GROUP ACTIONS AND AND COUNTABLE MODELS

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§1. Introduction. The field of Polish group actions has emerged in the last couple of years as a kind of sub-sub-discipline in its own right.

To some extent one can think of the study of countable models as being a special case of a Polish group action – since the infinite symmetric group, consisting of all permutations of the natural numbers, is a Polish group whose action on countable models gives rise to the isomorphism relation as its orbit equivalence relation. In these talks I will try to emphasize the connections between recent work on Polish groups and some basic concepts one might see in a first course in logic or model theory.

I will begin the discussion at a very general level, which hopefully might make some sense to almost any kind of logician. Of course this means that the initial pace will seem tedious to an expert. On the other hand as the paper progresses I want to increasingly mention connections to other areas and recent results and include stick figure sketches of their proofs. Perhaps the later parts of the paper will only be completely clear for someone who already knows much of this material or is willing to spend hours in the library digging up references. Thus the criticism may be raised that the level of exposition is inconsistent; and by the end perhaps no one will be completely happy. But short of turning this paper into a book that problem strikes me as an inevitable consequence of its goals.

My own outlook on the history of this sub-area would really be to place **The descriptive set theory of Polish group actions**, Becker-Kechris[1996], as being central. This book represented the intersection of two seemingly unrelated schools of thought.

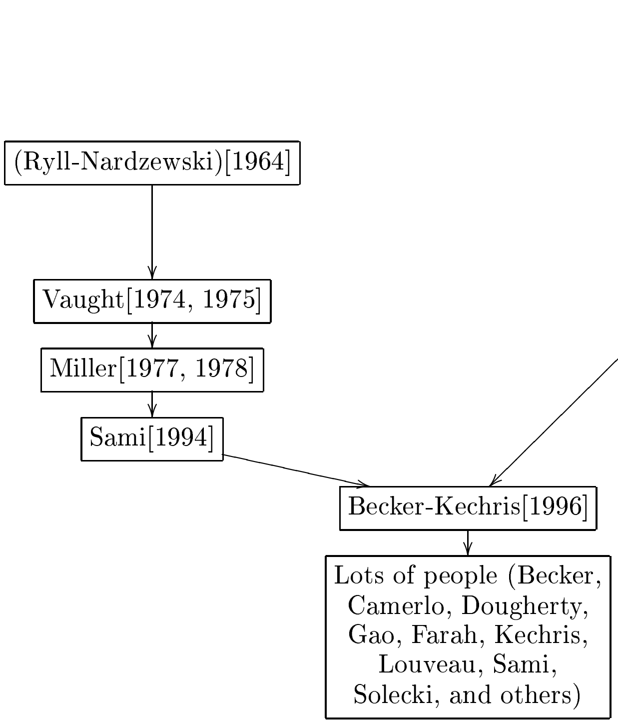
On the one hand was a sequence of papers treating topics in what one might call "invariant descriptive set theory." Perhaps the most well known papers here are those by Vaught, in which the development of the notion of Vaught transform led to unified proofs for results by Ryll-Nardzewski (every orbit is Borel) and Lopez-Escobar (invariant Borel sets in the space of countable structures are described by $\mathcal{L}_{\omega_1,\omega}$ sentences). For this Vaught was awarded the first Karp prize. A decade later, in work that was completed in the 1980's but not published until 1994, Sami used Vaught's work to obtain results on changing topologies for the "logic action" of S_∞ and showed that the topological Vaught conjecture holds for abelian Polish groups. (More on this below.)

The second sequence of papers comes from researchers in operator algebras and the study of infinite dimensional group representations. Although the topic of Polish group actions for its own sake was somewhat incidental to the main direction of their research, the group responsible for conjugacy on infinite dimensional representations is the infinite dimensional unitary group and thus Polish but not locally compact. By the time of Effros[1982] one has a completely general result for Polish groups acting on Polish spaces, largely divorced from the original context of Effros[1965], Glimm[1961], and Mackey[1957].

Both these traditions were very much in the mind of Becker and Kechris by the time of **The descriptive set theory of Polish group actions**. I have drawn up a small flow chart to try to illustrate how these papers might be viewed as interrelating, but it goes without saying that this interpretation is personal and subjective. Certainly there are papers off to one side of this chart (such as Friedman-Stanley[1989] and Burgess[1978]) which have proved extremely influential, but do not easily fall into the cartoon like summary of the history I have tried to give above.

§2. Dynamic changes in topologies. The main issue here is to discuss some of the tricks that enable us to change topologies and remain in the category of Polish spaces.

2.1. Polish spaces and Polish groups [definitions; the statement of the topological Vaught conjecture].



Definition 2.1. A topological space is said to be *Polish* if it is separable and there is some complete metric which generates its topology.

Examples (i) $\mathbb{R}$, the reals equipped with the topology generated by the open intervals. (The usual euclidean distance gives a complete metric.)

(ii) $C([0,1])$, the continuous functions from the unit interval to $\mathbb{R}$ with the topology generated by the sup norm, $d(f_1, f_2) = \sup_{0 \leq x \leq 1} |f_1(x) - f_2(x)|$.

(iii) $\mathbb{N}$ in the discrete topology. As a metric,

$$\begin{aligned} d(x_1, x_2) &= 1 \text{ if } x_1 \neq x_2, \\ &= 0 \text{ if } x_1 = x_2. \end{aligned}$$

(iv) $\mathbb{N}^{\mathbb{N}}$ in the product topology.

(v) More generally if $(X_i)_{i \in \mathbb{N}}$ is a sequence of Polish spaces, then their infinite product

$$\prod_{\mathbb{N}} X_i$$

is a Polish space in the product topology.

Proof. After possibly replacing each metric d_i with

$$\hat{d}_i(x, y) = \frac{d_i(x, y)}{d_i(x, y) + 1}$$

we may assume that each d_i is bounded by 1.

Then we can obtain a compatible complete metric on the product given by

$$d(\vec{x},\vec{y}) = \sum_{i\in\mathbb{N}} 2^{-i} d_i(x_i, y_i)$$

where

$$\vec{x} = (x_0, x_1, x_2, \dots, x_i, \dots)$$

$$\vec{y} = (y_0, y_1, y_2, \dots, y_i, \dots)$$

are both in $\prod_{\mathbb{N}} X_i$. ⊣

LEMMA 2.2. *Let X be a Polish space and $A \subset X$ a G_δ subset – that is to say, defined by a countable intersection of open sets. Then A in the subspace topology is again a Polish space.*

PROOF. First for open $\mathcal{O} \subset X$. Let d be complete metric for X, and find continuous

$$f : \mathcal{O} \to \mathbb{R}$$

with $f(x) \to \infty$ as $x \to X \setminus \mathcal{O}$. (For instance, $f(x) = \inf\{\frac{1}{d(x,c)} : c \notin \mathcal{O}\}$.) Then define $d_{\mathcal{O}}$ on $\mathcal{O}$ by

$$d_{\mathcal{O}}(x,y) = d(x,y) + |f(x) - f(y)|.$$

Suppose (x_i) is a sequence in $\mathcal{O}$ that is $d_{\mathcal{O}}$-Cauchy. From the definition of $d_{\mathcal{O}}$ it must be d-Cauchy, and hence by assumption that d witnesses X Polish, there is a limit x_∞. But we must also have that $(f(x_i))_{i\in\mathbb{N}}$ will be Cauchy in $\mathbb{R}$, and thus $x_\infty \in \mathcal{O}$.

For the general case

$$A = \bigcap_{i\in\mathbb{N}} \mathcal{O}_i$$

we can take

$$f_n : \mathcal{O}_n \to \mathbb{R}$$

as above and let

$$d_A(x,y) = d(x,y) + \sum_{n\in\mathbb{N}} 2^{-n} |f_n(x) - f_n(y)|.$$

⊣

DEFINITION 2.3. A topological group is said to be *Polish* if it is Polish as a space.

