A dichotomy theorem for being essentially countable

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Abstract. We introduce a definition to characterize when an orbit equivalence relation is Borel reducible to a countable equivalence relation. As a corollary we obtain a kind of low basis theorem – an orbit equivalence relation which arises from an *effective* action of a recursively presented Polish group on a recursive Polish space is Borel reducible to a countable Borel equivalence relation if and only if it is reducible to a Δ^1_1 countable equivalence relation using a Δ^1_1 function. We make some effort to draw out the consequences of the theorem for the model theoretic context, which in turn raises an open problem regarding first order logic.

1. Introduction

Definition We say that one equivalence relation, E, is Borel reducible to another, F, written $E \leq_B F$, if there is a Borel function θ such that

$$x_1 E x_2 \Leftrightarrow \theta(x_1) F \theta(x_2).$$

Definition An equivalence relation is *Borel* if it is Borel as a subset of the product of the space on which it is defined; it is *countable* if all its equivalence classes are countable. An equivalence relation on a standard Borel space X is *essentially countable* if there is a countable Borel equivalence relation E to which it is Borel reducible.

Example It is trivially seen that if E arises from the Borel action of a *countable* group then necessarily it is Borel and each equivalence class is countable. More surprisingly Kechris has shown in [**13**] that the orbit equivalence relations arising from Borel actions of locally compact Polish groups are necessarily essentially countable.

In very general terms the essentially countable equivalence relations might be thought of as those where there is some kind of notion of "finite rank".

Example Let σ be the sentence in $\mathcal{L}_{\omega_1,\omega}$ whose models are exactly the *finite rank* torsion free abelian groups – that is to say, those abelian, torsion free groups which have some finite sequence $g_1, g_2, ...g_n$ such that for every h there is some k such that $k \cdot h$ is in the subgroup generated by $g_1, g_2, ...g_n$. Let $\text{Mod}(\sigma)$ be the collection of models for σ with underlying set $\mathbb{N}$ and the usual Borel structure (as described in [**10**]); let $\cong_{\text{Mod}(\sigma)}$ be the equivalence relation of isomorphism on $\text{Mod}(\sigma)$. Then

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$\cong_{\mathrm{Mod}(\sigma)}$ is essentially countable. In fact, at each n we can let σ_n be the sentence whose models are exactly the rank n torsion free abelian groups, and we obtain the that $\cong_n (=_{\mathrm{df}} \cong_{\mathrm{Mod}(\sigma_n)})$ is Borel reducible to the action of $\mathbb{Q}^n$ by multiplication on the collection of all subgroups of $(\mathbb{Q}^n, +)$. Indeed recent years have witnessed an outpouring of papers on these equivalence relations – culminating in Simon Thomas' [**15**], where it was shown that at every n one has

$$\cong_n <_B \cong_{n+1} .$$

On the other hand it is known the isomorphism relation on countable torsion free abelian groups of infnite rank is not essentially countable in any of its customary Borel structures. (See e.g. [**9**], §5 [**11**]). Indeed a natural example of an equivalence relation which is not essentially countable is given by considering the coding of countable sets of reals; that is to say, for $f_1, f_2 \in \mathbb{R}^{\mathbb{N}}$ set

$$f_1 F_2 f_2$$

if for all $n \in \mathbb{N}$ there exists $m_1, m_2 \in \mathbb{N}$ with

$$f_1(n) = f_2(m_1),$$

$$f_2(n) = f_1(m_2).$$

One certainly find in [**11**] a proof that this equivalence relation is not essentially countable and in [**9**] a proof that isomorphism on general countable torsion free abelian groups Borel reduces F_2 – though in both cases the results are folklore and invoking theorems of these papers would represent a degree of overkill.

Example In the natural Borel structure, induced by looking at group operations placed on some fixed countable set, the isomorphism relation on finitely generated groups is essentially countable. (See [**10**].) This cannot be extended to general countable groups, since, as noted above, the isomorphism relation on countable abelian groups is already $\leq_B$ above F_2.

A special kind of essentially countable equivalence relation is formed by those which are smooth.

Definition An equivalence relation E is *smooth* if there is a Borel function θ from the space on which E lives to $\mathbb{R}$ (or, equivalently, any uncountable Polish space) with

$$x_1 E x_2 \Leftrightarrow \theta(x_1) = \theta(x_2).$$

Not all essentially countable equivalence relations, even among those given by the isomorphism relation on some natural class of structures, are smooth. For instance, §3.1 of [**6**] gives a gentle proof that $\cong_1$, isomorphism of rank one torsion free abelian groups, is not smooth. In [**10**] a model theoretic analysis of smoothness and essential countability is presented.

THEOREM 1.1. *(Hjorth, Kechris) Let L be a countable language, with the usual Borel structure generated by first order logic.*[1] *Let $\sigma \in \mathcal{L}_{\omega_1,\omega}$. (I) The following are equivalent: (a) $\cong_{\mathrm{Mod}(\sigma)}$ is smooth; (b) there is a countable fragment $F \subset \mathcal{L}_{\omega_1,\omega}$ such that for every $M \in \mathrm{Mod}(\sigma)$ we have that*

$$\mathrm{Th}_F M,$$

[1] So that for any ψ and $\vec{a}$ a finite sequence from $\mathbb{N}$, the set of M which satisfy $\psi(\vec{a})$ is Borel; this is indeed a standard Borel structure – see [**4**].

the F-theory of M, is $\aleph_0$-categorical; (c) there is a countable fragment $F \subset \mathcal{L}_{\omega_1,\omega}$ such that for every $M \in \mathrm{Mod}(\sigma)$ we have that

$$\mathrm{Th}_F M,$$

the F-theory of M, is atomic. (II) The following are equivalent: (a) $\cong_{\mathrm{Mod}(\sigma)}$ is essentially countable; (b) there is a countable fragment $F \subset \mathcal{L}_{\omega_1,\omega}$ such that for every $M \in \mathrm{Mod}(\sigma)$ we have some $\vec{a} \in M$ such that

$$\mathrm{Th}_F\langle M, \vec{a}\rangle,$$

the F-theory of the expansion of M obtained by adding names for all the elements of the $\vec{a}$ sequence, is $\aleph_0$-categorical; (c) there is a countable fragment $F \subset \mathcal{L}_{\omega_1,\omega}$ such that for every $M \in \mathrm{Mod}(\sigma)$ we have some $\vec{a} \in M$ such that

$$\langle M, \vec{a}\rangle,$$

is F-atomic.

Question Does there exist a first order theory T with $\cong_{\mathrm{Mod}(T)}$ essentially countable but not smooth?

