A dichotomy theorem for isomorphism

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December 24, 1999

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1 Introduction

This paper contains a dichotomy theorem for the isomorphism relation on a class of countable structures. The statement of the theorem is cleanest when the isomorphism relation is Borel and the class of models is defined by a sentence in $\mathcal{L}_{\omega_1,\omega}$, and therefore this is where we begin.

We display a canonical obstruction which must be present whenever we are unable to reduce the isomorphism relation to a countable Borel equivalence relation.

Theorem 1.1 *Let* $\mathrm{Mod}(\mathcal{L})$ *the space of countable* $\mathcal{L}$ *structures in either the (Polish) topology generated by quantifier free formulas or the topology generated by first order logic. Let* $\sigma \in \mathcal{L}_{\omega_1,\omega}$ *be such that* $\cong|_{\mathrm{Mod}(\sigma)}$ *is Borel and let* $\mathrm{Mod}(\sigma)$ *be the space of countable models of* σ*.*

Then exactly one of the following occurs:-

(I) there is a Borel function $\theta : \mathrm{Mod}(\sigma) \to X$*, some Polish space* X*, and there is a Borel equivalence relation* E *on* X *all of whose equivalence classes are countable, such that for all* $\mathcal{M}_0, \mathcal{M}_1 \in \mathrm{Mod}(\sigma)$

$$\mathcal{M}_0 \cong \mathcal{M}_1 \Leftrightarrow \theta(\mathcal{M}_0) E \theta(\mathcal{M}_1);$$

(II) there is a Borel function

$$\theta : (2^\omega)^\omega \to \mathrm{Mod}(\sigma)$$

such that for all $(x_n)_{n\in\omega}, (y_n)_{n\in\omega}$ *with each* $x_n, y_n : \omega \to \{0,1\}$*, we have*

$$\theta((x_n)_{n\in\omega}) \cong \theta)(y_n)_{n\in\omega})$$

if and only if for all $n \in \omega$ *there is* $K_n \in \omega$ *such that for all* $i > K_n$

$$x_n(i) = y_n(i).$$

By [13] and a result by Kechris at 5.1 of [11] we obtain that (I) is equivalent to each of the following:

(I') there is a Borel function

$$f : \mathrm{Mod}(\sigma) \to \mathbb{R}$$

such that

(a) f has countable image on each isomorphism type: $|\{f(\mathcal{N}) : \mathcal{N} \cong \mathcal{N}_0\}| \leq \aleph_0$ for each $\mathcal{N}_0 \in \mathrm{Mod}(\sigma)$;

(b) f assigns disjoint countable sets: for if $|\{f(\mathcal{N}) : \mathcal{N} \cong \mathcal{N}_0\}| \cap |\{f(\mathcal{N}) : \mathcal{N} \cong \mathcal{N}_1\}| \neq \emptyset$ then $\mathcal{N}_0 \cong \mathcal{N}_1$.

(I") there is a countable fragment $F \subset \mathcal{L}_{\omega_1,\omega}$ such that for every $\mathcal{M} \models \sigma$ we have some $\vec{a} \in \mathcal{M}$ with the model

$$\langle \mathcal{M}; \vec{a} \rangle$$

atomic over the fragment F;
(I"') there is a countable fragment $F \subset \mathcal{L}_{\omega_1,\omega}$ such that for every $\mathcal{M} \models \sigma$ we have some $\vec{a} \in \mathcal{M}$ with the F-theory of the model

$$\langle \mathcal{M}; \vec{a} \rangle$$

$\aleph_0$-categorical.

Very loosely one might think of (I), (I'), (I"), and (I"') as expressing the idea that the models of σ all have "finite rank." A specific case of this is when the models are all finitely generated, in the sense that every $\mathcal{M} \in \mathrm{Mod}(\sigma)$ has

$$\mathcal{M} = \{t(\vec{a}) : t \text{ is a term}\},$$

for some $\vec{a} \in \mathcal{M}$.