EXAMPLES 2.4. (i) $(\mathbb{R}, +)$, the reals equipped with its usual additive structure.
(ii) Any discrete group. Countable products of discrete groups, and more generally the class of Polish groups is closed under countable products.
(iii) S_∞ the group of all permutations of $\mathbb{N}$, equipped with the topology of pointwise convergence. So as a basis we may take all sets of the form

$$\{\pi \in S_\infty : \pi(\ell_1) = k_1, ..., \pi(\ell_n) = k_n\}.$$

(iv) U_∞ the unitary group of ℓ^2, infinite dimensional (separable) Hilbert space. Thus U_∞ is the group of all bijective linear operators

$$T : \ell^2 \to \ell^2$$

such that for all $\xi, \zeta \in \ell^2$

$$\langle \xi, \zeta \rangle = \langle T(\xi), T(\zeta) \rangle.$$

We give this the topology generated by the subbasis consisting of sets of the form

$$\{T \in U_\infty : |\langle T(\xi), \zeta \rangle - \langle T_0(\xi), \zeta \rangle| < \epsilon\}.$$

DEFINITION 2.5. Let G be a Polish group. We say that a Polish space X is a *Polish G-space* if it is a Polish space which is equipped with a continuous action by G. We then let E_G denote the orbit equivalence relation: $x_1 E_G x_2$ if

$$\exists g \in G(g \cdot x_1 = x_2).$$

For $x \in X$ let $[x]_G$ indicate the orbit of x:

$$\{y \in X : \exists g \in G(g \cdot x = y)\}.$$

X/G indicates the set of orbits.

EXAMPLES 2.6. (i) Let $\mathbb{Z}$ act on the complex unit circle $= \{e^{ix} : x \in \mathbb{R}\}$ in the following manner: For e^{ix} and $k \in \mathbb{Z}$ we let

$$k \cdot e^{ix} = e^{i(x + 2\pi k\sqrt{2})}.$$

In other words the action is induced by a rotation of $\sqrt{2}$ radians. In this way, among many others, we could view the complex unit circle as a Polish $\mathbb{Z}$-space.

(ii) Given a countable set X we can consider the space $(U_\infty)^X$ of function from X to the unitary group of Hilbert space; this space is Polish since it is naturally isomorphic to a countable product of U_∞ with itself. We can let U_∞ act on this space by point-wise conjugation:

$$(T \cdot f)(x) = T \circ f(x) \circ T^{-1}$$

for any $x \in X$, $f \in (U_\infty)^X$.

U_∞ is a Polish group and under this action $(U_\infty)^X$ is a Polish U_∞-space.

(iii) Perhaps the most natural example for a logician of a Polish group acting on a Polish space is the action of S_∞ on the space of countable models. There are a few fiddley points here related to precisely which topology we would want to place on this space, so I will postpone the details to section 2.2.

CONJECTURE 2.7. **Topological Vaught Conjecture**[1]: Whenever G is a Polish group and X a Polish G-space, then either

1. $X/G \leq \aleph_0$ (countably many orbits[2]), or
2. $X/G = 2^{\aleph_0}$ (continuum many orbits).

This conjecture should be compared with its counterparts for first order logic and countably infinitary logic.

[1] This conjecture is very much still open.

[2] In general I use "countably many" to mean either finitely many or exactly $\aleph_0$ many.

CONJECTURE 2.8. **Original Vaught Conjecture** Let T be a first order theory. Then either T has countably many countable models up to isomorphism or it has continuum many countable models up to isomorphism.

DEFINITION 2.9. For $\mathcal{L}$ a language we let $\mathcal{L}_{\omega_1,\omega}$ be the collection of formulas obtained by closing under countable disjunctions and conjunctions, as well as the usual first operations of existential and universal quantification, substitutions, and negation.

CONJECTURE 2.10. **Vaught Conjecture for $\mathcal{L}_{\omega_1,\omega}$** Let $\mathcal{L}$ be a countable language and σ a sentence in $\mathcal{L}_{\omega_1,\omega}$. Then either σ has countably many countable models up to isomorphism or it has continuum many countable models up to isomorphism.

As stated here, the topological Vaught conjecture is trivially true under CH (the continuum hypothesis). Frequently people use a reformulation of this conjecture that is equivalent under the negation of CH by Burgess' theorem 2.2 below. And in fact this equivalence is local, action by action, not just global: For each Polish group G and Polish G-space X, if CH fails then there are perfectly many orbits if and only if there are continuum many.

In fact I will find it convenient to refer to the following as *the* topological Vaught conjecture.

CONJECTURE 2.11. **Topological Vaught Conjecture** (Modern form): Whenever G is a Polish group and X a Polish G-space, then either

1. $X/G \leq \aleph_0$ (countably many orbits), or
2. there is a perfect set $P \subset X$ (that is to say, a set which is closed, nonempty, and has no isolated points) such that any two points in P are orbit inequivalent under the action of G.

Since any perfect set necessarily has size $2^{\aleph_0}$ it is easily seen that this alternative form of the topological Vaught conjecture implies the earlier. In fact it turns out that these two conjectures are equiprovable over ZFC – one can prove the first using only the axioms of ZFC if and only if one can prove the second using only the axioms of ZFC. This equiprovability follows by a short absoluteness argument. If the stronger form of the topological Vaught conjecture failed, then we can add lots of reals to blow up the size of the continuum; in the resulting model there must still be uncountably many orbits, by Shoenfield absoluteness; but equally there cannot be a perfect set of orbit inequivalent points, again by Shoenfield absoluteness; and thus we arrive at a contradiction with the local equivalence of the two forms of the topological Vaught conjecture under $\neg$CH. We sketch this argument due to Sami at 4.2 below. The proof of this equivalence for the model theoretic versions of the Vaught conjecture has also been written up precisely in Steel[1978].

One way to think of this alternate form of the topological Vaught conjecture is this: It is the conjecture that there cannot be "exactly $\aleph_1$ many orbits", where we understand "exactly" to mean that there are $\aleph_1$ many orbits but not perfectly many orbits.

It might be helpful before continuing with the discussion of the painfully open topological Vaught conjecture to look at some parallel problems. One can pose similar kinds of questions for other classes of equivalence relations, and in almost every case the answer is known.

I will need a couple more definitions from descriptive set theory. If you have not seen these before they may appear technical, but the motivation is a natural one: We wish to describe certain classes of equivalence relations which are not intractably complicated and compare them with the kind that arise for Polish group actions.

DEFINITION 2.12. A subset B of a Polish space X is said to be *Borel* if it appears in the smallest collection $\mathcal{B}$ of subsets of X such that

1. $\mathcal{B}$ includes the open sets in X;
2. $\mathcal{B}$ is an algebra – that is to say it is closed under complementation in X and finite unions and intersections;
3. moreover $\mathcal{B}$ is a σ-algebra – that is to say, an algebra which is also closed under countable intersections.

(Thus the collection of *the Borel sets* is the smallest σ-algebra containing the open sets.)

An equivalence relation $E \subset X \times X$ is *Borel* if it is Borel in the product topology.

A subset A of a Polish space X is said to be *analytic* or $\mathbf{\Sigma}^1_1$ if there is a Polish space Y and a Borel set $B \subset X \times Y$ (that is to say, Borel in the product topology) with

$$A = \{x \in X | \exists y \in Y((x,y) \in B)\}.$$

$P \subset X$ is $\mathbf{\Pi}^1_1$ or *co-analytic* if its complement is analytic.

Clearly every Borel set is analytic, and then since the Borel sets are closed under complementation we equally have that every Borel set is co-analytic. It is not hard to see that if a Polish group G acts continuously on a Polish space X then the set

$$\{(x_1, g, x_2) | g \cdot x_1 = x_2\}$$

is closed in $X \times G \times X$, and thus the resulting orbit equivalence relation E_G is analytic.

In the context of model theory Borel sets have an intuitive meaning. They correspond roughly to sets which can be defined by sentences in $\mathcal{L}_{\omega_1,\omega}$. (See corollary 2.18 below.) Analytic sets can be thought of as like a generalization of the set of models in a language which admit an expansion to a model of some given theory in a larger language.

THEOREM 2.1. *(Silver[1980]) Let E be a Borel (or even $\mathbf{\Pi}^1_1$) equivalence relation on a Polish space X. Then either*

1. *$X/E \leq \aleph_0$, or*
2. *$X/E = 2^{\aleph_0}$ (continuum many equivalence classes), and in fact in this case there is a perfect set of E-inequivalent points in X.*

This would be very nice for those of us who slave away trying to prove the topological Vaught conjecture if it were the case that the equivalence relations arising from Polish group actions were always Borel. Unfortunately while they must be analytic they can in general be non-Borel.

