density ideal do not admit classification by countable structures.)

9.2. Cardinality in $L(\mathbb{R})$

The customary notion of cardinality is to define the cardinality of a set to be the least ordinal with which it may be placed in a bijection. In some respects the resulting theory is unsatisfying, since then there seems no agreed method to settle so simple a question as the cardinality of the continuum, and indeed for many of our most fundamental notions, such as $\mathbb{R}$, there are no *definable* well orderings.

There is another approach to cardinality – not so much a rival definition as an alternative concept. It is completely immune to the independence results associated with the continuum hypothesis.

DEFINITION 9.10. Let A and B be sets in $L(\mathbb{R})$. We will say that the *$L(\mathbb{R})$-cardinality of A is less than that of B*, written

$$|A|_{L(\mathbb{R})} \leq |B|_{L(\mathbb{R})},$$

if there exists an injection $i \in L(\mathbb{R})$ with

$$i : A \hookrightarrow B.$$

The theory of $L(\mathbb{R})$ is canonical in the presence of sufficient large cardinal or determinacy assumptions, and there are no known interesting questions about its structure left unresolved by $\mathrm{AD}^{L(\mathbb{R})}$. Hence forth I will assume $\mathrm{AD}^{L(\mathbb{R})}$: the determinacy of all games of length ω on the integers are determined in $L(\mathbb{R})$; it implies a host of extraordinary regularity properties inside this model, including all sets Lebesgue measurable and the perfect set theorem for all subsets of $\mathbb{R}$.

Not only does $L(\mathbb{R})$ have an appealing internal structure, but it also includes anything we might want to consider *reasonably definable*. For instance, all Polish spaces up to homeomorphism, all Borel sets, all $\utilde{\Sigma}^1_1$, C-measurable, and so forth.

LEMMA 9.11. *(folklore; $\mathrm{AD}^{L(\mathbb{R})}$) In $L(\mathbb{R})$ every non-empty $\utilde{\Sigma}^2_1$ subset of $\mathcal{P}(\mathbb{R})$ has a $\utilde{\Delta}^2_1$ member.*

PROOF. (Compare [**65**].) If Ψ is a $\utilde{\Sigma}^2_1$ property on sets of reals, and α is least such

$$L_\alpha(\mathbb{R}) \models \exists R(\Psi(R)),$$

then α is $\utilde{\Sigma}^2_1$ definable, $L_\alpha(\mathbb{R})$ is the $\utilde{\Sigma}^2_1$ Skolem hull of the reals, and thus every element of $L_\alpha(\mathbb{R})$ is a $\utilde{\Sigma}^2_1$ singleton in some real parameter. □

THEOREM 9.12. *(Martin, Moschovakis, Steel; $\mathrm{AD}^{L(\mathbb{R})}$) Scale($\utilde{\Sigma}^2_1$) in $L(\mathbb{R})$.*

PROOF. See [**65**] or [**66**]. □

It is worth pointing out that for reasonably simple equivalence relations – certainly all Borel, but more generally all $\utilde{\Sigma}^1_1$, and even all $\utilde{\Delta}^2_1$ – the $L(\mathbb{R})$-analog of the $\leq_B$-reducibility relation holds if and only if there is some $L(\mathbb{R})$ injection from the quotient space provided by the first into the quotient associated to the second.

LEMMA 9.13. *($\mathrm{AD}^{L(\mathbb{R})}$) Let E, F be $\utilde{\Delta}^2_1$ equivalence relations on $2^{\mathbb{N}}$ in $L(\mathbb{R})$. Then*

$$|2^{\mathbb{N}}/E|_{L(\mathbb{R})} \leq |2^{\mathbb{N}}/F|_{L(\mathbb{R})}$$

if and only if there is some function

$$\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$$

in $L(\mathbb{R})$ *such that for all* $x, y \in 2^{\mathbb{N}}$

$$xEy \Leftrightarrow (\theta(x)F\theta(y)).$$

PROOF. The "if" direction of this lemma is trivial, so suppose instead that $|2^{\mathbb{N}}/E|_{L(\mathbb{R})} \leq |2^{\mathbb{N}}/F|_{L(\mathbb{R})}$. Thus there is some $S \subset 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ such that

$$\forall x \in 2^{\mathbb{N}} \exists y \in 2^{\mathbb{N}}((x, y) \in S),$$

$$\forall x_1, x_2 \in 2^{\mathbb{N}} \forall y_1, y_2 \in 2^{\mathbb{N}}((x_1, y_1), (x_2, y_2) \in S \Rightarrow (x_1 E x_2 \Leftrightarrow y_1 F y_2)).$$

This is a $\underset{\sim}{\Delta}{}^2_1$ property on S, so by 9.11 we may find some $S_0 \in \underset{\sim}{\Delta}{}^2_1$ as above, and then by 9.12 we can find a scale in $L(\mathbb{R})$ on S_0.

But since the scale property implies uniformization, we may actually find a function

$$\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$$

in $L(\mathbb{R})$ such that for all $x \in 2^{\mathbb{N}}$

$$(x, \theta(x)) \in S_0$$

and thus

$$\forall x_1, x_2 \in 2^{\mathbb{N}}(x_1 E x_2 \Leftrightarrow (\theta(x_1) F \theta(x_2))).$$

$\square$

THEOREM 9.14. *($\mathrm{AD}^{L(\mathbb{R})}$) Let G be a Polish group and X a Polish G-space. The following are equivalent.*

(I) $E^X_G \leq_{a\Delta^1_2} \mathrm{id}(\mathrm{HC})$;

(II) there is no turbulent Polish G-space Y and a continuous G-embedding $\pi : Y \to X$;

(III) $|X/G|_{L(\mathbb{R})} \leq |HC|_{L(\mathbb{R})}$.

PROOF. We had the equivalence of (I) and (II) from 9.7, and (I) implying (III) is trivial. Thus it suffices to show that if Y is a turbulent Polish G-space then there is no injection i in $L(\mathbb{R})$ from X/G to HC.