On the other side, (II) may be thought of expressing that the models of σ *interpret* – in some very generous notion of interpretation – a canonical theory which has infinite rank. For the purposes of illustrating that let us take an example.

Consider the language $\hat{\mathcal{L}}$ with a single binary relation $<$ and countably many unary predicates $(U_n)_{n\in\omega}$. Let $\varphi \in \hat{\mathcal{L}}_{\omega_1,\omega}$ be the sentence all of whose models are linearly ordered by $<$ with order type equal to $\mathbb{Z} \cdot \omega$ (in the reverse lexicographic ordering); that is to say, if we start with $\mathbb{Z} \cdot \omega$ the linear ordering given by ω-many $\mathbb{Z}$-blocks, then $\mathrm{Mod}(\varphi)$ consists in the the expansions of this structure by countably many unary predicates.

With all this in place, we can rephrase (II) as
(II') there is a Borel function

$$\theta : \mathrm{Mod}(\varphi) \to \mathrm{Mod}(\sigma)$$

such that for all $\mathcal{N}_0, \mathcal{N}_1 \in \mathrm{Mod}(\varphi)$ we have

$$\mathcal{N}_0 \cong \mathcal{N}_1 \Leftrightarrow \theta(\mathcal{N}_0) \cong \theta(\mathcal{N}_1).$$

The method of proof allows certain modifications. We can work on $\underset{\sim}{\Sigma}^1_1$ subsets of the space of all countable models, instead of just those defined by a countably infinitary formula. Thus in particular if F is a Borel equivalence relation, on a Polish space Y, and if F allows a Borel reduction to the isomorphism relation on some space of countable structures, then we have, as in the theorem above, exactly one of the following:
(I*) there is a Borel function $\theta : Y \to X$, some Polish space X, and there is a Borel equivalence relation E on X all of whose equivalence classes are countable, such that for all $y_0, y_1 \in Y$

$$y_0 F y_1 \Leftrightarrow \theta(y_0) E \theta(y_1);$$

(II*) there is a Borel function

$$\theta : (2^\omega)^\omega \to Y$$

such that for all $(x_n)_{n\in\omega}, (y_n)_{n\in\omega}$ with each $x_n, y_n : \omega \to \{0,1\}$, we have

$$\theta((x_n)_{n\in\omega}) F \theta((y_n)_{n\in\omega})$$

if and only if for all $n \in \omega$ there is $K_n \in \omega$ such that for all $i > K_n$

$$x_n(i) = y_n(i).$$

The situation for general, not necessarily Borel, isomorphism relations is more subtle. The first point to make is that the theorem *does* fail in this wider context. A counter example is given by the abelian p-groups, p a prime, as discussed in [12]. Here one has an explicit bijection between the isomorphism types and $2^{<\aleph_1}$; as may seem very plausible, and has been discussed in [6], there is no definable method of injecting $2^{<\aleph_1}$ into countable sets of reals, as would happen at (I) above; alternatively the equivalence relation described at (II) is certainly not reducible to $2^{<\aleph_1}$, as [12] even shows the equivalence relation E_0, of eventual agreement just on a single copy of 2^ω, to refuse subsets of the ordinals as complete invariants in any kind of definable, let alone Borel or $\utilde{\Delta}^1_2$, manner.

With this problem noted, and bearing in mind the extrapolations past Borel appearing in earlier papers such as [12], one can find the right statement for Σ^1_1 isomorphism relations. There are various ways of stating this. For instance bearing in mind (I') above:

Theorem 1.2 *Let $\sigma \in \mathcal{L}_{\omega_1,\omega}$ and let $\mathrm{Mod}(\sigma)$ be the space of countable models of σ.*

Then exactly one of the following occurs:-

($\hat{I}$) there is a $\utilde{\Delta}^1_2$ in the codes function $f : \mathrm{Mod}(\sigma) \to 2^{<\aleph_1}$,

(a) f has countable image on each isomorphism type: $|\{f(\mathcal{N}) : \mathcal{N} \cong \mathcal{N}_0\}| \le \aleph_0$ for each $\mathcal{N}_0 \in \mathrm{Mod}(\sigma)$;

(b) f assigns disjoint countable sets: for if $|\{f(\mathcal{N}) : \mathcal{N} \cong \mathcal{N}_0\}| \cap |\{f(\mathcal{N}) : \mathcal{N} \cong \mathcal{N}_1\}| \ne \emptyset$ then $\mathcal{N}_0 \cong \mathcal{N}_1$.