THEOREM 2.2. *(Burgess[1978]) Let E be a $\undertilde{\mathbf{\Sigma}}^1_1$ equivalence relation on a Polish space X.*

1. *$X/E \leq \aleph_1$, or*
2. *$X/E = 2^{\aleph_0}$, and in fact in this case there is a perfect set of E-inequivalent points in X.*

Burgess' theorem is optimal. If one considers the space $2^{\mathbb{N}}$ and sets $x_1 E x_2$ if $\omega_1^{\mathrm{ck}(x_1)} = \omega_1^{\mathrm{ck}(x_2)}$ (the supremum of the lengths of the x_1-recursive well orders of $\mathbb{N}$ equals the supremum of the lengths of the x_2-recursive well orders of $\mathbb{N}$) then we obtain an equivalence relation which has exactly $\aleph_1$ many classes and is in the complexity class $\undertilde{\mathbf{\Sigma}}^1_1$. By Cohen we know that $2^{\aleph_0}$ may be bigger than $\aleph_1$, and thus we cannot hope to extend Burgess' result in ZFC.

Thus any proof of the topological Vaught conjecture must use more than the simple fact that orbit equivalence relations being considered are analytic. It is known from Ryll-Nardzewski[1964] that the orbit equivalence relations arising from continuous actions of Polish groups on Polish spaces are a special kind of $\undertilde{\mathbf{\Sigma}}^1_1$ equivalence relation in which every orbit is equivalence class is Borel, but unfortunately the counterexample above with $\aleph_1$ many equivalence classes is also one in which every class is Borel.

It does not seem as if any easily stated fact about the kinds of equivalence relations arising from Polish group actions would allow some general descriptive set theoretic proof that there cannot be exactly $\aleph_1$ many orbits. The proof, if the conjecture is indeed true, would need to use very specific facts about the context of a Polish group action. It is unlikely to be soft.

2.2. The space of countable models [isomorphism of countable models viewed as a kind of Polish group action; different Polish topologies we may place on spaces of countable models].

DEFINITION 2.13. Let $\mathcal{L}$ be a countable language, and for notational simplicity assume that $\mathcal{L}$ is generated by a single binary relation R. We then let $\mathrm{Mod}(\mathcal{L})$ be the space of all $\mathcal{L}$-structures whose underlying set is $\mathbb{N}$, equipped with the topology generated by sets of the form

$$\{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}) : \mathcal{M} \models \varphi(k_1, \dots, k_n)\}$$

where φ is quantifier free and $k_1, \dots, k_n$ are in $\mathbb{N}$. There is then a natural homeomorphism

$$\mathrm{Mod}(\mathcal{L}) \to \{0,1\}^{\mathbb{N}\times\mathbb{N}}$$

$$\mathcal{M} \mapsto \chi_{R^{\mathcal{M}}}$$

associating to each $\mathcal{M}$ the characteristic function of its interpretation of R.

Thus $\mathrm{Mod}(\mathcal{L})$ is a Polish space.

We let S_∞ act on $\mathrm{Mod}(\mathcal{L})$ by

$$\pi \cdot \mathcal{M} \models R(k_1, k_2)$$

if and only if

$$\mathcal{M} \models R(\pi^{-1}(k_1), \pi^{-1}(k_2)).$$

Then $\mathrm{Mod}(\mathcal{L})$ is a Polish S_∞ space and the orbit equivalence relation is just the usual isomorphism relation.

Our next goal is to see that the topological Vaught conjecture really does generalize the model theoretic versions of the Vaught conjecture. To do this we need to consider alternative Polish topologies we may place on the space of countable models.

DEFINITION 2.14. (i) Let τ_{FO} be the topology on $\mathrm{Mod}(\mathcal{L})$ whose basic open sets have the form

$$\{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}) : \mathcal{M} \models \varphi(k_1, \dots, k_n)\}$$

where φ is first order and $k_1, \dots, k_n$ are in $\mathbb{N}$.

(ii) For $F \subset \mathcal{L}_{\omega_1,\omega}$ a countable fragment (closed under subformulas, instantiations, and first order operations), let τ_F be the topology on $\mathrm{Mod}(\mathcal{L})$ whose basic open sets have the form

$$\{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}) : \mathcal{M} \models \varphi(k_1, \dots, k_n)\}$$

where $\varphi \in F$.

All these provide Polish topologies on $\mathrm{Mod}(\mathcal{L})$. (Proof for (i) below). They are all finer that the topology of quantifier free formulas. They all give rise to the same Borel structure as the topology of quantifier free formulas, which I will continue to use as my default choice of topology on the space.

Thus for any theory T the set

$$\{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}) : \mathcal{M} \models T\}$$

is a closed S_∞-invariant subspace of $(\mathrm{Mod}(\mathcal{L}), \tau_{\mathrm{FO}})$, and hence a Polish S_∞ space in its own right. Sets of the form $\{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}) : \mathcal{M} \models T\}$ correspond exactly to the closed invariant subspaces; the subspace is minimal iff T complete.

A key step in the proof of Becker's theorem (which is coming up in section 3.3) is to pass from a not so good topology, like the quantifier free topology, to a better topology, like τ_{FO}.

LEMMA 2.15. *(Gregorczyk, Mostowski, Ryll-Nardzewski[1960])* $(\mathrm{Mod}(\mathcal{L}), \tau_{\mathrm{FO}})$ *is a Polish space.*

PROOF. Let B be the set of substitutions of the form $\varphi(k_1, \dots, k_n)$ where φ is a first order $\mathcal{L}$ formula. Let $X = \{0,1\}^B$, the space of all functions from B to $\{0,1\}$. We may naturally identify $(\mathrm{Mod}(\mathcal{L}), \tau_{\mathrm{FO}})$ with the functions in $f \in X$ which preserve the "inductive unraveling of truth" – for instance, $f(\varphi_1(k) \wedge \varphi_2(k)) = 1$ if and only if $f(\varphi_1(k)) = f(\varphi_2(k)) = 1$. Most of these requirements correspond to closed conditions in X, except for the rule governing existential quantifiers,

$$f(\exists x \varphi(k, x)) = 1 \Leftrightarrow \exists l \in \mathbb{N} f(\varphi(k, l)) = 1,$$

which is still no worse than G_δ.

Thus $(\mathrm{Mod}(\mathcal{L}), \tau_{\mathrm{FO}})$ is homeomorphic to a G_δ subspace of a Polish space and therefore Polish. $\dashv$

In many respects τ_{FO} would seem to be a nicer topology than the topology generated by quantifier free formulas. This becomes more apparent after we introduce the notion of *Vaught transform* below and note that τ_{FO} has a basis that is closed under the Vaught transforms.

The next subsection discusses an important theorem due to Becker and Kechris which states that among other things that for any Polish G-space there are "cofinally" many Polish topologies in which that action is still continuous and that are closed under the Vaught transforms. This was arguably the main result of Becker-Kechris[1996]. Sami[1994] had previously shown the same thing for a general class of continuous actions of S_∞, including its natural action on $(\mathrm{Mod}(\mathcal{L}), \tau_{\mathrm{FO}})$.

2.3. Becker-Kechris theorem [Vaught transforms; adding to the topology of a Polish G-space and keeping the action continuous].

DEFINITION 2.16. Let G be a Polish group and X a Polish G-space. For $V \subset G$ open and $B \subset X$ (preferably Borel), we let

$$B^{\Delta V}$$

be the set of $x \in X$ for which there is a non-meager[3] set of $g \in V$ for which $g \cdot x \in B$. We symmetrically define B^{*V} to be the set of $x \in X$ for which there is a relatively comeager set of $g \in V$ with $g \cdot x \in B$.

LEMMA 2.17. *(Vaught[1973]) For $\varphi \in \mathcal{L}_{\omega_1,\omega}$, B the set of $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ with $\mathcal{M} \models \varphi(k_1, \dots, k_n, a_1, \dots, a_m)\}$, and $V = \{\pi \in S_\infty : \pi(a_1) = b_1, \dots, \pi(a_m) = b_m\}$,*

$$B^{\Delta V} = \{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}) : \mathcal{M} \models \exists x_1 \dots \exists x_n \varphi(\vec{x}, \vec{b})\},$$

$$B^{*V} = \{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}) : \mathcal{M} \models \forall x_1 \dots \forall x_n \varphi(\vec{x}, \vec{b})\}.$$

Using this one can show by an induction on the Borel complexity of B that:

COROLLARY 2.18. *(Lopez-Escobar[1965]) If $B \subset \mathrm{Mod}(\mathcal{L})$ is Borel and invariant then there is a sentence $\varphi \in \mathcal{L}_{\omega_1,\omega}$ with*

$$B = \{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}) : \mathcal{M} \models \varphi\}.$$

THEOREM 2.3. *(Becker-Kechris[1996])*

Let G be a Polish group and X a Polish G-space. Let $B \subset X$ be Borel. Let $\mathcal{B}$ be a countable basis for G.