But here the set of such injections is $\underset{\sim}{\Sigma}{}^2_1$, and so by 9.11 if there is such an i then we may assume it to be $\underset{\sim}{\Sigma}{}^2_1$ in the codes. Next by 9.12 we can obtain an honest to goodness function $\theta \in L(\mathbb{R})$ with

$$\theta : X \to \mathrm{Mod}(\mathcal{L}(\in))$$

so that for all $x_1, x_2 \in X$

$$x_1 E^X_G x_2 \Leftrightarrow \theta(x_1) \cong \theta(x_2).$$

This contradicts 3.18 since all functions in $L(\mathbb{R})$ are Baire measurable. $\square$

THEOREM 9.15. *($\mathrm{AD}^{L(\mathbb{R})}$) Let G be a GE group and X a Polish G-space. Then the following are equivalent:*

(I) E^X_G admits classification by countable models;

(II) there is a countable language $\mathcal{L}$ with $E^X_G \leq_{FB} \cong|_{\mathrm{Mod}(\mathcal{L})}$;

(III) $x \mapsto \varphi_x$ provides a complete invariant for the orbit equivalence relation;

(IV) there is no turbulent Polish G-space Y and continuous G-embedding from Y to X;

(V) $|X/G|_{L(\mathbb{R})} \leq |HC|_{L(\mathbb{R})}$.

PROOF. 9.14 and 6.30. $\square$

LEMMA 9.16. *($\mathrm{AD}^{L(\mathbb{R})}$) In $L(\mathbb{R})$ all well ordered intersections of comeager sets are comeager.*

PROOF. By Kuratowski-Ulam, and [**67**] for all sets in $L(\mathbb{R})$ having Baire property. □

NOTATION 9.17. Let $\mathcal{L}$ be a language and κ an ordinal. S_κ denotes the permutation group on κ and $\mathrm{Mod}(\mathcal{L},\kappa)$ denotes the collection of all models whose underlying set is κ. We may write

$$E \leq_{L(\mathbb{R})} \cong|_{\mathrm{Mod}(\mathcal{L},\kappa)}$$

if there is a function θ in $L(\mathbb{R})$ so that for all x_1, x_2

$$x_1 E x_2 \Leftrightarrow L(\mathbb{R}) \models (\theta(x_1) \cong \theta(x_2)).$$

Here we specifically ask that the isomorphism between the models exist in $L(\mathbb{R})$.

THEOREM 9.18. *($\mathrm{AD}^{L(\mathbb{R})}$) Let G be a Polish group and X be a Polish G-space. If for some well ordered language $\mathcal{L}$ and ordinal κ*

$$E_G^X \leq_{L(\mathbb{R})} \cong|_{\mathrm{Mod}(\mathcal{L},\kappa)}$$

then the action is not turbulent.

PROOF. Suppose instead that X is turbulent. Work in $L(\mathbb{R})$.

The proof is akin to 3.18, only replacing the countability of $\mathbb{N}$ with the well orderability of $\mathcal{L}$ and κ and the use of 9.16. In the case of 3.18 we can simply enumerate the basic open sets in S_∞ in search of the sequence $\theta(x_i) \to g_\infty \cdot \theta(x)$. Here we need to stitch together a well ordered collection of basic open sets, meeting them not one after another according to a fixed enumeration, but strategically, in a construction that remembers ideas from forcing and applications of $\mathrm{MA}(\aleph_1)$.

Reintroduce all our usual notation for the group action. Let $\mathcal{B} = \{U_l : l \in \mathbb{N}\}$ be a countable basis for X, $\mathcal{B}_0$ for G, $G_0 = \{g_j : j \in \mathbb{N}\}$ a countable dense subgroup of G, $(V_n)_{n\in\mathbb{N}}$ a countable basis at 1_G each $V_n = V_n^{-1}$. Assume moreover that each $V_n \in \mathcal{B}_0$ and that $\mathcal{B}_0$ is closed under left and right translation by elements in G_0.

By the basis theorem at 9.11 we can assume that there is a $\underset{\sim}{\Delta}{}_1^2$ reduction

$$\theta : X \to \mathrm{Mod}(\mathcal{L},\kappa)$$

for $\kappa < \underset{\sim}{\delta}{}_1^2$ so that for all $g \in G$

$$\theta(x) \cong \theta(g \cdot x).$$

Then by the "coding lemma" of [**67**] we may code each element of S_κ by a real, and then gather from 9.12 that there is a function

$$\Gamma_0 : X \times G \to S_\kappa$$

so that for all $(x, g) \in X \times G$

$$\Gamma_0(x, g) : \theta(x) \cong \theta(g \cdot x).$$

For each basic open $U \subset X$ and $V \subset G$ let $A_0(U, V)$ be the set of $\alpha < \kappa$ such that there exists some $\beta < \kappa$ at which

$$\forall^* x \in U \forall^* g \in V(\Gamma_0(x, g)(\alpha) = \beta).$$

Note that $U' \subset U$ and $V' \subset V$ implies $A_0(U', V') \supset A_0(U, V)$.

Claim(1): Let $U, U' \subset X$ be basic open and $n \in \mathbb{N}$ with $V_n \cdot U' \subset U$, $h \in G_0$. Then for a comeager set of $x \in U'$ and $g \in V_n$ there exists

$$\pi : \theta(x) \cong \theta(g \cdot x)$$

$$\pi|_{A_0(U, hV_n)} = \text{id}$$

(i.e. $\pi(\alpha) = \alpha$ all $\alpha \in A_0(U, hV_n)$).

Proof of claim. Using 9.16 we have that there is some

$$\sigma : A_0(U, hV_n) \to \kappa$$

for whom $\forall^* x \in U \forall^* g \in hV_n$

$$\Gamma_0(x, g)|_{A_0(U, hV_n)} = \sigma.$$

For any sufficiently generic $x \in U'$ and $g \in V_n$ we may find $h' \in G$ so that $h'h, h'hg \in hV_n$ and

$$\Gamma_0(x, h'hg)|_{A_0(U, hV_n)} = \sigma$$

$$\Gamma_0(g \cdot x, h'hg)|_{A_0(U, hV_n)} = \sigma.$$

Then

$$\Gamma_0(g \cdot x, h'h) : g \cdot x \mapsto (h'hg) \cdot x,$$

$$\Gamma_0(x, h'hg) : x \mapsto (h'hg) \cdot x,$$

and so

$$(\Gamma_0(g \cdot x, h'hg))^{-1} \Gamma_0(x, h'hg)$$

is as needed. (Claim(1)$\square$)