(II) there is a Borel function

$$\theta : (2^\omega)^\omega \to \mathrm{Mod}(\sigma)$$

such that for all $(x_n)_{n\in\omega}, (y_n)_{n\in\omega}$ with each $x_n, y_n : \omega \to \{0,1\}$, we have

$$\theta((x_n)_{n\in\omega}) \cong \theta((y_n)_{n\in\omega})$$

if and only if for all $n \in \omega$ there is $K_n \in \omega$ such that for all $i > K_n$

$$x_n(i) = y_n(i).$$

The proof of the main theorem is fairly long and there are several parts. In section 7 we present the main definition which is used to define a split into cases. In some rough sense the idea here is to consider the Σ^1_1 sets which a given

$$\langle \mathcal{M}; \vec{a} \rangle$$

falls inside, and ask when we can produce a new type, over a given fragment F, realized by some

$$\langle \mathcal{N}; \vec{b} \rangle,$$

where $\mathcal{M}$ allows a "highly elementary" embedding into $\mathcal{N}$. The branching into three possibilities, at 11, 13, and 12, and the consequent rank function which appears when there is no such $\mathcal{N}$, constitutes the main idea.

In 8 we step through various standard reflection arguments, many of which are similar to parallel points in [8], and show that the various sets we need to be Σ^1_1 are indeed Σ^1_1.

Section 9 turns off into an aside. It uses ideas closely related to 7 to obtain a characterization of when an infinitary sentence allows a countable model; this characterization appears natural, to the author at least, given the arguments of section 10 and following. It can be viewed as the generalization of [7], which completely described when a countable model allows an uncountable model of its Scott's sentence and has as a consequence a result by Keisler and Shelah to the effect that a set of infinitary sentences allowing an uncountable model is Σ^1_1 in the codes.

10 returns to the main road. Here there is a fork into three possibilities. The first, case(I), corresponds to when we are appropriately able to add to the F-types for some suitable set of $\mathcal{M} \models \sigma$. The second and third, labeled (IIa) and (IIb), are when we are unable to add to the F-types for any countable Δ^1_1 fragment F, and therefore have a kind of rank function definable over the models of σ, which ranks the tree of partial attempts corresponding to case(I); this rank function can be finite everywhere, giving us case(IIa), or somewhere it may be infinite, giving (IIb).

In 11 we show that case(I) entails that we are in (II) of the theorem:

$$(E_0)^\omega \leq_B \cong|_{\mathrm{Mod}(\sigma)}.$$

The rough idea is to try to build a generic sequence of models, each one elementary in the next, each one realizing many more types than the last, and squeeze in between successive models a copy of E_0, the equivalence relation of eventual agreement on infinite binary sequences.

Section 12 handles case(IIb), when the rank function exists and for some models it is always infinite. Here we argue that we can assume that the rank function has as its supremum some countable recursive limit ordinal. Then we can try to stratify our generic model into countably many levels, classifying a tuple $\vec{b}$ from $\mathcal{M}$ on the basis of its rank. Once this is set in place we mimic the argument from 11, interpreting a different copy of E_0 on different levels.

Finally we are left with 13 and the discussion of case(IIa), which is in some ways the easiest case, or at least the most familiar, being almost identical to a parallel argument from [14]. This is the one place we use that $\cong|_{\mathrm{Mod}(\sigma)}$ is Borel. By assuming that the rank function is finite over all fragments, and iterating this observation through as many ordinal steps as the Borel complexity of the isomorphism relation, we finally obtain (I').

There are several sections of background material. Certainly 2, 5, and 4 with their discussion of set theoretical and model theoretic prerequisites, as well as the general theory of Borel equivalence relations, are standard for this kind of paper.