Then there is a Polish topology τ on X such that:

1. *τ is richer than the original topology on X;*
2. *for any $V \in \mathcal{B}$*

$$B^{\Delta V} \in \tau;$$

[3]A set is meager if it is included in a countable union of closed sets each of which contain no non-empty open subset.

3. *τ has a countable basis $\mathcal{C}$ such that for all $U_1, U_2 \in \mathcal{C}$ and $V_1, V_2 \in \mathcal{B}$,*

$$(U_1 \cap (X \setminus U_2)^{*V_1})^{\Delta V_2} \in \mathcal{C};$$

(This rather ominous looking condition plays a central role in the proof of Becker's theorem 3.6 below; it represents an approximation to the basis having the kind of closure conditions presented by the natural basis for τ_{FO} on $\mathrm{Mod}(\mathcal{L})$.)

4. *the action of G on X is still continuous with respect to τ (and thus (X, τ) is still a Polish G-space).*

In the case that B is invariant under the action of G we obtain that $B^{\Delta G} = B^{*G} = B$, and thus B is an invariant clopen subset of (X, τ). In particular, since for any sentence $\sigma \in \mathcal{L}_{\omega_1,\omega}$ we have that

$$\{\mathcal{M} : \mathcal{M} \models \sigma\}$$

is Borel and invariant, Vaught's conjecture for $\mathcal{L}_{\omega_1,\omega}$ is implied by the topological Vaught conjecture for S_∞.

It would take us too far afield to go into the details of their proof. Instead I will present a lemma, which by a transfinite induction can be used to prove the Becker Kechris theorem.

LEMMA 2.19. *X a Polish G-space, $V \subset G$ open and $C \subset X$ closed. Then there is a richer Polish topology on X for which the action is still continuous and $C^{\Delta V}$ is now open.*

PROOF. By a theorem of Birkhoff and Kakutani (see 1.1.1 Becker-Kechris[1996]) we may find a compatible right invariant metric d on G that is bounded by 1. Let $\mathcal{L}(G)$ be the space of $f : G \to [0,1]$ such that for all $g_1, g_2 \in G$

$$|f(g_1) - f(g_2)| \leq d(g_1, g_2).$$

Then $\mathcal{L}(G)$ is a Polish G-space under the natural action $(g \cdot f)(h) = f(hg)$.

For $x \in X$ let $f_x \in \mathcal{L}(G)$ be given by

$$f_x(g) = \inf\{d(g,h) : h \cdot x \notin C\}.$$

$X_C = \{(x, f_x) : x \in X\}$ is an invariant subset of $X \times \mathcal{L}(G)$ and $\pi_C : X \to X_C$

$$x \mapsto (x, f_x)$$

is a bijection which respects the G-action.

One then verifies, as in Hjorth[1999], that X_C is a G_δ subset of $X \times \mathcal{L}(G)$ and thus a Polish G-space. The pullback of the topology on X_C to X is as required. $\dashv$

§3. Smoothness and the Glimm-Effros dichotomy.

3.1. Smoothness [Effros lemma; G_δ orbits as the analogue of atomic models].

DEFINITION 3.1. An orbit equivalence relation E_G on Polish space X is said to be *smooth* if there is a Borel function[4] $f : X \to \mathbb{R}$ such that for all $x_1, x_2 \in X$

$$x_1 E_G x_2 \Leftrightarrow f(x_1) = f(x_2).$$

Burgess[1979] showed that for E_G given by the continuous action of a Polish group action we have E_G smooth if and only if there is a Borel set meeting every orbit in exactly one point.

EXAMPLE 3.2. For $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(\mathcal{L})$ set

$$\mathcal{M} \equiv \mathcal{N}$$

if they have the same (first order) theory.

Then $\equiv$ is smooth. To see this, let $(\varphi_n)_{n\in\mathbb{N}}$ enumerate the first order sentences of $\mathcal{L}$. For $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ associate the real $x_{\mathcal{M}} \in (0,1)$ in whose decimal expansion we have a 5 at the n^{th} position if $\mathcal{M} \models \varphi_n$ and 6 if it does not. $\mathcal{M} \mapsto x_{\mathcal{M}}$ is a Borel function witnessing smoothness.

EXAMPLE 3.3. Let $\mathbb{Z}_2^{<\mathbb{N}}$ be the infinite binary sequences which are eventually constantly 0. This is a countable set, and we give it the discrete topological structure. We further introduce the group structure obtained by pointwise addition mod 2; thus for $\vec{g}, \vec{h} \in \mathbb{Z}_2^{<\mathbb{N}}$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}(\vec{g} + \vec{h})(n) &= 0 \text{ if } \vec{h}(n) = \vec{g}(n) \text{ and}\\ &= 1 \text{ otherwise.}\end{aligned}$$

We then let it act on $2^{\mathbb{N}}$ in the natural manner: For $\vec{x} \in 2^{\mathbb{N}} =_{df} \{0,1\}^{\mathbb{N}}$, $\vec{g} \in \mathbb{Z}_2^{<\mathbb{N}}$, $n \in \mathbb{N}$, we again have $(\vec{g} \cdot \vec{x})(n) = 0$ if and only if $\vec{x}(n) = \vec{g}(n)$.

LEMMA 3.4. *The above orbit equivalence relation $E_{\mathbb{Z}_2^{<\mathbb{N}}}$ is non-smooth.*

PROOF. Suppose instead (by Burgess[1979]) B is a Borel selector. Let μ be the product ("coin flipping") measure on $2^{\mathbb{N}}$. Note that $(2^{\mathbb{N}}, \mu)$ is a probability space and $\mathbb{Z}_2^{<\mathbb{N}}$ acts by measure preserving transformations on $(2^{\mathbb{N}}, \mu)$.

Since countably many translates of B cover $2^{\mathbb{N}}$ we must have $\mu(B) > 0$. But then there are $\aleph_0$ many *disjoint* translates of B, each having the same measure, and hence a contradiction to $\mu(2^{\mathbb{N}}) = 1$. $\dashv$

There are many theorems giving necessary conditions for non-smoothness in terms of being able to *embed* this orbit equivalence relation $E_{\mathbb{Z}_2^{<\mathbb{N}}}$. For a wide variety of groups (but not S_∞!) one can show that E_G is smooth if and only if we cannot so embed this canonical example of a non-smooth equivalence relation.

For instance:

THEOREM 3.1. *(Becker-Kechris) Let G be a Polish group and X a Polish G-space, $x \in X$. Suppose that no orbit $[y]_G$ with the same closure as $[x]_G$ is comeager in*

$$\overline{[x]_G},$$

the closure of the orbit.

Then

[4]That is to say, for any open $O \subset \mathbb{R}$ the set $f^{-1}(O)$ appears in the σ-algebra on X generated by the open sets –i.e., $f^{-1}[O]$ is Borel.

1. E_G *is not smooth and*
2. *there is a continuous injection*

$$\rho : 2^{\mathbb{N}} \to X$$

such that for all $\vec{x}, \vec{y} \in 2^{\mathbb{N}}$

$$\vec{x} E_{\mathbb{Z}_2^{<\mathbb{N}}} \vec{y} \Leftrightarrow (\rho(\vec{x}) E_G \rho(\vec{y})).$$

In fact 2. implies 1. in the statement of the theorem.

After this Becker-Kechris theorem one might conclude that it would be nice to understand when an orbit is comeager in its own closure. Here a theorem was proved some 30 years earlier by Effros:

Theorem 3.2. *(Effros[1965]) Let G be a Polish group and X a Polish G-space. Then for* $x \in X$ *the following are equivalent:*

1. $[x]_G$ *is comeager in* $\overline{[x]_G}$*;*
2. $[x]_G$ *is* G_δ*;*
3. *the map*

$$G \to [x]_G$$

$$g \mapsto g \cdot x$$

is open.

In the case of the logic action of S_∞ on $(\mathrm{Mod}(\mathcal{L}), \tau_{\mathrm{FO}})$ – the $\mathcal{L}$-structures on $\mathbb{N}$ with the topology generated by first order logic – one has that the G_δ orbits correspond to atomic models, see Miller[1978]. (The omitting types theorem shows that the realization of any non-principal type is meager.)