Claim(2): For all $\alpha < \kappa$ the union of all $U \in \mathcal{B}$ such that for some $n \in \mathbb{N}$ and $h \in G_0$ we have

$$\alpha \in A_0(U, hV_n).$$

Proof of claim. Clearly this set is open. As for density, at each $\beta < \kappa$ let

$$D_\beta = \{(x, g) : (\Gamma_0(x, g))(\alpha) = \beta\}.$$

Then by 9.16 and separability of $X \times G$ there is some countable set $(\beta_i)_{i \in \mathbb{N}}$ such that

$$\bigcup_{i \in \mathbb{N}} D_{\beta_i}$$

is comeager in $X \times G$. Thus by Kuratowski-Ulam

$$\forall^* x \in X \forall^* g \in G((g, x) \in \bigcup_{i \in \mathbb{N}} D_{\beta_i}).$$

For each let

$$D_{\beta_i, x} = \{g \in G : (g, x) \in D_{\beta_i}\}.$$

By all sets having Baire property

$$\forall^* x \in X \forall^* h \in G \exists n \in \mathbb{N} \exists i \in \mathbb{N} (\forall^* g \in V_n((g, h \cdot x) \in D_{\beta_i}))$$

$$\therefore \forall^* x \in X \exists n \in \mathbb{N} \exists i \in \mathbb{N} (\forall^* g \in V_n((g, x) \in D_{\beta_i}))$$

by Kuratowski-Ulam.

For each x in this comeager set let (n_x, i_x) be the least pair in $\mathbb{N} \times \mathbb{N}$ under the lexicographic order such that

$$\forall^* g \in V_n((g, x) \in D_{\beta_i}).$$

Since $x \mapsto (n_x, i_x)$ is continuous on a comeager set we can find a collection of basic open subsets $\mathcal{U}$ whose union is dense in X and such that on any $U \in \mathcal{U}$ the function $x \mapsto (n_x, i_x)$ is constant on a relatively comeager subset. (Claim(2)$\square$)

Thus in particular the claim implies that for each $\alpha < \kappa$ the union of all open $U' \subset X$ such that there exists $U \in \mathcal{B}$, $n \in \mathbb{N}$, and $h \in G_0$, with

$$V_n \cdot U' \subset U,$$

and

$$\alpha \in A_0(U, hV_n)$$

will be open dense.

Claim(3): We can find $C_0' \subset X \times G$ comeager

$$\Gamma_1 : C_0' \times \mathbb{N}^4 \to S_\kappa$$

with

$$\forall (x,g) \in C_0'(\Gamma_1(x, g, (l,k,n,j)) : \theta(x) \cong \theta(g \cdot x))$$

and such that for all $(x,g) \in C_0'$ and $l,k,n,j \in \mathbb{N}$, if $x \in U_k$,

$$V_n \cdot U_k \subset U_l$$

$$g \cdot x \in \mathcal{O}(x, U_k, V_n)$$

then

$$\Gamma_1(x, g, (l,k,n,j))|_{A_0(U_l, h_j V_n)} = \mathrm{id}|_{A_0(U_l, h_j V_n)}.$$

Proof of claim. For sufficiently generic x, g with $g \in V_n$ we can certainly find

$$\sigma : \theta(x) \cong \theta(g \cdot x)$$

with

$$\sigma|_{A_0(U_l, h_j V_n)} = \mathrm{id}|_{A_0(U_l, h_j V_n)}.$$

More generally given $x \in U_n$ and $g \in G$ sufficiently generic with

$$g \cdot x \in \mathcal{O}(x, U_k, V_n)$$

we may find $g_0, g_1, ..., g_m \in V_n$ so that for

$$x_0 = x,$$

$$x_{i+1} = g_i \cdot x_i,$$

we have each $x_i \in U_k$ and each (x_i, g_i) meeting the comeager set shown to exist in claim(1) for $A_0(U_l, h_j V_n)$. Thus there is at each i some

$$\sigma_i : \theta(x_i) \cong \theta(x_{i+1})$$

with

$$\sigma_i|_{A_0(U_l, h_j V_n)} = \mathrm{id}_{A_0(U_l, h_j V_n)},$$

and we may find σ as their composition.

Uniformization for $\underset{\sim}{\Sigma}_1^2$ finds some Γ_1 producing σ as a function of x and g. (Claim(3)$\square$)

By claim (2) and 9.16 we can find comeager $C_0 \subset C_0'$ so that for all $(x,g) \in C_0$ and $\alpha < \kappa$ there is some $U, U' \in \mathcal{B}$, $n \in \mathbb{N}$, $h \in G_0$

$$x \in U'$$

$$V_n \cdot U' \subset U$$

$$\alpha \in A_0(U, hV).$$

Repeat: Let $A_1(U,V,(l,k,n,j))$ be the set of $\alpha<\kappa$ for which there is some β with

$$\forall^* x\in U\forall^* g\in V(\Gamma_1(x,g,(l,k,n,j))(\alpha)=\beta).$$

Iterating we obtain for each $i\in\mathbb{N}$ some comeager

$$C_i\subset X\times G$$

$$\Gamma_{i+1}:C_i\times(\mathbb{N}^4)^{i+1}\to S_\kappa$$

so that all $(x,g)\in C_i$ and $\vec{m}=((l(0),k(0),n(0),j(0)),...,(l(i),k(i),n(i),j(i))\in(\mathbb{N}^4)^{i+1}$ we have that if

$$x\in U_{k(i)}$$

$$g\cdot x\in\mathcal{O}(x,U_{k(i)},V_{n(i)})$$

$$V_{n(i)}\cdot U_{k(i)}\subset U_{l(i)}$$

then

$$\Gamma_{i+1}(x,g,\vec{m})|_{A_i(U_{l(i)},h_{j(i)}V_{n(i)},\vec{m}|_i)}=\text{ id};$$

and for all $\alpha<\kappa$, $(x,g)\in C_i$, $\vec{m}\in(\mathbb{N}^4)^i$, there are $U,U'\in\mathcal{B}$, $n\in\mathbb{N}$, $h\in G_0$

$$x\in U'$$

$$V_n\cdot U'\subset U$$

$$\alpha\in A_i(U,hV,\vec{m}).$$

Then define $A_{i+1}(U,V,\vec{m})$ be the set of α so that for some β

$$\forall^* x\in U\forall^* g\in V((\Gamma_{i+1}(x,g,\vec{m}))(\alpha)=\beta).$$

Claim(4): For $\vec{m}\in(\mathbb{N}^4)^{i+1}$

$$U\subset U_{k(i)},$$

$$V\subset V_{n(i)}$$

we have

$$A_{i+1}(U,V,\vec{m})\subset A_i(U_{l(i)},h_{j(i)}V_{n(i)},\vec{m}|_i).$$

Proof of claim: A tautology given the above specifications on Γ_{i+1}. (Claim(4)$\square$)