On the other hand 3 is a little bit more than the usual thumbnail sketch of the well known. Here there are several short proofs on the boundary of effective descriptive set theory and admissible set theory, all of which are commonplace but strangely difficult to find precisely in the literature. In the arguments of 8, 11, 12, 13, and even 7, it will be helpful to continually pass back and forward from the descriptive set theory of Σ^1_1 sets to the "inner model theory" of admissible sets, and while it would be theoretically possible to do everything just using effective descriptive set theory, the arguments would become unintuitive and hard to follow.

Given the length of the proof, the reader may already be planning how it might be possible to avoid reading most of it. This is a reasonable desire, so I should state forthrightly what is most peripheral.

Two sections which can be skipped are 9 and 6. The first, as remarked above, plays out the ideas of the proof for a result whose actual statement is unrelated; it is never referenced again.

The situation with the 6 is more ambiguous. It came about after considering arguments invoking Gandy-Harrington forcing. Often when becoming involved in calculations that use the "geometry" of Σ^1_1 sets it becomes easier if we fix a kind of "generic" member of the Σ^1_1, and perform our various calculations with repeated appeals to its genericity. The main point of 6 is the somewhat philosophical one, that the entire usefulness of a generic member of a Σ^1_1 set is that it is low, in the sense of having its Church-Kleene ordinal equal to ω_1^{ck}; for the rest of the paper we therefore refer only to reals that are low, never to reals which arise from a Gandy-Harrington generic filter. While it is true that there are isolated references back to 6, they are only made to 6.7, 6.10, and 6.11, all of which are well known.

2 Set theoretical notation

Notation For us the set of natural numbers is

$$\omega = \{0, 1, 2, 3, ...\}.$$

We identify each natural number with its set of predecessors:

$$n = \{0, 1, 2, ..., n-1\};$$

zero has no predecessors, and thus equals the empty set:

$$0 = \emptyset.$$

With this in hand $2 = \{0, 1\}$ and for any natural number n

$$2^n = \{f \mid f : \{0, 1, 2, ..., n-1\} \to \{0, 1\}\};$$

2^ω is the set of all functions from ω to $\{0, 1\}$, and in general given sets X and Y we let

$$X^Y = \{f : Y \to X\}.$$

Given a set X and $n \in \omega$, we use X^n for the set of all sequences of length n from X:

$$X^n = \{f \mid f : \{0, 1, 2, ..., n-1\} \to X\};$$

this may be naturally identified with the n-fold product of X:

$$X \times X \times \cdots [n \text{ times}] \times X.$$

Then $X^{<\omega}$ is the set of all finite sequences from X:

$$X^{<\omega} = \bigcup_{n \in \omega} X^n.$$

Definition Given sets $X_1, X_2, ...X_n$, *tree* on $X_1 \times X_2 \times ...X_k$ is a subset of

$$\bigcup_{n \in \omega} X_1^n \times X_2^n \times ...X_k^n$$

closed under subsequences in the natural way – so that given $(u_1, ..., u_k) \in X_1^n \times X_2^n \times ...X_k^n$ in the tree we require that for any $l < n$ we have that $(u_1|_l, u_2|_l, ..., u_k|_l)$ is again in the tree.

For T a tree on $X_1, X_2, ...X_n$, an *infinite branch through* T is some $(f_1, f_2, ..., f_k)$, each

$$f_i : \omega \to X_i$$

with $f_i|_l \in T$ all $l \in \omega$. We let $[T]$ denote the collection of all infinite branches.

Notation Given two finite sequences s and t, we use

$$s^\frown t$$

to indicate the concatenation, obtained by following s by t.

Thus if $s \in X^k$ and $t \in X^n$ then we have $s^\frown t \in X^{k+n}$ defined by

$$(s^\frown t)(i) = s(i)$$

for $i < k$ and

$$(s^\frown t)(k+i) = t(i)$$

for $i < n$.