3.2. Examples [algebraic characterization of discrete groups with smooth dual (Thoma); Bernoulli shifts up to isomorphism are smooth (Ornstein); but not general mpt's (Feldman)].

Example 3.5. As before let U_∞ be the infinite dimensional unitary group, consisting of all isomorphisms of Hilbert space. For H a countable (discrete) group we define $\mathrm{Rep}(H)$ to be the space of all homomorphism

$$\rho : H \to U_\infty.$$

This may naturally be identified with a closed subspace of the H-fold product of U_∞,

$$\prod_H U_\infty,$$

and thus is a Polish space. We let U_∞ act on $\mathrm{Rep}(H)$ by "pointwise conjugation",

$$(T \cdot \rho)(h) = T \circ \rho(h) \circ T^{-1}.$$

We then let $\mathrm{Irr}(H)$ be the *irreducible* representations in $\mathrm{Rep}(H)$ – that is to say those ρ for which there are no non-trivial closed subspaces of Hilbert space closed under the operators $\{\rho(h) : h \in H\}$. It turns out that $\mathrm{Irr}(H)$ is a G_δ subset of $\mathrm{Rep}(H)$, and thus a Polish U_∞-space in its own right.

Theorem 3.3. *(Thoma[1964]) The orbit equivalence* E_{U_∞} *on* $\mathrm{Irr}(H)$ *is smooth if and only if H is abelian-by-finite.*

One can extend the definition of $\mathrm{Rep}(H)$ and $\mathrm{Irr}(H)$ to a general locally compact group. Here one demands that the representations all be continuous, and we give these spaces the "compact-open" topology, generated by sets of the form

$$\{\rho : \rho[C] \subset O\},$$

where $C \subset H$ is compact and $O \subset U_\infty$ is open. Again we obtain that $\mathrm{Rep}(H)$ and $\mathrm{Irr}(H)$ are U_∞ Polish spaces.

THEOREM 3.4. *(Harish-Chandra[1953], more or less[5]) If H is a semi-simple connected Lie group, then the orbit equivalence E_{U_∞} on $\mathrm{Irr}(H)$ is smooth.*

EXAMPLE 3.6. Let M_∞ be the group of measure preserving transformations of the unit interval, subject to identification of transformations agreeing almost everywhere. We give this the topology generated by sets of the form

$$\{\pi \in M_\infty : \mu(\pi(A)\Delta\pi_0(A)) < \epsilon\},$$

for $\pi_0 \in M_\infty$, $A \subset [0,1]$ measurable, $\epsilon > 0$. In this topology and the operation of composition it becomes a Polish group.

It seems natural to think of measure preserving transformations as really being "isomorphic" if they are the same up to some isomorphism of the underlying standard Borel probability space. Thus we are led to the orbit equivalence relation of M_∞ acting on itself by conjugation:

$$\pi \cdot \rho = \pi \circ \rho \circ \pi^{-1}.$$

THEOREM 3.5. *(Feldman[6]) The orbit equivalence relation E_{M_∞} obtained above is not smooth.*

For some special classes of measure preserving transformations one obtains smoothness of the restriction of E_{M_∞}. For instance Ornstein showed that the isomorphism relation on *Bernoulli shifts* is smooth and that *entropy* provides a complete invariant.

3.3. Becker's theorem [Glimm-Effros dichotomy, and hence Vaught's conjecture, for solvable groups].

DEFINITION 3.7. (Becker) A Polish group G satisfies the *Glimm-Effros dichotomy* if whenever G acts continuously on a Polish space X either:

1. E_G is smooth; or
2. there is a continuous injection

$$\rho : 2^{\mathbb{N}} \to X$$

such that for all $\vec{x}, \vec{y} \in 2^{\mathbb{N}}$

$$\vec{x} E_{\mathbb{Z}_2^{<\mathbb{N}}} \vec{y} \Leftrightarrow (\rho(\vec{x}) E_G \rho(\vec{y})).$$

[5] I fib terribly. In fact he proved that the representations of semi-simple connected Lie groups are all "type I". While at the time it was apparently felt that this should turn into a proof of the smoothness of $E_{U_\infty}|_{\mathrm{Irr}(H)}$, this does not seem to have been confirmed until the proof of the "Mackey conjecture" a few years later by Glimm

[6] Actually something stronger and rather more specific is shown in Feldman[1974]. He obtains non-smoothness even for the property K transformations. See a Foreman[to appear] for a thorough discussion of this and related results.

It is not hard to see that the Glimm-Effros property for a group G implies that it does not provide a counterexample for the topological Vaught conjecture.

LEMMA 3.8. *Let G be a Polish group and X a Polish G-space. If G satisfies the Glimm-Effros dichotomy then X does not have exactly $\aleph_1$ many orbits.*

PROOF. Let X be a Polish G-space. We wish to show that either there are perfectly many orbits or (at most) countably many orbits.

Apply the definition of the Glimm-Effros dichotomy. Suppose first we are in case 2, and so we have a continuous embedding ρ of $E_{\mathbb{Z}_2^{<\mathbb{N}}}$ into E_G. We can consider the compact set

$$C = \{\vec{x} | \forall n (n \text{ not a prime power } \Rightarrow x(n) = 0)$$

$$\wedge \forall p \text{ prime } \forall k > 0 (x(p) = x(p^k))\}.$$

Any two elements of C are $E_{\mathbb{Z}_2^{<\mathbb{N}}}$-inequivalent. $\rho[C]$ is again compact and without isolated points. And thus $\rho[C]$ provides a perfect set of orbit inequivalent points.

Alternatively suppose we are in case 1. Then there is Borel G-invariant

$$f : X \to \mathbb{R},$$

such that for all $x_1, x_2 \in X$

$$x_1 E_G x_2 \Leftrightarrow f(x_1) = f(x_2).$$

A routine calculation shows that E_G must consequently be Borel, and we can finish by applying Silver's theorem. ⊣

Ever expanding classes of groups have been shown to satisfy the Glimm-Effros dichotomy:

1. Locally compact Polish groups (Effros, after Glimm).
2. Abelian Polish groups (Solecki).
3. Nilpotent (Hjorth).
4. Solvable (Becker).

Notably S_∞ fails this dichotomy.

THEOREM 3.6. *(Becker) Solvable Polish groups satisfy the Glimm-Effros dichotomy.*

PROOF. Let G be a solvable Polish group and X a Polish G-space. Assume that 2. fails, that is to say, we are unable to continuously inject $E_{\mathbb{Z}_2^{<\mathbb{N}}}$ into E_G. Since G is solvable it has a *complete* compatible right invariant metric (Gao[1998], Hjorth-Solecki[1999]), call it d_G.

Following theorem 2.3 we may assume that we have a basis $\mathcal{C}$ for the topology τ on X and $\mathcal{B}$ on G such that:

1. for all $U_1, U_2 \in \mathcal{C}$ and $V_1, V_2 \in \mathcal{B}$,
$$(U_1 \cap (X \setminus U_2)^{*V_1})^{\Delta V_2} \in \mathcal{C};$$
2. the action of G on X is still continuous with respect to τ (and thus (X, τ) is still a Polish G-space).

It is not hard to show that there is a Borel function

$$\theta : X \to \mathbb{R}$$

such that for all $x_1, x_2 \in X$

$$\theta(x_1) = \theta(x_2) \Leftrightarrow \overline{[x_1]_G} = \overline{[x_2]_G}.$$

(We may code the closure of an orbit by a real number, and the coding can be completed in a Borel manner.) Thus it suffices to show that $\overline{[x]_G}$ is a complete invariant of $[x]_G$; that is to say, $\overline{[x_1]_G} = \overline{[x_2]_G}$ if and only if $[x_1]_G = [x_2]_G$.

Fix some $x \in X$.

Now by theorem 3.1, the Becker-Kechris theorem on embedding $E_{\mathbb{Z}_2^{<\mathbb{N}}}$, we know that there must an $[y]_G$ with the same closure as $[x]_G$ which is comeager in $\overline{[x]_G} = \overline{[y]_G}$.