Use 9.16 to produce a comeager set C on which θ is continuous in the sense that for any $R\in\mathcal{L}$ and $\vec{\alpha}\in\kappa^{<\mathbb{N}}$

$$\theta^{-1}[\{\mathcal{M}:\mathcal{M}\models R(\vec{\alpha})\}]\cap C$$

is relatively clopen. Let C_∞ be the set of $x\in C$ so that

$$\forall g\in G_0\forall i\in\mathbb{N}((x,g)\in C_i,g\cdot x\in C).$$

It suffices to pick x and y in C_∞ and demonstrate an isomorphism between $\theta(x)$ and $\theta(y)$.

For this, let $\mathcal{Z}(x,y)$ be the space of all infinite sequences

$$((h_i)_{i\in\mathbb{N}},(h_i')_{i\in\mathbb{N}},((l(i),k(i),n(i),j(i))_{i\in\mathbb{N}})$$

in $G_0^{\mathbb{N}}\times G_0^{\mathbb{N}}\times(\mathbb{N}^4)^{\mathbb{N}}$ such that for

$$x=x_0,x_1,x_2,...$$

and

$$y_0=y,y_1,y_2,...$$

defined by

$$x_{i+1}=h_i\cdot x_i$$

and

$$y_{i+1} = h'_i \cdot y_i$$

we have:

(i) $h_i \approx^{x_i}_{k(2i),n(2i)} 1_G$ (and therefore $x_{i+1} \in \mathcal{O}(x_i, U_{k(2i)}, V_{n(2i)})$);
(ii) $h'_i \approx^{y_i}_{k(2i+1),n(2i+1)} 1_G$ (and therefore $y_{i+1} \in \mathcal{O}(y_i, U_{k(2i+1)}, V_{n(2i+1)})$);
(iii) $U_{k(i+1)} \subset U_{l(i+1)} \subset U_{k(i)} \subset U_{l(i)}$;
(iv) $x_i \in U_{k(2i)}$;
(v) $y_i \in U_{k(2i+1)}$;
(vi) $V_{n(i+1)} \subset V_{n(i)}$;
(vii) $\mathcal{O}(y_i, U_{k(2i+1)}, V_{n(2i+1)})$ is dense in $U_{l(2i+2)}$;
(viii) $\mathcal{O}(x_i, U_{k(2i)}, V_{n(2i)})$ is dense in $U_{l(2i+1)}$;
(ix) $V_{n(i)} \cdot U_{k(i)} \subset U_{l(i)}$.

The definition indicates $\mathcal{Z}(x,y)$ to be closed in the space $G_0^{\mathbb{N}} \times G_0^{\mathbb{N}} \times (\mathbb{N}^4)^{\mathbb{N}}$ and so Polish in the product topology (where here we consider G_0 as a discrete space). We need a well ordered collection of open dense subsets, parallel to 3.18:

For R in $\mathcal{L}$ and $\vec{\alpha} \in \kappa^{<\mathbb{N}}$ let $D_{R,\vec{\alpha}}$ be the set of $(\vec{h}, \vec{h}', \vec{m}) \in \mathcal{Z}(x,y)$ so that for some i either

$$\forall^* z \in U_{l(i)}(\theta(z) \models R(\vec{\alpha}))$$

or

$$\forall^* z \in U_{l(i)}(\theta(z) \models \neg R(\vec{\alpha})).$$

For $\alpha < \kappa$ let $D^0_{\theta,\alpha}$ be the set of $(\vec{h}, \vec{h}', \vec{m}) \in \mathcal{Z}(x,y)$ so that at some i if we let

$$\beta_{i,\alpha} =_{df} (\Gamma_{2i-2}(x_{i-1}, h_{i-1}, \vec{m}|_{2i-2})) \circ (\Gamma_{2i-4}(x_{i-2}, h_{i-2}, \vec{m}|_{2i-4})) \circ ...(\Gamma_0(x_0, h_0))(\alpha)$$

then

$$\beta_{i,\alpha}, \alpha \in A_{2i-1}(x_i, U_{l(2i)}, g_{j(2i)} V_{n(2i)}, \vec{m}|_{2i});$$

note that from this, (i), (ix), and the assumptions on the Γ_i functions give that

$$(\Gamma_{2i}(x_i, h_i))(\alpha) = \alpha,$$

$$(\Gamma_{2i}(x_i, h_i))(\beta_{i,\alpha}) = \beta_{i,\alpha}.$$

Similarly let $D^1_{\theta,\alpha}$ be the set of $(\vec{h}, \vec{h}', \vec{m}) \in \mathcal{Z}(x,y)$ for which there is some i such that if

$$\gamma_{i,\alpha} =_{df} (\Gamma_{2i-1}(y_{i-1}, h'_{i-1}, \vec{m}|_{2i-1})) \circ (\Gamma_{2i-3}(y_{i-2}, h'_{i-2}, \vec{m}|_{2i-3})) \circ ...(\Gamma_0(y_0, h'_0))(\alpha)$$

then

$$\gamma_{i,\alpha}, \alpha \in A_{2i}(y_i, U_{l(2i+1)}, g_{j(2i+1)} V_{n(2i+1)}, \vec{m}|_{2i+1});$$

again from this, (ii), (ix), and the assumptions on the Γ_i's we obtain

$$(\Gamma_{2i+1}(y_i, h'_i))(\alpha) = \alpha,$$

$$(\Gamma_{2i+1}(y_i, h'_i))(\gamma_{i,\alpha}) = \gamma_{i,\alpha}.$$

Once we obtain some i with

$$\beta \in A_{2i-1}(x_i, U_{l(2i)}, g_{j(2i)} V_{n(2i)}, \vec{m}|_{2i})$$

or γ with

$$\gamma \in A_{2i}(y_i, U_{l(2i+1)}, g_{j(2i+1)} V_{n(2i+1)}, \vec{m}|_{2i+1})$$