In the case that $t = \langle a \rangle$ has length 1 we may write

$$s^\frown a$$

instead of $s^\frown \langle a \rangle$.

Notation Frequently I will write

$$\vec{s}$$

to indicate that $\vec{s}$ is a sequence. In this case

$$s_j$$

will refer to the jth term.

For $\vec{s}$ a sequence we can use $lh(\vec{s})$ to indicate the length of s. Thus if $\vec{s} \in X^k$ then $lh(\vec{s}) = k$.

Notation For R a well founded relation or $<$ a well ordering, we let $||R||$ or $|| < ||$ denote the ordinal corresponding to their ranks. Thus for R a relation on a set X and assuming that there is no infinite descending chain $(x_i)_{i \in \omega}$ such that at each i we have

$$x_i R x_{i+1}$$

we let $||R||$ be the least ordinal γ such that there is some

$$\rho : X \to \gamma$$

such that whenever xRy we have $\rho(y) \in \rho(x)$.

I will assume basic notions of constructibility. In particular for X a recursive Polish space and $x \in X$ I need to ask that the reader can make sense the constructible hierarchy built over x – that is to say, the collection of transitive structures of the form

$$L_\alpha[x]$$

where α is an ordinal.

This is easily understood when X is a space such as 2^ω or ω^ω but requires more creativity when X is a recursive space whose elements cannot be viewed as infinite subsets of the hereditarily finite sets. For these less friendly spaces we need to work out some method of coding, and since the precise details will be irrelevant, I will leave them undiscussed. Alternatively the reader may choose to view all our statements about recursive spaces as holding only for those that are rudimentarily definable subsets of

$$\mathcal{P}(\mathrm{HF})$$

and whose elements are therefore subject to relative constructibility in the usual way. Indeed this would be an appropriate attitude, since we will be primarily interested in the recursive Polish spaces 2^ω, ω^ω, $\mathrm{Mod}(\mathcal{L})$ (see section 5 below) and their products; in all these cases the individual points of the space are subsets of the hereditarily finite sets.

Definition LO is the subset of $2^{\omega\times\omega}$ consisting of x such that

$$<^x = \{(n,m) : x(n,m) = 1\}$$

defines a linear ordering. WO is the subset of LO consisting of x with $<^x$ defining a linear ordering of ω which is a well ordering, in the sense that there is no infinite sequence $(n_i)_{i\in\omega}$ such that at each i

$$n_{i+1} <^x n_i.$$

3 Descriptive set theoretical prerequisites

We presume a robust familiarity with low level effective descriptive set theory, as applying to Σ^1_1, Π^1_1, and Δ^1_1 subsets of recursive Polish spaces; a development of recursive spaces adequate to our needs can be found in [20]; a presentation specific to the space ω^ω is given in [16], which should already be good enough since our recursive spaces can naturally be thought of as recursive closed subsets of ω^ω.

We will also use the theory of admissible sets. This is developed [1] and to a lesser extent [19].

While [1] completely covers the theory of admissible sets and while [20] thoroughly chronicles the effective theory for Σ^1_1, there seems to be no canonical reference for the connections between these two topics. Thus we sketch a few isolated proofs below.

Definition For $x \in 2^\omega$ we let $\omega_1^{\mathrm{ck}(x)}$ be the supremum of the ranks of the well orders of ω recursive in x. ω_1^{ck} indicates the supremum of the ranks of the well orderings that are outright recursive.

Theorem 3.1 *Any $\Sigma^1_1(x)$ well founded relation on 2^ω has rank less than $\omega_1^{\mathrm{ck}(x)}$.*