Using the Effros theorem, we can obtain for each open

$$V_n = \{g \in G : d_G(1_G, g) < 2^{-n}\}$$

some open $U_n \in \mathcal{C}$ containing y with diameter less than 2^{-n} such that

$$V_n \cdot y \supset U_n \cap [y]_G.$$

Then using the closure assumptions on the the basis $\mathcal{C}$ for the topology and a calculation which we do not present here, it can be argued that whenever $U \in \mathcal{C}$ and

$$U \cap U_n \cap [y]_G \neq \emptyset$$

$$\hat{x} \in U_n \cap [x]_G$$

then there are g arbitrarily close to the identity with

$$g \cdot \hat{x} \in U^{\Delta V_n^2},$$

and thus in particular we may find $g, h \in V_n^2$ with $hg \cdot \hat{x} \in U$.

Repeating this observation infinitely often we may find a sequence $(h_n)_{n\in\mathbb{N}}$ in G and $(x_n)_{n\in\mathbb{N}}$ in $[x]_G$ with:

1. $x_n \in U_n$;
2. $h_n \in V_n^4$, and thus $d_G(h_n, 1_G) < 2^{-n+2}$;
3. $x_{n+1} = h_n \cdot x_n$.

We find h_{n+1} by applying the preceding observation with U_{n+1} taking the place of the set U.

Thus if we let $g_n = h_n h_{n-1} \dots h_1 h_0$ then we have $x_n = g_n \cdot x_0 \in [x]_G$ and by 1. we have

$$x_n \to y.$$

On the other hand 2. and right invariance of the metric gives

$$d_G(g_n, g_{n-1}) < 2^{-n+2}$$

and thus (g_n) is Cauchy. But for g_∞ the limit we obtain from continuity of the action,

$$g_\infty \cdot x_0 = y.$$

Thus we have established that $[x]_G = [y]_G$ is comeager in $\overline{[x]_G}$.

Generalizing we therefore have that whenever $\overline{[x_1]_G} = \overline{[x_2]_G}$ we must have both $[x_1]_G$ and $[x_2]_G$ are comeager in this common closure. Hence the orbits overlap. Hence they are equal. $\dashv$

EXAMPLE 3.9. S_∞ does not satisfy the Glimm-Effros dichotomy.

Let $\mathcal{L}$ be generated by binary function $+$, unary function $-(\ )$, and constant symbol 0. Let AG be the subset of $\mathrm{Mod}(\mathcal{L})$ consisting of $\mathcal{G}$ in which $(+^{\mathcal{G}}, -(\ )^{\mathcal{G}}, 0^{\mathcal{G}})$ defines an abelian group structure on $\mathbb{N}$. Certainly this corresponds to a first order definable property on $\mathcal{G}$, and so AG is a closed invariant subspace of $\mathrm{Mod}(\mathcal{L})$.

Then for p a prime let TAG_p be the set of all p-groups – that is to say, $\mathcal{G}$ such that for all $g \in \mathcal{G}$ there exists $n > 0$ with

$$\mathcal{G} \models p^n \cdot g = 0.$$

TAG_p is an invariant G_δ set, and hence a Polish S_∞-space in its own right.

The Ulm analysis of abelian p-groups shows that we may find a *reasonably nice* (e.g. absolutely Δ^1_2 in the codes, in the sense of Hjorth[2000]) function $U : \mathrm{TAG}_p \to 2^{<\omega_1}$ assigning bounded subsets of $\aleph_1$ as complete invariants.

Claim: TAG_p/S_∞ is not smooth.

Roughly this follows because in general there can be no *reasonably simply definable* (e.g. absolutely Δ^1_2!) ω_1 sequence of reals.

Claim: We cannot embed $E_{\mathbb{Z}_2^{<\mathbb{N}}}$ into TAG_p/S_∞.

Otherwise by postcomposing with U we would obtain an assignment $\hat{U}$ of bounded subsets of $\aleph_1$ as complete invariants for $2^{\mathbb{N}}/\mathbb{Z}_2^{<\mathbb{N}}$. Then we would obtain some fixed $\alpha < \omega_1$ and μ-measure[7] one set on which the reduction $\hat{U}$ takes values inside 2^α. (To see this, we can take $(M; \in)$ some countable transitive model containing the real parameter used in the embedding of $E_{\mathbb{Z}_2^{<\mathbb{N}}}$; then there will be a μ-measure 1 set of $x \in 2^{\mathbb{N}}$ that are "random" generic for M – and for any such x there will be some $\alpha < \omega_1^M = \omega_1^{M[x]} < \omega_1^V$ with $\hat{U}(x) \in 2^\alpha$).

But then since 2^α is naturally isomorphic to $2^{\mathbb{N}}$, and hence Borel isomorphic to $\mathbb{R}$, we obtain a contradiction to our proof before that $2^{\mathbb{N}}/\mathbb{Z}_2^{<\mathbb{N}}$ is non-smooth on any μ-measure one set.

(Here we have skipped over technicalities relating to the fact that $\hat{U}$ is not necessarily Borel. The issue here however is that it can be chosen to be universally measurable in the codes; hence on a measure one set it will equal a Borel function, or alternatively the proof of 3.4 can be seen to work in the more general context of μ-measurable functions.)

§4. Vaught's conjecture on analytic sets.

4.1. Variations on the conjecture.

NOTATION 4.1. For H a Polish group let $\mathrm{TVC}(H)$ be the statement that whenever X is a Polish H-space then either there are only countably many orbits or there is a perfect set $P \subset X$ of orbit inequivalent points:

$$\forall x_1, x_2 \in P (x_1 \neq x_2 \Rightarrow [x_1]_H \neq [x_2]_H).$$

[7] μ the "coin flipping" measure from before

Let WVC(H) be the statement that when X is a Polish H-space then either there are only countably many orbits or there are $2^{\aleph_0}$ many orbits.

LEMMA 4.2. *(Sami) If H is a Polish group and X is a Polish H-space witnessing the failure of TVC(H), then X continues to witness the failure of TVC(H) through all generic extensions.*

PROOF. **Claim:** In no generic extension can X contain a perfect set of orbits.

Proof of Claim: The statement that there are perfectly many orbits may be expressed as:

$$\exists P \subset X(P \text{ perfect} \wedge$$

$$\forall x_1, x_2 \in P \forall g \in G(x_1 \neq x_2 \Rightarrow g \cdot x_1 \neq x_2)).$$

This is $\underset{\sim}{\mathbf{\Sigma}}^1_2$, and hence absolute by Shoenfield. (Claim□)[8]

Claim: In every generic extension X/H is uncountable.

Proof of Claim: For conceptual simplicity let us do the case that

$$X = \{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}) : \mathcal{M} \models T\}$$

and $H = S_\infty$. Then for any $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ we have that $[\mathcal{M}]_{S_\infty}$ is uniformly Borel in (any code for) the Scott sentence of $\mathcal{M}$. Moreover, in some natural parameter space Y we have that

$$\mathcal{S} = \{(\mathcal{M}, y) : y \text{ codes the Scott sentence of } \mathcal{M}\}$$

is $\underset{\sim}{\mathbf{\Pi}}^1_1$. Then the statement that the models of T has uncountably many isomorphism types among its countable models becomes the statement that for all sequences $(\mathcal{M}_i), (y_i)$, either:

1. there exists i with $(\mathcal{M}_i, y_i) \notin \mathcal{S}$; or
2. there exists $\mathcal{N} \models T$ with $\mathcal{N}$ not satisfying any of the sentences coded by the $\{y_i : i \in \mathbb{N}\}$.

Thus it is $\underset{\sim}{\mathbf{\Pi}}^1_2$ and subject to Shoenfield absoluteness.

The general case for arbitrary Polish group actions follows by working not with the Scott sentence, but the *stabilizer* of the points in question; again $[x]_H$ is uniformly Borel in (any code for) the stabilizer of x (see the proof of 1.2.4, 7.1.2 Becker-Kechris[1996]). (Claim□) ⊣

COROLLARY 4.3. *It is provable in ZFC that every Polish group H satisfies WVC(H) (i.e. the original topological Vaught conjecture) if and only it is provable that every Polish group satisfies TVC(H).*

PROOF. $\Leftarrow$ since any perfect set has size $2^{\aleph_0}$.

$\Rightarrow$ since if we have a group H and a Polish H-space X showing the failure of TVC(H) then we may go to a forcing extension in which $2^{\aleph_0} = \aleph_2$. In the forcing extension Sami's lemma and Burgess' theorem from 2.2 on analytic equivalence relations imply that we must have $|X/H| = \aleph_1$. ⊣

[8] This proof also works to show that if a $\underset{\sim}{\mathbf{\Sigma}}^1_1$ A does not contain a perfect set of orbits then in no generic extension it will contain a perfect set of orbits. The point here is that if P is a perfect subset of A then by say an appeal to the Baire category theorem and the Jankov-von Neumann uniformization we may find a perfect $P_0 \subset P$ and a continuous function f on P_0 such that for all $x \in P_0$ we that $f(x)$ witnesses $x \in A$.