then for all $i' > i$

$$\beta \in A_{2i'-1}(x_{i'}, U_{l(2i')}, g_{j(2i')} V_{n(2i')}, \vec{m}|_{2i'})$$

or

$$\gamma \in A_{2i'}(y_{i'}, U_{l(2i'+1)}, g_{j(2i+1)}(2V_{n(2i'+1)}, \vec{m}|_{2i'+1})$$

respectively by (iii), (vi), and claim(4). Thus we respectively would have either

$$(\Gamma_{2i'}(x_{i'}, h_{i'}))(\alpha) = \alpha,$$
$$(\Gamma_{2i'}(x_{i'}, h_{i'}))(\beta_{i,\alpha}) = \beta_{i,\alpha}$$

or

$$(\Gamma_{2i'+1}(y_{i'}, h'_{i'}))(\alpha) = \alpha,$$
$$(\Gamma_{2i'+1}(y_{i'}, h'_{i'}))(\gamma_{i,\alpha}) = \gamma_{i,\alpha}.$$

Each $D_{R,\vec{a}}$ is dense by assumption of θ continuous on C. All the $D^0_{\theta,\alpha}$ and $D^1_{\theta,\alpha}$ are dense by assumption on the C_i's.

By 9.16 obtain some sequence $(\vec{h}, \vec{h}', \vec{m}) \in \mathcal{Z}(x, y)$ meeting all these comeager sets. Then if we set

$$\sigma_i = \Gamma_{2i}(x_i, h_i, \vec{m}|_{2i})$$
$$\sigma'_i = \Gamma_{2i+1}(y_i, h'_i, \vec{m}|_{2i+1})$$

and

$$\mathcal{M}_i = \theta(x_i)$$
$$\mathcal{N}_i = \theta(y_i)$$

then by the definitions of the Γ_i functions

$$\sigma_i(\mathcal{M}_i) = \mathcal{M}_{i+1}$$
$$\sigma'_i(\mathcal{N}_i) = \mathcal{N}_{i+1}.$$

By meeting all the $D_{R,\vec{a}}$ sets and the fact that the x_i's and y_i's stay inside the comeager set of points on which θ is continuous we have some $\mathcal{M}_\infty$ with

$$\mathcal{M}_i \to \mathcal{M}_\infty$$
$$\mathcal{N}_i \to \mathcal{M}_\infty.$$

From the $D^0_{\theta,\alpha}$ and $D^1_{\theta,\alpha}$ there are $\sigma_\infty \in S_\kappa$ and $\sigma'_\infty \in S_\kappa$ with

$$\sigma_i \circ \sigma_{i-1} \circ \ldots \sigma_0 \to \sigma_\infty$$
$$\sigma'_i \circ \sigma'_{i-1} \circ \ldots \sigma'_0 \to \sigma'_\infty.$$

Then

$$\sigma_\infty(\mathcal{M}_0) = \sigma'_\infty(\mathcal{N}_0).$$

□

Corollary 9.19. *Let G be a Polish group, X a Polish G-space, κ an ordinal, and $\mathcal{L}$ a well ordered language. If*

$$E^X_G \leq_{L(\mathbb{R})} \cong|_{\mathrm{Mod}(\mathcal{L},\kappa)}$$

then for some countable language $\mathcal{L}'$

$$E^X_G \leq_{L(\mathbb{R})} \cong|_{\mathrm{Mod}(\mathcal{L}')};$$

in fact for suitably chosen $\mathcal{L}'$

$$E^X_G \leq_{a\Delta^1_2} \cong|_{\mathrm{Mod}(\mathcal{L}')}.$$

Proof. By 9.7 and 9.18. □

CHAPTER 10

Looking ahead

I will finish with some questions and conjectures, everyone of which I consider interesting and important to this direction. Several – such as 10.4, 10.5, and 10.6 – I am inclined to consider difficult.

In some ways the main results of this manuscript could be a considered as a reaction to the *failure* to find an analogue of Harrington-Kechris-Louveau for the notion of turbulence. Here there are two sibling theorems, proved by similar ideas.

THEOREM 10.1. *Let G be a Polish group and X a Polish G-space. Then the following are equivalent:*

(I) E_G^X is reducible to $\mathrm{id}(2^{<\omega_1})$ *by a function that is absolutely Baire measurable in the codes (in the sense 3.12);*

(II) there is no properly generically ergodic Polish G-space Y and continuous G-embedding $\pi : Y \hookrightarrow X$.

THEOREM 10.2. *Let G be a Polish group and X a Polish G-space. Then the following are equivalent:*

(I) E_G^X is reducible to $\mathrm{id}(\mathrm{HC})$ *by a function that is absolutely Baire measurable in the codes (in the sense of 6.18);*

(II) there is no turbulent Polish G-space Y and continuous G-embedding $\pi : Y \hookrightarrow X$.

In both cases we thus have that in some sense the presence of a certain dynamical property is a necessary and sufficient condition for non-classifiability. For actually calculating the classification difficult of a given problem the conditions " properly generically ergodic" and "turbulent" almost always appear at least implicitly through some reduction, and 10.1 and 10.2 seem to provide an explanation for this fact.

It should be admitted that the proofs of both these results are completely abstract and use non-trivial set theoretical machinery. One simply sets into place a transfinite process of changing the topology around a point in the space, and shows that the operation must terminate before ω_1 steps for any orbit, and it can only terminate by either displaying a properly generically ergodic or turbulent space embedded into X or by producing an invariant of the required type.

The main defect in these arguments is that they give no further information when E_G^X is Borel. For instance, by combining 10.1 with [**30**] one has:

COROLLARY 10.3. *Let G be a Polish group and X a Polish G-space with E_G^X Borel. Then the following are equivalent:*

(I) $E_G^X \leq_B$ $\mathrm{id}(\mathbb{R})$*;*

(II) there is no properly generically ergodic Polish G-space Y and continuous G-embedding $\pi : Y \hookrightarrow X$.

But the point is that the *proof* of 10.1 does *not* give this result on its own. Similarly in the proof of 10.2 one obtains no benefit by assuming E_G^X is Borel. Thus we are left with questions such as:

QUESTION 10.4. Let G be a Polish group and X a Polish G-space with E_G^X Borel. Must one of the following hold?