Proof We let R be our relation which is well founded in the sense of there being no sequence $(y_i)_{i\in\omega}$ with

$$y_i R y_{i+1}$$

at each i. We let

$$T \subset \bigcup_{n\in\omega} 2^n \times \omega^n \times 2^n$$

be a recursive in x tree with

$$yRz$$

if and only if there is $f \in \omega^\omega$ such that at all n

$$(y|_n, f|_n, z|_n) \in T.$$

With this set up, we can just go ahead and apply the usual proof of Kunen-Martin as found at 2G [20]. We let H be the set of all triples of the form

$$(s_1, u_1, s_2, u_2, ...u_n, s_{n+1})$$

where $n \in \omega$ and each $s_i \in 2^n$, each $u_i \in \omega^n$, and each $(s_i, u_i, s_{i+1}) \in T$. This gives rise to a well founded relation by defining

$$(s_1, u_1, s_2, u_2, ...u_n, s_{n+1}) S (s'_1, u'_1, s'_2, u'_2, ...u'_n, s'_{m+1})$$

if $m \geq n$ and for each $j \leq n$ we have $u_j \subset u'_j$ and $s_j \subset s'_j$. We may embed the well founded relation R into S in the usual way: For any $(y_1, y_2, ..., y_{n+1})$ and $(f_1, ..., f_n)$ with

$$(y_i, f_i, y_{i+1}) \in [T]$$

all $i \leq n$, we can associate

$$(y_1|_n, f_1|_n, ..., f_n|_n, y_{n+1}|_n) \in S$$

and observe that it must have rank at least equal to y_{n+1}.

Thus the rank of S is at least that of R. The usual Kleene Brower ordering on S given at 4A of [20] gives a well ordering of the sequences in S with rank at least that of R. Since H is a recursive in x subset of the hereditarily finite sets, we obtain a well ordering of ω which is recursive in x and has rank at least that of R.

□

Recall that no ordinal between ω and $\omega_1^{\mathrm{ck}(x)}$ can be admissible, since any ω-model of Kripke-Platek set theory can provide ordinals for all its orderings of ω which are actually well orderings in V.

Corollary 3.2 $L_{\omega_1^{\mathrm{ck}(x)}}[x]$ *is admissible.*

Proof The main issue is to verify the characteristic Σ_1 reflection axiom. So suppose that $A \in L_{\omega_1^{\mathrm{ck}(x)}}[x]$ and

$$\forall a \in A \exists \alpha < \omega_1^{\mathrm{ck}(x)} (L_\alpha[x] \models \varphi(x, a))$$

for some formula φ of set theory.

Unraveling the usual recursive definition of the well ordering of $L[x]$ and using that A appears at a level coded by a recursive in x well ordering, we may assume that $A = \omega$. We then define a well ordering on $\omega \times \omega$ by

$$(n, m) < (n', m')$$

if and only if either:

(i) $n < n'$; or

(ii) $n = n'$ and for $<_n$ the first recursive in x well ordering of ω such that

$$L_{||<_n||}[x] \models \varphi(x, n)$$

we have $m <_n m'$.

It is easily checked that this is a $\Sigma^1_1(x)$ well ordering of $\omega \times \omega$, and thus it has rank less than $\omega_1^{\mathrm{ck}(x)}$. If we let γ be its rank, then the definitions at once give

$$\forall n \in \omega \exists \alpha < \gamma (L_\alpha[x] \models \varphi(x, a)).$$

□

This granted we can take a further step with the notation.

Definition For a a set, $\omega_1^{\mathrm{ck}(a)}$ is the least ordinal $\alpha > \omega$ for which

$$L_\alpha[a]$$

is admissible.

This is not our original definition, but equivalent to it in the case that $a \subset \omega$. We can then go on and let $\omega_2^{\mathrm{ck}(a)}$ be the next admissible ordinal for a and in general let $\omega_\delta^{\mathrm{ck}(a)}$ be the δth ordinal $\alpha > \omega$ with

$$L_\alpha[a]$$

admissible.