NOTATION 4.4. For H a Polish group let $\mathrm{TVC}(H, \underset{\sim}{\mathbf{\Sigma}}^1_1)$ be the statement that whenever X is a Polish H-space and $A \subset X$ is $\underset{\sim}{\mathbf{\Sigma}}^1_1$ then either

$$A/H \leq \aleph_0$$

or there is perfect $P \subset A$ such that for all $x_1, x_2 \in P$

$$x_1 \neq x_2 \Rightarrow [x_1]_H \neq [x_2]_H.$$

Sami's argument also shows absoluteness of $\mathrm{TVC}(H, \underset{\sim}{\mathbf{\Sigma}}^1_1)$.

LEMMA 4.5. *(H. Friedman[1973]) Up to isomorphism there are exactly $\aleph_1$ many possible order types for the ordinals in countable ω-models of Kripke-Platek set theory.*

COROLLARY 4.6. *$TVC(S_\infty, \underset{\sim}{\mathbf{\Sigma}}^1_1)$ fails.*

4.2. Some sketches around the proof for characterizing Vaught's conjecture on analytic sets.

THEOREM 4.1. *A Polish group H satisfies $TVC(H, \underset{\sim}{\mathbf{\Sigma}}^1_1)$ if and only if S_∞ does not divide H.*

PROOF. $\Rightarrow$: It was previously known by results of Mackey and Becker and Kechris (see 2.3.5 Becker-Kechris[1996]) that if K is a closed subgroup of a Polish group H and Y is a Polish K-space, then we can in some sense lift the action of K on Y up to an action of H on a Polish space X, and obtain many similarities between X/H and Y/K. In particular this "Mackey" lift suffices to show that $\neg\mathrm{TVC}(K, \underset{\sim}{\mathbf{\Sigma}}^1_1)$ implies $\neg\mathrm{TVC}(H, \underset{\sim}{\mathbf{\Sigma}}^1_1)$.

But now if we have that S_∞ is the continuous surjective image of K then certainly we can obtain that any orbit equivalence relation induced by a continuous action of S_∞ is induced by the continuous action of K.

$\Leftarrow$: We suppose that X is a Polish H-space which provides a counterexample to $\mathrm{TVC}(H, \underset{\sim}{\mathbf{\Sigma}}^1_1)$. Let A be the $\underset{\sim}{\mathbf{\Sigma}}^1_1$ set which has uncountably many but not perfectly many orbits. We need to find a closed subgroup of H which has S_∞ as its continuous homomorphic image.

Let us first consider the argument for the following special case. Let $\mathcal{L}_0$ be a countable language and $\mathcal{M}_0$ a $\mathcal{L}_0$-model with underlying set $\mathbb{N}$ and let us suppose that H is the automorphism group of $\mathcal{M}_0$. Then let us suppose for some countable $\mathcal{L}_1 \supset \mathcal{L}_0$ we have that X is space of all expansions of $\mathcal{M}_0$ to an $\mathcal{L}_1$-structure satisfying some given theory T.

(Actually this case is not quite as special as it seems. Any closed subgroup of S_∞ is the automorphism group of a countable structure, and Becker and Kechris have shown that if $H = \mathrm{Aut}(\mathcal{M}_0)$ then any Polish H-space may be naturally identified with a subspace of the expansions of $\mathcal{M}_0$ to some richer language $\mathcal{L}_1$.)

We need to show that the automorphism group of $\mathcal{M}_0$ is very complicated. In fact we will find some expansion $\mathcal{N} \in A$ with S_∞ dividing $\mathrm{Aut}(\mathcal{N})$.

From the proof of Sami's lemma we obtain that through all generic extensions A has uncountably many orbits but not perfectly many. Thus in particular if we force to make the continuum of the ground model countable then there must be some new equivalence class in A.

So let $\mathbb{P} = \mathrm{Coll}(\omega, 2^{\aleph_0})$ be the forcing to collapse the continuum of V. By the completeness lemma for forcing we may find some $\mathbb{P}$-term σ such that

$$\mathbb{P} \Vdash \sigma[\dot{G}] \in A \wedge \forall \mathcal{N} \in A(\mathcal{N} \ncong \sigma[\dot{G}]).$$

Step 1: There exists $p \in \mathbb{P}$ such that

$$(p,p) \Vdash_{\mathbb{P}\times\mathbb{P}} \sigma[\dot{G}_l] \cong \sigma[\dot{G}_r],$$

where here $\dot{G}_l \times \dot{G}_r$ names the generic object on $\mathbb{P} \times \mathbb{P}$. In other words, the isomorphism type of the model $\sigma[G]$ does not depend on the choice of the generic filter G.

The proof of the claim goes back to some early arguments by Harrington and Silver. The rough idea is that if it failed then in $V^{\mathrm{Coll}(\omega,|\mathcal{P}(\mathbb{P}\times\mathbb{P})|)}$ we could find a "perfect set" of generic filters, such that any two distinct filters pass through some pair of conditions forcing that their evaluations of the term σ leads to non-isomorphic models.

More precisely, we let (D_n) enumerate in $V^{\mathrm{Coll}(\omega,|\mathcal{P}(\mathbb{P}\times\mathbb{P})|)}$ the dense open subset of $\mathbb{P} \times \mathbb{P}$ found in V and we choose conditions $\{p_s : s \in 2^{<\mathbb{N}}\}$ such that for $t \subset s$ we have $p_s \leq p_t$ and for $s, t \in 2^n$ (some $n \in \mathbb{N}$), $s \neq t$, we have

$$(p_s, p_t) \in \bigcap_{k<n} D_k,$$

$$(p_s, p_t) \Vdash \sigma[\dot{G}_l] \ncong \sigma[\dot{G}_r].$$

For $w \in 2^{\mathbb{N}}$ we let G_w be the filter generated by $\{p_{w|n} : n \in \mathbb{N}\}$. It is not hard to see that

$$w \mapsto \sigma[G_w]$$

is a continuous injection and that for $w \neq w'$ we have

$$\sigma[G_w] \ncong \sigma[G_{w'}].$$

Step 2: For $(\mathbb{P}, p, \sigma)$ as above, there exists some $\varphi \in V$ (already in the ground model) such that

$$p \Vdash_{\mathbb{P}} \varphi \text{ is Scott sentence of } \sigma[\dot{G}].$$

This follows from general facts about forcing. Let τ be a term for an object that exists in the generic extension but whose value does not depend on the generic:

$$(p,p) \Vdash_{\mathbb{P}\times\mathbb{P}} \tau[\dot{G}_l] = \tau[\dot{G}_r].$$

Then one can argue by induction on the set theoretical rank of τ that there must exist some $a \in V$ with

$$p \Vdash_{\mathbb{P}} \tau[\dot{G}] = a.$$

Step 3: Iterate step 1 to find some sequence $(\mathbb{P}_\alpha, p_\alpha, \sigma_\alpha)$ such that at each $\alpha \neq \beta$

$$p_\alpha \Vdash_{\mathbb{P}_\alpha} \sigma_\alpha[\dot{G}] \in A,$$

$$(p_\alpha, p_\alpha) \Vdash_{\mathbb{P}_\alpha \times \mathbb{P}_\alpha} \sigma_\alpha[\dot{G}_l] \cong \sigma_\alpha[\dot{G}_r],$$

$$(p_\alpha, p_\beta) \Vdash_{\mathbb{P}_\alpha \times \mathbb{P}_\beta} \sigma_\alpha[\dot{G}_l] \ncong \sigma_\beta[\dot{G}_r].$$

Step 4: Apply step 2 repeatedly to now obtain a sequence (φ_α) of infinitary formulas[9] such that at each α

$$p_\alpha \Vdash_{\mathbb{P}_\alpha} \varphi_\alpha \text{ is Scott sentence of } \sigma_\alpha[\dot{G}].$$

At each α let $\gamma(\alpha)$ be the Scott height of any model of φ_α. Let

$$(\psi^\alpha_\beta)_{\beta<\delta(\alpha)}$$

enumerate without repetitions the canonical $\gamma(\alpha)$-types appearing in any model of φ_α.