In light of 1.1 the question of whether smoothness and essential countability coincide for first order structures has a purely model theoretic reading. We are asking for a countable first order theory T for which there exists a countable fragment F over which every $M \models T$ has some $\vec{a}$ with

$$\langle M, \vec{a}\rangle$$

F-atomic, but for which there is no fragment F' for which every model $M \models T$ is F'-atomic; notice that this purely model theoretic formulation makes no explicit reference to countable models. In practically all the standard cases, say for instance the kinds of first order theories one finds in [**2**], it is easily determined that $\mathrm{Mod}(T)$ is either smooth or not essentially countable.

While I would conjecture that an example of a countable first order theory with $\cong_{\mathrm{Mod}(T)}$ essentially countable but non-smooth should exist, but simply be very hard to conceive, it does pay to seriously consider the possibility that there may be a structure theorem which denies this. This would represent some kind of basic obstruction or dichotomy theorem for $L_{\omega,\omega}$ equivalence relations which is entirely missing for $L_{\omega_1,\omega}$. Here it is worth pointing out that any such dichotomy theorem could not be proved using purely Baire category methods.

Example Let G be a countable infinite group and L the language which contains function symbols F_g for each $g \in G$ along with unary predicates $(U_n)_{n\in\mathbb{N}}$. Let T be the first order theory which asserts that the composition of the functions exactly mimics the group (so for each g_1, g_2 we have an axiom asserting $F_{g_1} \circ F_{g_2} = F_{g_1 g_2}$), the identity element acts trivially ($\forall a(F_1(a) = a)$), and that otherwise the action of G is free (for each $g \neq 1$ we have $\forall a(F_g(a) \neq a)$). We can equip with $\mathrm{Mod}(T)$, the space of T-structures with underlying set $\mathbb{N}$, the usual topological structure, under which for every quantifier free $\psi(\cdot)$ and $\vec{a}$ we have

$$\{M : M \models \psi(\vec{a})\}$$

becomes clopen. In this topology it is a Polish space, and it follows easily from the usual kinds of arguments that on no comeager set C is

$$\cong|_C$$

smooth. (Compare the arguments from say §3.1[**6**].) A failure to be essentially countable is not yet detectable at the level of Baire category. If we take C to be the comeager set of models on which the group action is transitive, that is to say, the structure satisfies the $L_{\omega_1,\omega}$ sentence

$$\forall a, b \bigvee_{g\in G} (F_g(a) = b),$$

then we have that

$$\cong |_C$$

is essentially countable. This can be observed by (II)(c) 1.1 since on this set we have that any $M \in C$ is the functional closure of *any* $a \in M$. However the isomorphism relation of the theory is not essentially countable. One way to see this is actually to use forcing, starting with a fragment F in the ground model and showing its failure to achieve (II)(c) in a suitable generic extension. We begin with an action of G on $\mathbb{N}$ which is free but has infinitely many orbits and interpret the function symbols $(F_g)_{g\in G}$ accordingly. Then we force to obtain mutually Cohen generic $(x_a)_{a\in\mathbb{N}}$ elements of $2^{\mathbb{N}}$; for each x_a we define the indicated pattern of unary predicates at a, with

$$U_n(a) \Leftrightarrow x_a(n) = 1.$$

It is a routine exercise in forcing to see that if $a_0, a_1 ..., a_\ell = \vec{a}$ is a *finite* sequence from $\mathbb{N}$ then

$$\mathrm{Th}_F\langle M, \vec{a}\rangle$$

can be uniformly defined in

$$V^{[(x_{F_g(a_i)})_{i\leq\ell,g\in G}]}$$

from F and $(x_{F_g(a_i)})_{i\leq\ell,g\in G}$; then considering some $b \notin \{F_g(a_i) : i \leq \ell, g \in G\}$ we have that the F-type of x_b will not be in $V^{[(x_{F_g(a_i)})_{i\leq\ell,g\in G}]}$, and hence certainly not F-atomic over $\mathrm{Th}_F\langle M, \vec{a}\rangle$.

In this note we introduce a new definition to analyze essential countability.

Definition Let G be a Polish group acting continuously on a Polish space X. We then say that X is a *stormy* G-space if for every non-empty, open $V \subset G$ and $x \in X$ we have that the natural map

$$g \mapsto g \cdot x$$

from V to $V \cdot x$ is not an open function.

For future reference note that this definition has a kind of stability and upward persistence.

LEMMA 1.2. *If X is a stormy Polish G-space with respect to the topology τ, then for any stronger Polish G-space topology $\tau_0 \supset \tau$ we can find a $G_\delta(X, \tau)$ subspace $X_0 \subset X$ such that (X_0, τ_0) is still a stormy Polish G-space.*

Proof. It is an entirely classical fact that we can find an X_1 which is a G_δ subspace of (X, τ) such that the induced inclusion map

$$(X_1, \tau) \hookrightarrow (X, \tau_0)$$

is continuous; this follows since any Borel map is continuous on a dense G_δ (see [**12**]). Then we let

$$X_0 = \{x \in X : \forall^* g \in G(g \cdot x \in X_1)\},$$

the set of points for which a comeager collection of group elements place us in X_1. Using the continuity of the action of G on (X, τ_0) we still have that the induced map

$$(X_0, \tau) \hookrightarrow (X, \tau_0)$$

continuous; so the topology τ on X_0 equals τ_0 on X_0. So then the structure of the definition of stormy, with its insistence that something happens at *every* orbit, guarantees that

$$(X_0, \tau_0)$$

is still stormy. $\square$ An especially important instance of this definition is

provided by model theory. Here let us consider that S_∞ acts in the usual manner on the space of all models of some infinitary sentence $\sigma \in L_{\omega_1,\omega}$; cofinally many of the Polish topologies are of the form τ_F for some countable fragment F – that is to say, they have a basis consisting of sets of the form

$$\{M : M \models \varphi(\vec{a})\}$$

for some $\varphi \in F$, $\vec{a} \in \mathbb{N}^{<\omega}$. (Again, see [**10**]). Here we may assume without loss of generality that we restrict ourselves to open sets V of the form

$$\{\pi \in S_\infty : \forall i \leq \ell(\pi(a_i) = a_i)\}$$

for some chosen $\vec{a} = a_0, a_1, ...a_\ell$ in $\mathbb{N}$. Then the action of S_∞ on $\langle$ Mod(σ), $\tau_F\rangle$ is stormy if and only if for every $M \models \sigma$ there is *no* $\vec{a} \in M$ with $\langle M, \vec{a}\rangle$ being F-atomic. Building on the methods of [**8**]:

THEOREM 1.3. *Let G be a Polish group and X a Polish G-space with the induced orbit equivalence relation E^X_G Borel. Then exactly one of the following holds: (I) E^X_G is essentially countable; (II) there is a stormy Polish G-space Y and a continuous G-embedding*

$$\rho : Y \to X.$$

Note in (II) that since ρ is one to one, the image $\rho[Y]$ is Borel in X. Thus we may without loss of generality assume that Y is a Borel invariant subset of X and that we have found a Polish topology on Y which refines the original X-topology under which the action remains continuous and becomes stormy. I have no concrete instances of an equivalence relation whose essential countability can only be affirmed or refuted by appeal to 1.3. However it can be said that the argument is effective; it uses the "geometry" of Σ^1_1 sets in the now usual way. In particular the following represents a corollary of the method, which would presumably have not been obvious hitherto.