1. E_G^X is reducible to id(HC) by a function that is Borel in the codes;
2. there is a turbulent Polish G-space Y and continuous G-embedding $\pi : Y \hookrightarrow X$.

Indeed one can even ask:

QUESTION 10.5. Let G be a Polish group and X a Polish G-space. Must one of the following hold?

1. $E_G^X \leq_{FB} E_{S_\infty}^Y$ for some Polish S_∞-space Y;
2. there is a turbulent Polish G-space Y and continuous G-embedding $\pi : Y \hookrightarrow X$.

In the context of the smooth/non-smooth distinction, it is exactly understood which Polish groups only give rise to smooth orbit equivalence relations: Clearly the orbits under the continuous action by a compact group are all closed, and following say proofs of §7.1 we have reducibility to the identity relation on $\mathbb{R}$; conversely Solecki showed that any *non*-compact Polish group can act continuously on a Polish space with non-smooth orbit equivalence relation.

The situation for classification by countable models is far less clear. From **[4]** and §7.4 these include S_∞, its closed subgroups, the $\aleph_0$ products of locally compact groups, and all their closed subgroups. We might then hope that this class of groups allowing countable models for invariants should be in some natural way generated from these examples, but in fact I think there is at least one more kind of group that needs to be considered:

CONJECTURE 10.6. Whenever $\mathrm{Hom}^+([0,1])$ acts continuously on a Polish space the orbit equivalence relation admits classification by countable models.

Apart from the above problems – all of which are more or less directly connected with trying to understand which equivalence relations may be simply subsumed under the study of countable models – we can also consider a number of parallel issues. Somewhat more general than the equivalence relations induced by S_∞ are those arising from continuous actions of the unitary group, U_∞. Perhaps the most interesting problem in this regard is a hopelessly vague one: Does there exist a canonical transfinite sequence of invariants that serve U_∞ as well as the φ_α invariants from §6.1? It is hard to make this question more mathematically exact, but one way might be to ask if there is an analogue 6.11 assigning a privileged kind of invariant when the orbit equivalence relation has low Borel complexity.

QUESTION 10.7. Let X be a Polish U_∞-space with $E_{U_\infty}^X$ in $F_{\sigma\delta}$. Is $E_{U_\infty}^X$ necessarily Borel reducible to probability measures on the unit interval considered up to absolute continuity? That is to say must there exist a Borel assignment $x \mapsto \mu_x$ such that $x_0 E_{U_\infty}^X x_1$ if and only if μ_{x_0} and μ_{x_1} have the same null sets?

It is known from the spectral theorem as found in **[10]** or **[83]** that we do indeed obtain such a reduction for the equivalence relation of conjugacy in the unitary group U_∞.

Of course there are many problems with this flavor, and ultimately one can ask about invariants for arbitrary Polish groups. It is not clear what form such an analysis might take, so we might propose the following test. The case for $G = S_\infty$ was established by Harvey Friedman using the Scott analysis:

QUESTION 10.8. For which Polish groups G do we have the following: Whenever X is a Polish G-space and E a Borel equivalence relation with $E \leq_B E_G^X$ there must exist a Polish G-space Y with $E \leq_B E_G^Y$ and E_G^Y Borel?

As far as I am aware, this could be all Polish groups. The proof of the affirmative answer for $G = S_\infty$ is given in [**19**].

We can also ask about a structure for reduction to group actions. Very tepidly and on the basis of wafer thin evidence the authors of [**39**] conjecture a positive answer to:

QUESTION 10.9. For E a Borel equivalence relation must one of the following hold?

1. $E \leq_B E_G^X$ for some Polish group G and G-space X; or
2. $E_1 \leq_B E$.

From 8.2 at most one. In a restricted case [**77**] upholds this:

THEOREM 10.10. *(Solecki) For I a Borel ideal on $2^{\mathbb{N}}$ exactly one of the following:*

1. *$2^{\mathbb{N}}/I \leq_B E_G^X$ for some Polish group G and Polish G-space X; or*
2. *$E_1 \leq_B 2^{\mathbb{N}}/I$.*

Even in specific situations the conjecture is not known to hold, in as much as frequently it is not clear when we can reduce to a group action. For instance:

QUESTION 10.11. In some suitable standard Borel structure, is biholomorphism on complex manifolds $\leq_B E_G^X$ for some Polish group G and some Polish G-space X?

There are also two incidental problems which appeared in the course of chapter 7:

QUESTION 10.12. (Kechris; [**49**]) If G is a Polish group for which E_G^X is always Borel reducible to a countable equivalence relation, must it be the case that G is locally compact?

QUESTION 10.13. (Becker; [**3**]) Are the CLI groups the only ones with the weak Glimm-Effros property?

APPENDIX A

Ordinals

Ordinals arise from the need to keep counting through infinitely many stages. For instance, if we define the $\underset{\sim}{\Sigma}_1^0$ subsets of $\mathbb{R}$ to be the open sets and $\underset{\sim}{\Pi}_1^0$ to be the closed, and more generally $\underset{\sim}{\Sigma}_{n+1}^0$ to be the countable unions of $\underset{\sim}{\Pi}_n^0$ and $\underset{\sim}{\Pi}_{n+1}^0$ to be the complements of $\underset{\sim}{\Sigma}_{n+1}^0$ then we obtain an initial segment of the Borel sets consisting of those with *finite* rank. The issue here is that not all Borel sets are in these classes. A Borel set of the form

$$B = \bigcup_{n \in \mathbb{N}} B_n$$

with each $B_n \in \underset{\sim}{\Sigma}_n^0$ may provide a counterexample.

So we are led to a notation for the infinite rank Borel sets. The first infinite ordinal is ω and we can let $\underset{\sim}{\Sigma}_\omega^0$ be the countable unions of finite rank Borel sets, $\underset{\sim}{\Pi}_\omega^0$ *their* complements, $\underset{\sim}{\Sigma}_{\omega+1}^0$ to be the countable unions of $\underset{\sim}{\Pi}_\omega^0$, and so on.