Since $\varphi_\alpha \neq \varphi_\beta$ for all $\alpha \neq \beta$ we have

$$\gamma(\alpha), \delta(\alpha) \to \infty$$

as

$$\alpha \to \infty.$$

Step 5: Choose some $\delta(\alpha) \geq \beth_{\aleph_1}$.

Step 6: Choosing some very large λ, obtain an ω-model

$$(N; \hat{\varphi}, \hat{H}, \hat{X}, \mathcal{M}_0, \hat{A}, (\hat{\mathbb{P}}, \hat{p}, \hat{\sigma}), \hat{\in})$$

for the theory of the structure

$$(V; \varphi_\alpha, H, X, \mathcal{M}_0, A, (\mathbb{P}_\alpha, p_\alpha, \sigma_\alpha), \in),$$

which is in turn generated by the indiscernibles $(\hat{\psi}_q)_{q\in\mathbb{Q}}$, where for each $q \in \mathbb{Q}$ the structure N satisfies that $\hat{\psi}'_q$ is a canonical $\hat{\gamma}$ type for any model (appearing in any generic extension of N) satisfying $\hat{\varphi}$.

We can do this by the proof of $\beth_{\aleph_1}$ being the Hanf number of $\mathcal{L}_{\omega_1,\omega}$.

Step 7: Use the generating indiscernibles $(\hat{\psi}_q)_{q\in\mathbb{Q}}$ to inject $\mathrm{Aut}(\mathbb{Q}; <)$ into $\mathrm{Aut}(N; \hat{\varphi}_\alpha, \dots)$.

Step 8: Let $G \subset \hat{\mathbb{P}}$ be N-generic below $\hat{p}$ and let $\mathcal{A} = \hat{\sigma}[G]$. Then show that each automorphism from step 7 can in some sense be lifted into the automorphism group of $\mathcal{A}$.

This uses a kind of back and forth argument and the fact that $N[G]$ believes that $\hat{\varphi}$ is the Scott sentence of $\mathcal{A}$.

[9] In $\mathcal{L}\infty, \omega$.

Step 9: Note that since $N[G]$ is an ω-model we must that $\mathcal{A}$ is an expansion of $\mathcal{M}_0$. Hence we can obtain $\mathrm{Aut}(\mathbb{Q};<)$ as the continuous image of a closed subgroup of $\mathrm{Aut}(\mathcal{M}_0)$. It is not hard to show that S_∞ divides $\mathrm{Aut}(\mathbb{Q},<)$, and thus S_∞ appears as the continuous image of a closed subgroup of $\mathrm{Aut}(\mathcal{M}_0)$.

Clearly the above argument depended heavily on the existence of the Scott analysis and our ability to play some kind of combinatorial game with the various "virtual Scott sentences." In general there does not seem to be a perfect analog of the Scott sentence for arbitrary Polish group actions[10]; instead it turns out that the Becker-Kechris theorem 2.3 can be used entirely to supplant the combinatorial analysis above.

Remarks about the general case (See Hjorth[to appear] for details).

Suppose now that we have again that $A \subset X$ provides a $\underset{\sim}{\mathbf{\Sigma}}^1_1$ counterexample to $\mathrm{TVC}(H, \underset{\sim}{\mathbf{\Sigma}}^1_1)$.

By the same arguments as before find some sequence $(\mathbb{P}_\alpha, p_\alpha, \sigma_\alpha)$ such that at each $\alpha \neq \beta$

$$p_\alpha \Vdash_{\mathbb{P}_\alpha} \sigma_\alpha[\dot{G}] \in A,$$

$$(p_\alpha, p_\alpha) \Vdash_{\mathbb{P}_\alpha \times \mathbb{P}_\alpha} \sigma_\alpha[\dot{G}_l] E_H \sigma_\alpha[\dot{G}_r],$$

$$(p_\alpha, p_\beta) \Vdash_{\mathbb{P}_\alpha \times \mathbb{P}_\beta} \neg\sigma_\alpha[\dot{G}_l] E_H \sigma_\beta[\dot{G}_r].$$

Now one can apply a kind of variation of the Becker-Kechris theorem to obtain some ordinal length sequence (τ_α) of "virtual Polish topologies," such that at each α

$$p_\alpha \Vdash_{\mathbb{P}_\alpha} \sigma_\alpha[\dot{G}] \text{ is } G_\delta \text{ in } (X, \tau_\alpha).$$

In some sense we can do this so that a complete description of τ_α exists in V – this is parallel to φ_α being in the ground model V for the special case above.

Then the back and forth argument from step 8 and its appeal to the basic properties of the Scott sentence is replaced by a kind of generalized back and forth argument (applied to the group H) that uses the Effros lemma on G_δ orbits.

$\dashv$

4.3. Extensions and speculations. The method of proof allows some other uses. For instance virtually word for the word the same argument as above yields:

THEOREM 4.2. *Let $\sigma \in \mathcal{L}_{\omega_1,\omega}$ and suppose $A \subset \mathrm{Mod}(\sigma)$ is analytic and has exactly $\aleph_1$ many models up to isomorphism.*[11] *Then there is a countable model $\mathcal{M}$ of σ for which S_∞ divides $\mathrm{Aut}(\mathcal{M})$.*

Rather less clear is whether we could hope to characterize other properties of the equivalence relations of Polish group actions in terms of the algebraic properties of those groups. For instance one can hope for some criteria of when a Polish group satisfies some version of the Glimm-Effros. For this I suspect there

[10] But chapter 6 of Hjorth[2000] gives an extended discussion of what we can still hope for.

[11] As in previous usage, here "exactly" is intended to indicate that there is no perfect set of non-isomorphic models.

are technical reasons why insisting on Borel functions witnessing smoothness may be too restrictive, so I will ask the question in the following way:

QUESTION 4.7. Is there a Polish group H such that the following two properties are equivalent for an arbitrary Polish G?

1. There is a closed subgroup G_0 of G and a continuous homomorphism $\pi : G_0 \twoheadrightarrow H$ onto H.
2. There is a Polish G-space X such that:
   there is no continuous injection
   $$\rho : 2^{\mathbb{N}} \to X$$
   with all $\vec{x}, \vec{y} \in 2^{\mathbb{N}}$ having
   $$\vec{x} E_{\mathbb{Z}_2^{<\mathbb{N}}} \vec{y} \Leftrightarrow (\rho(\vec{x}) E_G \rho(\vec{y}));$$
   there is no absolutely $\underset{\sim}{\boldsymbol{\Delta}}^1_2$ function (in the sense of say Hjorth[2000]) $\pi : X \to \mathbb{R}$ with
   $$\forall x_1, x_2 \in X (x_1 E_G x_2 \Leftrightarrow \pi(x) = \pi(y)).$$

Finally while we have a counterexample for the Glimm-Effros dichotomy for $\mathcal{L}_{\omega_1,\omega}$ given above, and hence a counterexample for continuous actions of S_∞, a strictly first order example seems to be missing.

QUESTION 4.8. For a countable first order theory T we can consider $\mathrm{Mod}(T)$, the space of all models for T with underlying set $\mathbb{N}$, equipped with the topology τ_{FO} given by first order formulas.

Does there exist a countable first oder T and an absolutely $\underset{\sim}{\boldsymbol{\Delta}}^1_2$ in the codes function
$$\pi : \mathrm{Mod}(T) \to 2^{<\omega_1}$$
such that the following both occur?

1. π is onto;
2. for all $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(T)$ we have
   $$\mathcal{M} \cong \mathcal{N} \Leftrightarrow \pi(\mathcal{M}) = \pi(\mathcal{N}).$$

If no such example can be found we might ask more optimistically:

QUESTION 4.9. Does first order logic satisfy the Glimm-Effros dichotomy? That is to say, if T is a countable first order theory, is it that case that one of the following must hold?

1. There is a Borel function θ from $\mathrm{Mod}(\mathcal{L})$ to $\mathbb{R}$ such that for $\mathcal{M}, \mathcal{N} \models T$
   $$\mathcal{M} \cong \mathcal{N} \Leftrightarrow \theta(\mathcal{M}) = \theta(\mathcal{N}).$$
2. There is a continuous injection
   $$\rho : 2^{\mathbb{N}} \to \mathrm{Mod}(\mathcal{L})$$
   such that for all $\vec{x}, \vec{y} \in 2^{\mathbb{N}}$
   $$\vec{x} E_{\mathbb{Z}_2^{<\mathbb{N}}} \vec{y} \Leftrightarrow (\rho(\vec{x}) \cong \rho(\vec{y})).$$

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