THEOREM 1.4. *Let G be a recursive Polish group and X a recursive Polish G-space with the induced orbit equivalence relation $E^X_G \in \Delta^1_1$. If E^X_G is essentially countable, then there is a countable Δ^1_1 equivalence relation on an effective Polish space to which E^X_G is reducible by a Δ^1_1 function.*

It might also be worth thinking about the model theoretic consequences of this theorem. Again, for a logician the isomorphism relations on classes of countable structures, while not quite Borel, comprise the central examples to which all others should be compared.

In the next corollary we bear in mind that $\cong_{\mathrm{Mod}(\sigma)}$ is Borel if and only if there is some countable ordinal which bounds the Scott heights of models of σ. This result from [**1**] makes possible to state the following corollary in largely model theoretic terminology.

COROLLARY 1.5. *Let $\sigma \in L_{\omega_1,\omega}$ for some countable language L. Assume that there is some countable ordinal which bounds the Scott heights of the models of σ. Then exactly one of the following holds. (I) there is a countable fragment $F \subset \mathcal{L}_{\omega_1,\omega}$ such that for every $M \in \mathrm{Mod}(\sigma)$ we have some $\vec{a} \in M$ such that*

$$\langle M, \vec{a}\rangle,$$

is F-atomic; (II) there is a countable fragment F and some $\varphi \in L_{\omega_1,\omega}$ which has a countable model, such that for every

$$M \models \varphi$$

there is no $\vec{a} \in M$ with

$$\langle M, \vec{a}\rangle$$

F-atomic.

To see that the corollary indeed follows from the theorem we observe first of all that (I) 1.5 is a direct restatement of (I) 1.3 in light of 1.1. This leaves (II). Note at this point we actually obtain in applying 1.3 a Borel S_∞-invariant Borel subspace Y and a Polish topology τ_Y on Y under which the action becomes stormy. We may find a countable fragment F such that $\tau_F \supset \tau_Y$ by [**14**], and then by 1.2 find a $Y_0 \subset Y$ so that (Y_0, τ_F) is a stormy Polish S_∞-space; membership in Y_0, by [**14**] again, is given by some sentence $\varphi \in L_{\omega_1,\omega}$.

2. Setting Sail

In all the lemmas below we assume that G is a Polish group and X is a Polish G-space. E^X_G is the orbit equivalence relation. We assume there are fixed countable bases for X and G and that the basis for G contains a neighborhood basis of symmetric sets around 1_G.

LEMMA 2.1. *Suppose for every $x \in X$ there is a non-empty open $V \subset G$ with*

$$V \to V \cdot x$$

$$g \mapsto g \cdot x$$

an open mapping. Then E^X_G is essentially countable.

Proof. Let us say that a non-empty basic open set $V \subset G$ is *x-safe* if for all $W \subset V$ open, non-empty, $\overline{W \cdot x}$ includes a non-empty relatively open subset of $\overline{V \cdot x}$. Our assumption in the lemma yields that for each $x \in X$ there is such an x-safe point. Note moreover that any non-empty open subset of an x-safe set is again x-safe. If V is x-safe and $U \subset X$ is a non-empty basic open set, then we say that U is a *V-x-harbor* if there is some symmetric basic open neighborhood W of 1_G such that

$$\forall h \in W \forall y \in U \cap \overline{Vh \cdot x}(y \in \overline{V \cdot x}).$$

Claim(I): For each x there are U and V as above. **Proof of claim:** Choose open

$\hat{V}$ with

$$\hat{V} \mapsto \hat{V} \cdot x$$

an open map. Let $g_0 \in \hat{V}$. Appealing to the continuity of the group operations we may choose $V \subset \hat{V}$ an open neighborhood of g_0 and W a symmetric open neighborhood of 1_G such that

$$VW \subset \hat{V}.$$

V is clearly x-safe, since the mapping

$$V \to V \cdot x$$

is open. Let us choose U basic open with

$$\overline{V \cdot x} \supset U \cap \overline{\hat{V} \cdot x}.$$

Then we have for all $h \in W$ and $y \in U \cap \overline{Vh \cdot x}$

$$y \in U \cap \overline{\hat{V} \cdot x}$$

$$\therefore y \in \overline{V \cdot x},$$

just as required. (Claim(I)$\square$) For each x, the collection of such (U, V) as above

is uniformly $\underset{\sim}{\Pi}^1_1$ in x. The bases from which they are chosen are countable. Hence we may assign

$$x \mapsto (U_x, V_x)$$

in a Borel manner so that each V_x is x-safe and each U_x is a V_x-x-harbor. We can then define a Borel function

$$X \to \mathcal{F}(X),$$

from X to the standard Borel space of closed subsets of X, given by

$$x \mapsto \overline{U_x \cap V_x \cdot x}.$$

By Kechris' criterion for being essential countable, as presented at §5 of [**7**], it is enough to show that this map has countable image on each equivalence class (claim(II)) and that images of distinct equivalence classes are disjoint (claim(III)). **Claim(II):** For each x, the set $\{\overline{U_y \cap V_y \cdot y} : y \in [x]_G\}$ is countable. **Proof of**

claim: Otherwise we could find an uncountable set $(y_\alpha)_{\alpha \in \aleph_1}$ of elements of $[x]_G$ with each y_α having the same $U_{y_\alpha} = U, V_{y_\alpha} = V$ and the same W witnessing that U is a V-y_α-harbor, and yet

$$U \cap \overline{V \cdot y_\alpha} \neq U \cap \overline{V \cdot y_\beta}$$

for $\alpha \neq \beta$. Considering the separability of the group G we can find two y, y' in this uncountable set with $y \in Wy'$ and $y' \in Wy$. Then for all $z \in U \cap \overline{V \cdot y} \subset U \cap \overline{VW \cdot y'}$ we must have by assumptions on U and V that $z \in U \cap \overline{V \cdot y'}$, and conversely. Thus