The above provides the motivation. The formal definition is:

Definition A.1. A set α is an *ordinal* if

(i) α is transitive (so for all $\beta \in \alpha$ and $\gamma \in \beta$ we have $\gamma \in \alpha$);

(ii) α is linearly ordered by the $\in$-relation (so for all $\beta, \gamma \in \alpha$, either $\beta \in \gamma$, or $\gamma \in \beta$, or $\gamma = \beta$).

α is a *successor* ordinal if it has a largest element, and otherwise it is a *limit*.

Example A.2. The first ordinal is the empty set, $\emptyset$, which we may identify with 0. Then

$$1 = \{0\}$$

the set whose only member is $0 (=_{df} \emptyset)$. Then

$$2 = \{0, 1\}$$

and more generally

$$n + 1 = \{0, 1, 2, ..., n\}.$$

The first infinite ordinal is

$$\omega = \{0, 1, 2, 3, ...\}$$

and we keep counting with

$$\omega + 1 = \{0, 1, 2, 3, ...\omega\},$$

$$\omega + 2 = \{0, 1, 2, 3, ...\omega, \omega + 1\},$$

reaching the next limit at

$$\omega + \omega = \{0, 1, 2, 3, ...\omega, \omega + 1, \omega + 2, ...\}.$$

Strictly speaking 0 is a limit ordinal, but it is frequently put in a class by itself. In any case, with the exception of this minor point, we can divide the ordinals into the classes of successor and limit.

At once from the definition at A.1 we have that any member of an ordinal is again an ordinal.

DEFINITION A.3. A set A is said to be *well ordered* by a relation $<$ if

(i) $<$ is well founded, in the sense that every non-empty subset of A has a $<$-least element:

$$\forall B \subset A(B \neq \emptyset \Rightarrow \exists b \in B(\forall a \in A(a < b \Rightarrow a \notin B)));$$

(ii) A is linearly ordered by the $<$-relation.

Two immediate consequences of the usual formalizations of mathematical reasoning are:

THEOREM A.4. *Every ordinal is well ordered by* $\in$.

THEOREM A.5. *If* $<$ *well orders the set* A *then there is some ordinal* α *so that* $(A,<)$ *and* $(\alpha, \in)$ *are isomorphic as linear orderings.*

For our purposes we may as well take A.4 and A.5 as axioms.

A.4 provides the foundation for arguments by *transfinite induction*. Suppose $C \subset \alpha$ includes 0, and for all $\beta \in \alpha$ it includes $\beta+1$, and whenever $\lambda \in \alpha$ is a limit ordinal with $\lambda \subset C$ we have $\lambda \in C$, then we may conclude that C equals α: Otherwise there will be some $\in$-least $\gamma \in (\alpha \setminus C)$ by the assumption $\in$ provides a well ordering on α; but then this least γ can be neither successor nor limit by the assumptions on C.

DEFINITION A.6. A set is countable if there is an onto map

$$\pi : \mathbb{N} \twoheadrightarrow A.$$

ω_1 is the first uncountable ordinal.

Thus we allow that finite sets are countable. A set is said to have cardinality $\aleph_0$ if it is infinite and countable.

Since there is a bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$, countable unions of countable sets are countable.

LEMMA A.7.

$$\text{Borel} = \bigcup_{\alpha \in \omega_1} \underset{\sim}{\Pi}^0_\alpha$$

PROOF. $\supset$ is clear from the definition of Borel as the smallest superset of open forming a σ-algebra.

For the other direction we have that open sets are $\underset{\sim}{\Pi}^0_2$ and so should only be concerned with showing $\bigcup_{\alpha \in \omega_1} \underset{\sim}{\Pi}^0_\alpha$ forms a σ-algebra. Suppose that $B_n \in \underset{\sim}{\Pi}^0_{\alpha(n)}$ for each $n \in \mathbb{N}$. Note that since ω_1 is not the countable union of countable sets there will be some $\delta < \omega_1$ with each $\alpha(n) < \delta$. But then

$$\bigcup_{n \in \mathbb{N}} B_n \in \underset{\sim}{\Sigma}^0_\delta \subset \underset{\sim}{\Pi}^0_{\delta+1}.$$

□

For a thorough treatment see [**45**] or [**68**].

APPENDIX B

Notation

NOTATION B.1. **Sets** $\emptyset$ is the empty set, uniquely defined by the description that it has no members.

For A, B both sets, $A \subset B$ indicates that A is included in B; $A \subset B$ is *not* taken to mean that A is necessarily unequal to B, so for instance it is true that

$$\emptyset \subset \emptyset.$$

$A \setminus B$ is the set of all points in A and not in B. $A \cup B$ is the set of points in either A or B and $A \cap B$ is the set of points in both. $A \Delta B$ indicates the *symmetric difference* of these sets:

$$A \Delta B = (A \setminus B) \cup (B \setminus A).$$

For A some set and $Q(\cdot)$ some property, we use

$$\{x \in A : Q(x)\}$$

and

$$\{x \in A | \ Q(x)\}$$

interchangeably to mean the set of all points in A with the property Q. This allows some deviations, for instance if f is a function, then

$$\{f(x) : Q(x)\}$$

would indicate the set of all images under f of x's which have Q. In the case of functions it would look strange to write

$$\{f : X \to Y : Q(f)\},$$

and so we use

$$\{f : X \to Y | \ Q(f)\}$$

to indicate the set of functions from X to Y which have property Q.

$\mathcal{P}(A)$ is the set of all subsets of A.

NOTATION B.2. **Sequences** A sequence – that is to say a function from $\mathbb{N}$ or some finite initial segment of $\mathbb{N}$ – is commonly denoted by vector notation, as in $\vec{x}$; here the reader should expect that the nth term of this sequence is x_n. Alternative notations for sequences include $(\cdot, \cdot, ...)$ and $\langle \cdot, \cdot, ... \rangle$. For instance the sequence $(a_0, a_1, ..., a_{k-1})$ has k elements and its second term is a_1.

For s, t finite sequences, we denote the concatenation by $s^\frown t$. So for l the length of s and $i < l$ we have

$$(s^\frown t)(i) = s(i),$$

whilst for $j \geq l$ we let

$$(s^\frown t)(j) = t(j - l).$$

There is a variant for the case that one of the sequences has length one, where we write $s^\frown a$ or $a^\frown s$ instead of $s^\frown\langle a\rangle$ or $\langle a\rangle^\frown s$. Thus for $l =$ length of s,

$$(s^\frown a)(i) = s(i), \qquad i < l,$$
$$(s^\frown a)(l) = a,$$
$$(a^\frown s)(0) = a,$$
$$(a^\frown s)(i+1) = s(i), \qquad i < l.$$

We also write $(x_n)_{n\in\mathbb{N}}$ to indicate the sequence $(x_0, x_1, ...)$. For S a set we may write $(B_s)_{s\in S}$ for the function which assigns B_s to each $s \in S$.