Notation One easily sees that for φ a formula of set theory there are corresponding $P_\varphi \in \Pi^1_1$ and $A_\varphi \in \Sigma^1_1$ such that for all $x \in 2^\omega$ and $e \in \omega$ with $\{e\}^x$, the eth recursive in x relation on ω, providing a well ordering of ω with rank $\gamma < \omega_1^{\mathrm{ck}(x)}$ one has

$$L_\gamma[x] \models \varphi(x)$$

if and only if

$$(x, e) \in P_\varphi$$

if and only if

$$(x, e) \in A_\varphi.$$

Theorem 3.3 *(Spector-Gandy)*

(1) A set $P \subset \omega^\omega$ is Π^1_1 if and only if there is a recursive closed set $C \subset \omega^\omega \times \omega^\omega$ such that

$$x \in P \Leftrightarrow \exists y \in \Delta^1_1(x)((x, y) \in C).$$

(2) A set $P \subset \omega^\omega$ is Π^1_1 if and only if there is a formula φ of set theory such that

$$x \in P \Leftrightarrow \exists \alpha < \omega_1^{\mathrm{ck}(x)}(L_\alpha[x] \models \varphi(x)).$$

Proof (1) is given at 4F.3 [20], so this leaves us facing down (2).

So suppose that φ is given. Then we may find $P_\varphi \in \Pi^1_1$ as above so that for all $x \in 2^\omega$ and $e \in \omega$ with $\{e\}^x$ presenting a recursive in x well ordering of ω of rank $\gamma < \omega_1^{\mathrm{ck}(x)}$ one has

$$L_{||\{e\}^x||}[x] \models \varphi(x)$$

if and only if

$$(x, e) \in P_\varphi.$$

Thus $x \in P$ if and only if there is $e \in \omega$ with $\{e\}^x$ a well ordering and $(x, e) \in P_\varphi$.

The converse direction follows since given

$$P = \{x \in 2^\omega : \forall f \in \omega^\omega \exists n((x|_n, f|_n) \notin T)\}$$

for T some recursive tree on x, we have $x \in P$ if and only if the least admissible structure containing x can provide a ranking function for the recursive in x tree

$$T_x = \{u : (x|_{lh(u)}, u) \in T\}.$$

□

(2) implies that a subset of ω is Π^1_1 if and only if it is Σ_1 over $L_{\omega_1^{\mathrm{ck}}}$.

Theorem 3.4 *(Boundedness for Σ^1_1.) Assume:*

1. *$A \subset \omega^\omega$ is Σ^1_1;*
2. *$\psi(v_0, v_1)$ is a Σ_1 formula of set theory;*
3. *for all $x \in A$ there exists $d \in L_{\omega_1^{\mathrm{ck}(x)}}[x]$ with*

$$L_{\omega_1^{\mathrm{ck}(x)}}[x] \models \psi(x, d).$$

Then there is some $\gamma_0 < \omega_1^{\mathrm{ck}}$ such that for all $x \in A$ there exists $d \in L_{\gamma_0}[x]$ with

$$L_{\gamma_0}[x] \models \psi(x, d).$$

Proof We let B be the set of $(x, e) \in 2^\omega \times \omega$ such that $\{e\}^x$ is a recursive linear ordering on ω and for all other $e^* \in \omega$ we have either
(i) $\{e^*\}^x$ is not a well ordering of ω; or
(ii) $(x, e^*) \in A_{\neg\exists d\psi(v_0,d)}$; or
(iii) there is an ordering preserving map of $\{e\}^x$ into $\{e^*\}^x$.

By our assumptions this implies that $(x, e) \in B$ if and only if $\{e\}^x$ is a well ordering of ω with rank no greater than α where α is least such that

$$L_\alpha[x] \models \exists d\psi(x, d).$$

And then we can obtain a well founded Σ^1_1 relation on $B \times B$ by setting

$$(x, e, n) > (x', e', m)$$

if and only if $x = x'$ and $e = e'$ and n occupies a higher position in the linear of ω provided by $\{e\}^x$ than does m.

Now the result follows by 3.1.

□

Note that the property indicated for γ_0 in the conclusion of 3.4 is uniformly Π^1_1 in any recursive well ordering of length γ_0. Thus the property will be uniformly in any index for the Σ^1_1 set Σ_1 over $L_{\omega_1^{\mathrm{ck}}}$.