$$U \cap \overline{V \cdot y} = U \cap \overline{V \cdot y'}$$

after all. (Claim(II)$\square$) **Claim(III):** If V is x-safe, then $V \cdot x$ is comeager in $\overline{V \cdot x}$. **Proof of claim:** Suppose not. Then there is some open $U \subset X$ with $V \cdot x$ meager in $U \cap \overline{V \cdot x}$. And then we can find some open $V_0 \subset V$ and open $U_0 \subset U$ with $V_0 \cdot x$ dense in $U_0 \cap \overline{V \cdot x}$; which would place us in the situation that V_0 is now our x-safe set and $V_0 \cdot$ is meager in $U_0 \cap \overline{V_0 \cdot x}$. But then if we cover $V_0 \cdot x$ by countably many closed sets, $F_0, F_1, ...F_n, ...$, each of which is nowhere dense in $U_0 \cap \overline{V_0 \cdot x}$, then there must be some open non-empty $W \subset V_0$ with $W \cdot x \subset F_n$, at some $n \in \mathbb{N}$. And so $W \subset V_0 \subset V$ contradicts the definition of safety. (Claim(III)$\square$) $\square$ This lemma has as an immediate corollary a result previously proved by Alexander Kechris by other means:

COROLLARY 2.2. *(Kechris) If G is a locally compact Polish group and X is a Polish G-space, then E_G^X is essentially countable.*

Proof. Fix $x \in X$ and consider some non-empty precompact $V \subset G$. $\overline{V}/G_x$ is compact[2], and hence the natural map

$$\overline{V}/G_x \twoheadrightarrow \overline{V} \cdot x$$

maps closed sets to closed sets and is one-to-one and onto, thus it is open. In particular

$$V/G_x \twoheadrightarrow V \cdot x$$

and

$$V \twoheadrightarrow V \cdot x$$

are both open, and we fall under the case assumptions of 2.1. $\square$

LEMMA 2.3. *If E_G^X is essentially countable, then for a comeager collection of $x \in X$ there exists $V \subset G$ open and non-empty with*

$$V \to V \cdot x$$

$$g \mapsto g \cdot x$$

open.

Proof. Let

$$\theta : X \to Z$$

be a Borel function reducing E_G^X to some countable equivalence relation on a Polish space Z. Following 4.1 [**3**] (or see §18 of [**12**], along with references tracing back to earlier work by Burgess) we have that the range of θ is Borel and there is a Borel function

$$\rho : \mathrm{Ran}(\theta) \to X$$

[2]Here as elsewhere we follow the notation of [**1**]; so G_x is the *stabilizer* of the point x: that is to say, the subgroup $\{g \in G : g \cdot x = x\}$.

such that for all $z \in \mathrm{Ran}(\theta)$ we have $\theta(\rho(z)) = z$. For each $x \in X$ let

$$p_x = \rho(\theta(x));$$

thus this $x \mapsto p_x$ is a Borel function which in effect "selects" a countable set from each orbit. Following 7.1.2 of [**1**] (the orbit equivalence relation is Borel, and thus too is the stabilizer function $x \mapsto G_x$ from X to $\mathcal{F}(X)$) we may in a Borel manner choose for each x a group element g_x with

$$g_x \cdot x = p_x.$$

For the moment let us say that a point $y \in X$ is *good* if there is some basic open neighborhood W of 1_G such that $\forall^* h \in W(p_y = p_{h\cdot y})$. **Claim:** The set of good points is comeager. **Proof of claim:** Otherwise use Kuratowski-Ulam to find some $y \in X$ and open non-empty $V \subset G$ such that $\forall^* g \in V(g \cdot y$ is not good). Then since $\{p_{y'} : y' E_G^X y\}$ is countable, we may find some single y_0 and non-empty open $W \subset V$ such that

$$\forall^* g \in W(p_{g\cdot y} = y_0).$$

Then we find some basic open neighborhood $\hat{W}$ of 1_G and $g_0 \in G$ with $\hat{W} g_0 \subset W$; after possibly nudging g_0 slightly, we may assume it falls into the relatively comeager sets $\{g \in W : g \cdot y$ not good $\}$ and $\{g \in W : p_{g\cdot y} = y_0\}$. But this gives

$$\forall^* g \in \hat{W}(p_{g\cdot(g_0\cdot y)} = y_0 = p_{g_0\cdot y}),$$

and hence $g_0 \cdot y$ must be good after all. (Claim$\Box$) Let $(W_\ell)_{\ell\in\mathbb{N}}$ be a neighborhood basis at 1_G. We may apply the above claim to choose a Borel function

$$\gamma : X \to \mathbb{N}$$

such that

$$\forall^* y \in X \forall^* h \in W_{\gamma(y)}(p_y = p_{h\cdot y}).$$

Finally choose a comeager set $C \subset X$ such that (a) $\gamma|_C$ is continuous on C; (b) $x \mapsto g_x$ and $x \mapsto p_x$ are continuous on C; (c) for all $x \in C$ we have $\forall^* g \in W_{\gamma(x)}(p_x = p_{g\cdot x})$. Let $x \in X$ be such that $\forall^* g \in G(g\cdot x \in C)$. We can then find some non-empty open set $\hat{V} \subset G$ and a point $x_0 \in [x]_G$ such that

$$\forall^* g \in \hat{V}(p_{g\cdot x} = x_0).$$

We then find open U_0 having non-empty intersection with $\hat{V} \cdot x$ such that for some single ℓ_0

$$\forall y \in U_0 \cap C(\gamma(y) = \ell_0).$$

Finally we pass to $V = \{g \in \hat{V} : g \cdot x \in U_0\}$. I now want to show that $g \mapsto g \cdot x$ is open from V to $V\cdot x$. So consider some $y \in V\cdot x$ and V_0 an open neighborhood of 1_G; I want to show that there is an open U_1 containing y such that for all $y_1 \in V\cdot x \cap U_1$ we have $y_1 \in V_0 \cdot y$. First choose V_1 an open neighborhood of 1_G with

$$(V_1)^{-1} = V_1,$$

$$(V_1)^4 \subset V_0,$$

$$V_1 \cdot y \subset V \cdot x,$$

$$V_1 \subset W_{\ell_0}.$$

And then take $g_0 \in V_1$ with

$$y' = g_0 \cdot y \in C$$

and

$$p_{y'} = x_0.$$

And then let $U' \subset U_0$ be a basic neighborhood of y' such that for all $\hat{y} \in U' \cap C$ we have

$$g_{\hat{y}} \in g_{y'} V_1.$$

I claim that $U_1 = U_0 \cap (U')^{\Delta V_1}$ succeeds. For this purpose, let us suppose $y_1 \in V \cdot x \cap U_1$; we can choose $g_1 \in V_1$ with