For a set A, we let $A^{\mathbb{N}}$ denote the collection of all infinite sequences from A and $A^{<\mathbb{N}}$ denote the collection of all finite sequences.

Given the identification of each natural number with its predecessors, we thus have that 2^n equals the set of functions from $\{0, 1, ..., n-1\}$ to $\{0, 1\}$. So in this context, 2^1 does *not* equal 2, but rather the set $\{v_0, v_1\}$, where v_0 is the sequence of length 1 whose only term is 0 and v_1 is the sequence whose only term is 1.

Of course this may seem an insane convention, but it makes for compacting expressions which would otherwise be verbose. In context it will never create confusion, but the reader should note that 2^{-n} *always* equals the inverse of 2 to the nth power. In particular 2^{-1} equals

$$\frac{1}{2},$$

and not some exotic attempt to invert the set $\{v_0, v_1\}$ from above.

NOTATION B.3. **Functions** Given a set A and a function f with domain A and a subset $B \subset A$, we use $f|_B$ to denote the restriction of the function f to B. In the special case that

$$f : \mathbb{N} \to C$$

is a function from $\mathbb{N}$ to some set C we use $f|_n$ to denote the restriction of f to the set $\{0, 1, ..., n-1\}$. Since we understand $\mathbb{N} = \{0, 1, 2, ...\}$ and each $n = \{0, 1, ..., n-1\}$, this is entirely consistent. A^B is used to indicate the set of all functions from B to A.

In general $f[A] = \{f(x) : x \in A\}$ whenever the set A is a subset of the domain of f, and $f^{-1}[B] = \{x : f(x) \in B\}$ when B is a subset of the range.

$$f : X \to Y$$

indicates that f is a function with domain X and range Y. An expression such as

$$f : x \mapsto \frac{x}{5} + 7$$

indicates that f is the function which takes x, divides by 5, adds seven, and spits out

$$\frac{x}{5} + 7.$$

If X is a set and $A \subset X$ a subset, then we define the characteristic function of A on X by

$$\chi_A : X \to \{0, 1\},$$
$$\chi_A(x) = 1 \text{ if } x \in A,$$
$$\chi_A(x) = 0 \text{ if } x \notin A.$$

If $X \times Y$ is a product of two spaces and $f : X \times Y \to B$ is a function whose domain is the product, we prefer the informal $f(x, y)$ to denote the value of f applied to the pair $(x, y) \in X \times Y$, rather than the fussier but more correct $f((x, y))$. Given $x \in X$ we let

$$\begin{aligned} f(x, \cdot) : Y &\to B \\ y &\mapsto f(x, y). \end{aligned}$$

If A is a set, then $(x_a)_{a \in A}$ indicates the function with domain A which assumes the value x_a for any $a \in A$. So in particular in the case that $A = \mathbb{N}$ this leads to one of the usual notations , $(x_n)_{n \in \mathbb{N}}$, for a sequence.

NOTATION B.4. **Topological notions** We customarily identify a topological space with its underlying set.

If X is a topological space and $A \subset X$, then $\overline{A}$ denotes its closure, A^o denotes its interior (that is the largest open set included in A), and

$$\partial A = \overline{A} \cap \overline{X \setminus A}$$

indicates the boundary.

In the case that X is a metric space and d is the metric, we let

$$d(B) = \sup \{d(b_1, b_2) : b_1, b_2 \in B\}.$$

For $b \in X$

$$d(b, B) = \inf \{d(b, b_1) : b_1 \in B\}.$$

NOTATION B.5. **Logical quantifiers** $\exists x(P(x))$ is used to indicate that there exists some x with $P(x)$, while $\forall x(P(x))$ indicates that every x has $P(x)$. These "quantifiers" are embellished in a number of ways. For instance

$$\exists^\infty x(P(x))$$

indicates that there are infinitely many different x's with P, while

$$\forall^\infty x(P(x))$$

is used when all but finitely many x's have P – that is to say, $\forall^\infty x(P(x))$ is equivalent to

$$\neg\exists^\infty \neg x(P(x)).$$

Alternatively

$$\exists x \in X(P(x))$$

is used to assert that there is some element x of the set X with $P(x)$ whilst

$$\forall x \in X(P(x))$$

asserts that no element of X fails to have P.

The categoricity quantifiers $\exists^*$ and $\forall^*$ are defined at 2.45.

NOTATION B.6. **Unions of sets** If A is a set of sets we let

$$\bigcup A$$

be the union of all sets contained in A. If

$$x \mapsto A_x$$

is an assignment of sets to elements of a space X, then for any $Y \subset X$

$$\bigcup_{x \in Y} A_x$$

indicates the union of the set of sets $\{A_x : x \in Y.\}$.

Of course one could iterate the union operation, and we have

$$\bigcup\bigcup A$$

as the union of all sets contained in $\bigcup A$, and so on. The end result of iterating this process infinitely often and taking all sets arising this way is denoted $\mathrm{TC}(A)$. Thus $\mathrm{TC}(A)$, the *transitive closure* of A, is the collection of all sets appearing as an element somewhere in the hierarchy

$$\begin{gathered} A, \\ \bigcup A, \\ \bigcup\bigcup A, \\ \bigcup\bigcup\bigcup A, \\ \vdots \end{gathered}$$

Alternatively one may define the transitive closure of A to be the smallest transitive set including A.

Thus if we have $A = \{\{1, \{2,4\}\}, 6, \{1,3,4\}\}$, the set whose members are exactly $\{1, \{2,4\}\}, 6, \{1,3,4\}$, then $\mathrm{TC}(A)$ would be

$$\{0, 1, 2, 3, 4, 5, 6, \{2,4\}, \{1,3,4\}, \{1, \{2,4\}\}\}.$$

The hereditarily countable sets, HC, are all sets A for which $\mathrm{TC}(A)$ is countable.

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