$$y_2 =_{\mathrm{df}} g_1 \cdot y_1 \in V \cdot x \cap U_1 \cap C$$

and $p_{y_2} = x_0$. We may then appeal to the definition of U_1 to find $g_2 \in V_1$ with

$$y_2' =_{\mathrm{df}} g_2 \cdot y_2 \in U' \cap C.$$

Since $y_2 \in U_0 \cap C$ and $g_2 \in V_1 \subset W_{\ell_0}$ we can appeal to the assumption on U_0 to find a sequence of group elements $(\epsilon_n)_{n \in \mathbb{N}}$ which approach 1_G and have the property that

$$p_{\epsilon_n \cdot y_2'} = p_{y_2} = x_0$$

and each $\epsilon_n \cdot y_2' \in C$; since the function $\hat{y} \mapsto p_{\hat{y}}$ is continuous on C we must have $p_{y_2'}$ equals x_0 as well. By assumption on U' we may find some $h \in V_1$ with

$$g_{y_2'} = g_{y'} \cdot h.$$

But this gives

$$g_{y_2'} \cdot y_2' = x_0 = g_{y'} \cdot y'$$

$$\therefore g_{y'} h \cdot y_2' = g_{y'} \cdot y'$$

$$\therefore h \cdot y_2' = y'$$

$$\therefore h g_2 g_1 \cdot y_1 = y' = g_0 \cdot y$$

$$\therefore y_1 = g_1^{-1} g_2^{-2} h^{-1} g_0 \cdot y \in (V_1)^4 \cdot y \subset V_0 \cdot y.$$

□

Definition Let G be a Polish group and X a Polish G-space. We say that X is a *stormy* G-space if

$$V \to V \cdot x$$

$$g \mapsto g \cdot x,$$

as V ranges over non-empty open subsets of the group and x ranges over points in X, is never an open mapping.

PROPOSITION 2.4. *If $E_G^X \leq_B E$, and X is a stormy Polish G-space, then E is not essentially countable.*

Proof. This is a corollary of 2.3. □ In the remainder of the note we wish to start working towards a kind of converse to this proposition. If E is induced by continuous action of a Polish group G, then E is essentially countable if and only if there is no stormy Polish G-space Y and continuous G-embedding from Y to X. It might be worthwhile first to pause in order to point out a couple of facts in the neighborhood of this definition.

Example It is not hard to see that if the canonical map from V to $V \cdot x$ is open, then $V \cdot x$ is comeager in its closure. This was implicitly proved at claim(III) of 2.1. The reverse implication fails. For instance let $G = S_\infty$, the infinite symmetric group, and consider the space B_2, consisting of all one-to-one sequences $\vec{x} = (x_0, x_1, ...) \in ([0,1])^{\mathbb{N}}$. The responsible group acts by permutation of the indices in the obvious manner:

$$(\pi \cdot \vec{x})_n = x_{\pi^{-1}(n)}.$$

Consider the point $\vec{y} = (y_0, y_1, ...)$ defined by

$$y_0 = 0,$$

but

$$y_n = \frac{1}{n}$$

for $n > 0$. We can then let

$$V_0 = \{\pi \in S_\infty : \pi(0) = 0\},$$

and for $k > 0$ we let

$$V_k = \{\pi \in S_\infty : \pi(k) = 0, \pi(0) = k\}.$$

It is not hard to see that at each $k > 0$ we have $V_k \cdot \vec{y}$ comeager in its closure; and thus for

$$V = \bigcup_{k \geq 0} V_k$$

we have that $V \cdot \vec{y}$ comeager in $\overline{V \cdot \vec{y}}$. In fact the set $V \cdot \vec{y}$ is a G_δ. A point $\vec{x}$ is in $V \cdot \vec{y}$ if and only if

(i) $\forall n, m \exists \ell (|x_\ell - 1/n| < 1/m)$;
(ii) $\forall n, \ell_1, \ell_2 (\ell_1 \neq \ell_2 \Rightarrow (|x_{\ell_1} - 1/n)| \geq (n+1)^{-2} \vee |x_{\ell_2} - 1/n)| \geq (n+1)^{-2})$;
(ii) $\forall \ell, m \exists n (|x_\ell - 1/n| < 1/m)$;
(iv) $\forall m \exists k \neq 0 ((|x_0| < 1/m) \vee (|x_k| < (k+1)^{-2} \wedge |x_0 - 1/k| < (m(k+1))^{-2}))$.

However the map

$$V \to V \cdot \vec{y}$$

$$\pi \mapsto \pi \cdot \vec{y}$$

is not open, since $V_0 \cdot \vec{y}$ is contained in a closed nowhere dense subset of $\overline{V \cdot \vec{y}}$.

Question Suppose that for every $x \in X$ there is *some* V such that $V \cdot x$ is non-meager in $\overline{V \cdot x}$. Must E_G^X be essentially countable? (must the hypotheses of 2.1 hold?)

Proposition 2.5. *Let G be a Polish group acting continuously on a Polish space X. Let $\mathcal{B}$ be a countable basis for G. Suppose*

$$\forall^* x \in X \forall W \in \mathcal{B}(W \neq 0 \Rightarrow W \cdot x \text{ is meager in } \overline{W \cdot x}).$$

Then E_G^X is not essentially countable.

Proof. First of all we can take a dense G_δ set $C \subset X$ such that for all $x \in C$ and all $W \in \mathcal{B}, W \neq 0$ we have $W \cdot x$ is meager in $\overline{W \cdot x}$. Then we can let

$$X_0 = C^{*G},$$

the Vaught transform of C. This is then an invariant G_δ set, and it suffices to show $E_G^{X_0}$ is not essentially countable. For this it suffices to observe the following claim: **Claim:** X_0 is a stormy Polish G-space. **Proof of Claim:** Fix $x \in X_0$ and

$V \subset G$ open. We may then choose $\epsilon \in G$ with $\epsilon \cdot x \in C$. Then we may choose some non-empty $W \in \mathcal{B}$ with $W\epsilon \subset V$. $W \cdot (\epsilon \cdot x)$ is meager in $\overline{W \cdot (\epsilon \cdot x)}$, and thus the map

$$W \to W \cdot (\epsilon \cdot x)$$

$$g \mapsto g\epsilon \cdot x$$

is not open, and hence

$$W\epsilon \to W\epsilon \cdot x$$

$$g\epsilon \mapsto g\epsilon \cdot x$$

is not open, and thus, since $W\epsilon \subset V$,

$$V \to V \cdot x$$

$$g \mapsto g \cdot x$$

is also not an open mapping. (Claim$\square$) $\square$

Example There is a canonical example of a Polish group giving rise to an orbit equivalence relation which is not essentially countable. The orbit equivalence relation, F_2, corresponds in some rough sense to countable sets of reals. It arises on the B_2, the space of one-to-one sequences of reals in the unit interval, as a result of the natural action of S_∞ permuting the indices, as introduced in the previous example. There are several ways to prove that this equivalence relation is not essentially countable. One can even use forcing and give a metamathematical proof. However the most direct method, as sketched in the exercises from chapter 2 of [**6**], is in fact to show that for $\mathcal{B}$ the canonical basis of S_∞, we have for each $W \in \mathcal{B}$ non-empty

$$\forall^* \vec{x} \in B_2(W \cdot \vec{x} \text{ is meager in } \overline{W \cdot \vec{x}}),$$

and then to combinatorially verify that no function continuous on a comeager set can reduce F_2 to a countable equivalence relation. The interest of 2.5 is to show that this "standard proof" fits naturally into a more general context.

3. Organizing the Cases

We from now on assume that G is a Polish group, which is recursive as a Polish space, with the group operations of multiplication and inversion both recursive and continuous. We assume that X is a recursive Polish space, equipped with an action of G on X which is recursive and continuous as a function

$$G \times X \to X.$$

We further assume $E_G^X \in \Delta_1^1$. We wish to work towards showing that either E_G^X may be reduced in to a countable Borel equivalence relation or we may continuously and in an action respectful manner embed into X a stormy Polish G-space. In the first case one can in fact obtain that X is reducible to a countable Borel equivalence relation which is actually Δ_1^1 using a function which is actually Δ_1^1; for clarity of exposition we will not stress the various small calculations which must be checked to obtain this extra effectivity, but instead push on with the proof of the main result at 1.3. At the very end we make a few remarks about how it may be refined. In general of course the Polish group, the Polish space, and the action of the group on the space might only be recursive in z, for some parameter $z \in \mathbb{N}^{\mathbb{N}}$. The proof of this more general case is a relativized version of the specific one we consider below. We assume that G and X come equipped with some suitably recursive countable bases in the sense of [**8**]. At some point it will be helpful to make some simplifying assumptions: We ask that the basis for G be closed under finite intersection and inversion; that the basic open sets in X and G be closed under translation by elements from the subgroup G_0 consisting of recursive elements of G.

Definition For $A \subset X$ be Σ_1^1, $W_0, V_0 \subset G$ basic open with $1_G \in V_0$, $V_0^{-1} = V_0$, we say that (A, W_0) is V_0-*damming* if for all $V_1 \subset G$ with $1_G \in V_1$, $V_1^{-1} = V_1$, all $x \in A^{\Delta W_0}$, and $y_0 \in V_0 \cdot x$, there is $B \in \Delta_1^1$ and there is $W_1 \subset G$ basic open, with

$$y_0 \in B^{\Delta W_1}$$

such that

$$\forall y_1, y_2 \in V_0 \cdot x \cap B^{\Delta W_1}(y_1 \in V_1 \cdot y_2).$$

The property of being V_0-damming is Π_1^1 in the indices. The point is that if E_G^X is Δ_1^1 then it follows from [**1**] §7.1 that the stabilizer function $x \mapsto G_x$ is Δ_1^1, in some appropriately understood effective presentation of the collection of closed subsets of G. From this we obtain that the set of (y_1, y_2, V_1) such that

$$y_1 \in V_1 \cdot y_2$$

is Δ_1^1. This granted, it all routinely unravels to the level of Π_1^1. Thus by Π_1^1-reflection[3], if (A, W_0) is V_0-damming then we may find some Δ_1^1 set $D \supset A$ with (D, W_0) again V_0-damming. We are here confronted with a split in cases.

Definition We let X_{bad} be the set of all $x \in X$ such that for all $A \in \Sigma_1^1$, all basic open $W_0, V_0 \subset G$ with $1_G \in V_0$, $V_0^{-1} = V_0$, if $x \in A^{\Delta W_0}$ then (A, W_0) is not V_0-damming.

The set X_{bad} is Σ_1^1 by the reflection argument indicated above. From the structure of the definitions one can see that X_{bad} is invariant under the action of G_0: if $g_0 \cdot x \in A^{\Delta W_0}$ for some $g_0 \in G_0$, some V_0-damming (A, W_0), then $x \in A^{\Delta W_0 g_0}$ and $(A, W_0 g)$ is $g_0^{-1} V_0 g_0$-damming. But then we see that it must be invariant under

[3]See [**5**].

the entire action of G, since if $g_0 \cdot x \in A^{\Delta W_0}$ then for any g sufficiently close to g_0 we have $g \cdot x \in A^{\Delta W_0}$. Thus X_{bad} is invariant under the action of the entire group G. Whether this set X_{bad} is empty provides the split in cases.

Notation From now on let us understand that $V_0, V_1, V_2, ..., V, \hat{V}, V', W, ..., W_0, W_1, W_2 ...$ and such like always refer to basic open subsets of the group G.

4. Case(1): X_{bad} is Empty

Claim 4.1. *For $x \in X$ there exists V_0 non-empty such that for all $y \in V_0 \cdot x$ and V_1 symmetric containing 1_G, we have some $B \in \Delta^1_1$ and $W \subset G$ such that*

$$y \in B^{\Delta W},$$

$$\forall y_1, y_2 \in B^{\Delta W} \cap V_0 \cdot x (y_1 \in V_1 \cdot y_2).$$

Proof. The case assumption implies that every such x will be included in some $A^{\Delta W_0}$ which necessitates some V_0 to succeed. □

But then we may find a countable algebra $\mathcal{B}$ of Δ^1_1 subsets of X with the following properties: (i) $\mathcal{B}$ is closed under finite boolean operations; (ii) $\mathcal{B}$ is closed under translation by elements in G; (iii) for any $B \in \mathcal{B}$ and V basic open we have

$$C^{\Delta V}, C^{*V} \in \mathcal{B};$$

(iv) $\forall x \in X \exists V_0 \neq 0 \forall y \in V_0 \cdot x \forall V_1$, if V_1 is a symmetric neighborhood of 1_G then there exists $B \in \mathcal{B}$ and $W \subset G$ such that

$$y \in B^{\Delta W} \wedge \forall y_1, y_2 \in B^{\Delta W} \cap V_0 \cdot x (y_1 \in V_1 \cdot y_2);$$

(v) we may decompose $\mathcal{B}$ into some directed union

$$\mathcal{B} = \bigcup_{\alpha \in \delta} \mathcal{B}_\alpha,$$

some $\delta \leq \omega_1^{\text{ck}}$, such that each element of $\mathcal{B}_0$ is either closed or open, and for $\gamma > 0$ we have that each element in $\mathcal{B}_\gamma$ is either the countable union or intersection of elements in

$$\bigcup_{\alpha < \gamma} \mathcal{B}_\alpha.$$

Definition We let $\tau_{*,\mathcal{B}}$ be the topology generated by taking as a basis all sets of the form $B^{\Delta W}$ where $B \in \mathcal{B}$ and $V \subset G$ basic open.

Claim 4.2. *$(X, \tau_{*,\mathcal{B}})$ is a Polish G-space giving rise to the same Borel structure as the original topology on X.*

Proof. (v) implies by an appeal to transfinite induction on δ that there is a Polish topology having $\mathcal{B}$ as its basis. This granted, the claim follows from the Becker-Kechris theorem on changing topologies from chapter 5 of [**1**].

□

Claim 4.3. *For each $x \in X$ there is some $V \subset G$ non-empty such that*

$$g \mapsto g \cdot x$$

is an open mapping from V to $(V \cdot x, \tau_{*,\mathcal{B}})$.

Proof. For any x we appeal to the listed properties of $\mathcal{B}$; $V = V_0$ from (iv) succeeds by the definition of the topology $\tau_{*,\mathcal{B}}$.

□ But then by 2.1 we have that E_G^X is essentially countable.

5. Case(2): X_{bad} is Non-Empty

Following [**8**], we let

$$X_{*\text{low}} = \{x \in X : \forall^* g \in G(\omega_1^{\text{ck}(g\cdot x)} = \omega_1^{\text{ck}})\}$$

and consider the topology τ^*_∞ whose basis consists of all sets of the form

$$X_{*\text{low}} \cap A^{\Delta W},$$

$A \in \Sigma^1_1$ in X, W basic open in G. As shown there, this is a Polish topology, and in fact $(X_{*\text{low}}, \tau^*_\infty)$ is a Polish G-space in the action inherited from X. As at [**8**], we will use the following characterization of when $\omega_1^{\text{ck}(x)} = \omega_1^{\text{ck}}$: x is low exactly when for every Σ^1_1 set A, either $x \in A$ or there is a Σ^1_1 set B with $x \in B$ and $A \cap B = 0$.

Definition We let $X_{*\text{bad}} = X_{*\text{low}} \cap X_{\text{bad}}$, equipped with the subspace topology induced by τ^*_∞.

This $X_{*\text{bad}}$ is the intersection of two invariant sets, and hence G-invariant. It is relatively open in $X_{*\text{low}}$, and hence again a Polish G-space. We want to show that it is in fact a stormy Polish G-space. So fix any $x \in X_{*\text{bad}}$. If one point in the orbit of x fails to have the natural map $V \to V \cdot x$ open for all $V \subset G$ then so does every point in the orbit of x. Thus after possibly nudging x by a sufficiently generic group element $g \in G$ we may assume first of all that

$$\omega_1^{\text{ck}(x)} = \omega_1^{\text{ck}}$$

and secondly that for all $A \in \Sigma^1_1$, if $x \in A$ then there is a basic open neighborhood W of 1_G such that

$$x \in A^{*W}.$$

The assumption that $x \in X_{\text{bad}}$ implies that for any $A_0 \in \Sigma^1_1, W_0, V_0 \subset G$, with $1_G \in V_0$, $V_0^{-1} = V_0$, $x \in A_0^{\Delta W_0}$ there exists V_1 with $1_G \in V_1$, $V_1^{-1} = V_1$, and there exist $x' \in A_0^{\Delta W_0}, y' \in V_0 \cdot x'$ such that for all $B \in \Delta^1_1$, $W_1 \subset G$,

$$y' \in B^{\Delta W_1} \Rightarrow (\exists y'_1, y'_2 \in B^{\Delta W_1} \cap V_0 \cdot x'(y'_1 \notin V_1 \cdot y'_2)).$$

Claim 5.1. *For all $V \subset G$ with $1_G \in V$, $V^{-1} = V$, there exist V_1, with $1_G \in V_1$, and $y \in V \cdot x$ such that for all $B \in \Delta^1_1$, $W_1 \subset G$,*

$$y \in B^{\Delta W_1} \Rightarrow (\exists y_1, y_2 \in B^{\Delta W_1} \cap V \cdot x(y_1 \notin V_1 \cdot y_2)).$$

Proof. Otherwise the characteristic property of low reals implies that there is a Σ^1_1 set A containing x such that for all $x' \in A$, for all $y' \in V \cdot x'$, for all V_1 containing 1_G, there exists $B \in \Delta^1_1$, $W_1 \subset G$ such that

$$y' \in B^{\Delta W_1},$$

$$\forall y'_1, y'_2 \in V \cdot x' \cap B^{\Delta W_1}(y'_1 \in V_1 \cdot y'_2).$$

Choose V_0 containing 1_G with $V_0^2 \subset V$, $V_0^{-1} = V_0$. Choose $W_0 \subset V_0 \cap W$ such that

$$x \in A^{*W_0} \subset A^{\Delta W_0}.$$

Then for any $\hat{x} \in A^{\Delta W_0}$ and any $y' \in V_0 \cdot \hat{x}$ and V_1 with $1_G \in V_1$, $V_1^{-1} = V_1$, we may choose

$$x' \in W_0 \cdot \hat{x} \cap A \subset V_0 \cdot \hat{x} \cap A.$$

We have $y' \in V_0 V_0^{-1} \cdot x' \subset V \cdot x'$, and so assumption on A yields some $B^{\Delta W_1}$ containing y' such that

$$\forall y_1', y_2' \in V \cdot x' \cap B^{\Delta W_1}(y_1' \in V_1 \cdot y_2').$$

But now note that for all $y_1', y_2' \in B^{\Delta W_1} \cap V_0 \cdot \hat{x}$ we have

$$y_1', y_2' \in V_0 V_0^{-1} \cdot x' \subset V \cdot x'$$

$$\therefore y_1' \in V_1 \cdot y_2'.$$

Thus A_0 and W_1 as given here contradict the assumption on x.

□ The next claim states that we can obtain the same result even for $B \in \Sigma^1_1$. Since our topology is based around the Σ^1_1 sets, not just the Δ^1_1, this is critical.

Claim 5.2. *For all $V \subset G$ with $1_G \in V$, $V^{-1} = V$, there exist V_1, with $1_G \in V_1$, and $y \in V_1 \cdot x$ such that for all $B \in \Sigma^1_1$, $W_1 \subset G$,*

$$y \in B^{\Delta W_1} \Rightarrow (\exists y_1, y_2 \in B^{\Delta W_1} \cap V \cdot x(y_1 \notin V_1 \cdot y_2)).$$

Proof. Fix V. Choose y and V_1 as in the last claim. Since the property indicated for y is $\Sigma^1_1(x)$, we may assume

$$\omega_1^{\mathrm{ck}(x,y)} = \omega_1^{\mathrm{ck}(x)} = \omega_1^{\mathrm{ck}}$$

by the Gandy low basis theorem. With this extra assumption on y we will prove that y satisfies the conclusion of the stronger claim presently before us. For this purpose consider some $B \in \Sigma^1_1$ and $W_1 \subset G$. Let

$$f_B : X \to \mathrm{LO}$$

be a rank, assigning to each $z \in X$ a linear ordering $f_B(z) \subset \mathbb{N} \times \mathbb{N}$ in a Δ^1_1 manner, such that $f_B(z)$ is a well ordering if and only if $z \notin B$. For each

$$\alpha < \omega_1^{\mathrm{ck}} (= \omega_1^{\mathrm{ck}(x,y)})$$

we let $B_{(\alpha)}$ be the set of $z \in X$ such that $(\alpha, \in)$ allows an order preserving injection into $(\mathbb{N}, f_B(z))$. These sets are all Δ^1_1. One possibility is that for some $\alpha < \omega_1^{\mathrm{ck}}$ we have $y \notin (B_{(\alpha)})^{\Delta W_1}$. But then if we let $\hat{B}$ be the set of $z \in B$ such that $(\mathbb{N}, f_B(z))$ does not contain α in its wellfounded part, then we have that $\hat{B}$ is a Δ^1_1 set included in B and $y \in \hat{B}^{\Delta W_1}$; and then we are immediately done by the assumptions on y. The other possibility is that for each $\alpha < \omega_1^{\mathrm{ck}}$ we have $y \in (B_{(\alpha)})^{\Delta W_1}$, and hence appealing to assumptions on y we may find $y_1^\alpha, y_2^\alpha \in (B_{(\alpha)})^{\Delta W_1}$ with $y_1^\alpha \notin V_1 \cdot y_2^\alpha$. But then appealing to boundedness we can find an illfounded linear ordering $(\mathbb{N}, <^*)$ such that there exists y_1^∞, y_2^∞ with

$$\exists^* g \in W_1((\mathbb{N}, <^*) \text{ embeds into} f_B(g \cdot y_1^\infty)),$$

$$\exists^* g \in W_1((\mathbb{N}, <^*) \text{ embeds into} f_B(g \cdot y_2^\infty)),$$

$$y_1^\infty \notin V_2 \cdot y_2^\infty.$$

But if $(\mathbb{N}, <^*)$ embeds into $f_B(g \cdot y_i^\infty)$ then $f_B(g \cdot y_i^\infty)$ is illfounded and so $g \cdot y_i^\infty \in B$. Thus we are done. □ But since this last claim holds for any $x \in X_{*\mathrm{bad}}$, we have that the action of G on $X_{*\mathrm{bad}}$ is indeed stormy.

6. Remarks on Effectivity

Little effort was made in case(1) to indicate how we can produce a lightfaced reduction to a lightfaced Δ^1_1 equivalence relation. On the surface, 2.1 is phrased so that it only looks like a bold faced result and even the statement of Kechris' criterion at 5.2 of [**7**] is only as a boldfaced result. First of all, inspecting the proof of Kechris' criterion one actually obtains:

Proposition 6.1. *(Kechris) Let E be a Δ^1_1 equivalence relation on a recursive Polish space $\mathcal{X}$ and*

$$\theta : \mathcal{X} \to \{0,1\}^{\mathbb{N}}$$

a Δ^1_1 function such that (i) θ has countable image on each equivalence class; (ii) θ maps distinct equivalence classes to disjoint sets. Then there is a Δ^1_1 equivalence relation F on $\{0,1\}^{\mathbb{N}}$ all of whose equivalence classes are countable such that θ witnesses $E \leq_B F$.

We are still faced with the brutishness of $\mathcal{B}$ and boldfacedness of 2.1. These problems are both dealt with in the same way. The main claim is this:

Claim 6.2. *In the proof from case(1), we may assume that there is a $D \subset X \times \mathbb{N}$ in Δ^1_1 such that each $B \in \mathcal{B}$ has the form*

$$B = D_k = \{x : (x, k) \in D\}$$

for some suitable k.

This claim is proved by the boundedness theorem for Σ^1_1 subsets of ω_1^{ck}; in some sense, we can "bound" all of the needed Δ^1_1 sets in some manageable object. To see this recall that every Δ^1_1 set has the form

$$\{y : L_\alpha[y] \models \varphi(y)\}$$

for some $\alpha < \omega_1^{\mathrm{ck}}$ and formula $\varphi(\cdot)$ of set theory. If $V_x \subset G$ satisfies the statement of 4.1, then this is a Π^1_1 property on the basic open set V. If $x \in X, y \in V_x \cdot x$, then the statement that $W_0 \subset G$ and

$$B = \{y : L_\alpha[y] \models \varphi(y)\}$$

satisfy the assumption of 4.1 is $\Pi^1_1(x, y)$. Thus we have in effect a Π^1_1 function from X to the basic open sets in G, and then composing a Π^1_1 function from $X \times X$ to

$$\omega_1^{\mathrm{ck}} \times \mathcal{L}(\in).$$

Π^1_1 functions into the natural numbers are always Δ^1_1. Thus by boundedness we can assume that there is a single $\delta < \omega_1^{\mathrm{ck}}$ such that we can always choose $\alpha \leq \delta$. In this way, granting certain routine technical details about effectivizing the enumeration, we can prove the claim. But then if $(\hat{V}_\ell)_{\ell \in \mathbb{N}}$ is a basis for G, we have a Π^1_1, and hence Δ^1_1, function

$$X \to \mathbb{N}$$

$$x \mapsto n_x$$

assigning to each x some n_x such that $\hat{V}_{n_x}$ fulfills 4.1. Then we can let (m_x, k_x) be chosen in a Δ^1_1 way so that

$$U_x = (D_{k_x})^{\Delta \hat{V}_{m_x}}$$

along with $V_x = \hat{V}_{n_x}$ completes the construction of 2.1. We can then finish by letting $\theta(x)$ be the characteristic function of the following set of natural numbers:

$$\{2^k 3^n : V_x \cdot x \cap U_x \cap (D_k)^{\Delta \hat{V}_n} \neq 0\};$$

in this manner we have assigned to x in a Δ^1_1 way an element $\theta(x)$ such that

$$\theta(x_1) = \theta(x_2)$$

if and only if

$$\overline{U_{x_1} \cap V_{x_1} \cdot x} = \overline{U_{x_2} \cap V_{x_2} \cdot x_2}.$$

By the proof of 2.1, this function satisfies Kechris' criterion.

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