4 Equivalence relations

Definition For X and Y Polish spaces and E and F we write

$$E \leq_B F$$

if there is a Borel function

$$\theta : X \to Y$$

such that for all $x_1, x_2 \in X$

$$x_1 E x_2 \Leftrightarrow \theta(x_1) F \theta(x_2).$$

Here we say that E is *Borel reducible* to F and that θ is a reduction If we can find a reduction θ which is not only Borel but continuous, then we write $E \leq_c F$.

We write $E \sqsubseteq_B F$ if we may find a Borel reduction which is one to one and $E \sqsubseteq_c F$ if we can find a continuous reduction which is one to one.

Definition An equivalence relation E on a Polish space X is *Borel* if it appears in the σ-algebra generated by the open sets in the product topology on $X \times X$; in other words, it is Borel with respect to the canonical Polish topology on $X \times X$.

An equivalence relation is *countable* if every equivalence class is countable.

Perhaps the simplest of the Borel equivalence relations are the identity relations on various Polish spaces.

Notation For X a Polish space we let $\mathrm{id}(X)$ be the identity relation on X, so that $x_1 \mathrm{id}(X) x_2$ if and only if $x_1 = x_2$.

These simplest of equivalence relations are the subject of Silver's theorem from [22], restated below in modern notation. This theorem is also sometimes called the *first dichotomy theorem* for Borel equivalence relations.

Here it should be understood as implicit in its statement that we view ω as a discrete space, so that it is certainly Polish, and we view 2^ω as a Polish space in the product topology.

Theorem 4.1 *(Silver) Let E be a Borel equivalence relation. Then exactly one of the following happens:*

(I) $E \leq_B \mathrm{id}(\omega)$;
(II) $\mathrm{id}(2^\omega) \leq_B E$.

In fact the statement of the theorem can be slightly embellished. In (II) we at once have that the reduction must be one to one, and so $\mathrm{id}(2^\omega) \sqsubseteq_B E$; then one can obtain a comeager subset of 2^ω on which the reduction is continuous, since every Borel function is continuous on a comeager set, and this comeager set must in turn contain a homeomorphic copy of the Cantor space, and thus in fact $\mathrm{id}(2^\omega) \sqsubseteq_c E$.

Just beyond the identity relations one has the Vitali equivalence relation.

Definition We let E_v, the *Vitali equivalence relation,* be the equivalence relation on $\mathbb{R}$ arising from the cosets of $\mathbb{Q}$; thus xE_vy if and only if $x-y\in\mathbb{Q}$.

We let E_0 be the equivalence relation of eventual agreement on 2^ω. Thus for $f_0, f_1 \in 2^\omega$ we let $f_0E_0f_1$ if there is some $N\in\omega$ such that for all $k>N$

$$f_0(k)=f_1(k).$$

Fact 4.2 *(See [3])* $E_v \leq_B E_0 \leq E_v$.

Here one has the *second dichotomy theorem:*

Theorem 4.3 *(Harrington-Kechris-Louveau; see [8]) Let E be a Borel equivalence relation. Then exactly one of the following happens:*
(I) $E\leq_B \mathrm{id}(2^\omega)$;
(II) $E_0\leq_B E$.

In fact in (II) they obtain $E_0 \sqsubseteq_c E$. In any case a short Baire category argument shows that for all equivalence relations F, if $E_0\leq_B F$ then $E_0\sqsubseteq_c F$.

Among the Borel equivalence relations an important type are those induced by the continuous action of a Polish group on a Polish space. In the case that a Polish group G acts continuously on a Polish space X we use E^X_G to denote the orbit equivalence relation.

Here there is essentially only one *known* Borel equivalence relation not reducible to some such E^X_G.

Definition We view $(2^\omega)^\omega$ as a Polish space in the product topology. For $\vec{x}=\langle x_0,x_1,x_2,...\rangle, \vec{y}=\langle y_0,y_1,...\rangle\in(2^\omega)^\omega$ we set

$$\vec{x}E_1\vec{y}$$

if there is some $N\in\omega$ such that

$$\forall k>N(x_k=y_k).$$

Theorem 4.4 *(Kechris-Louveau; see [18]) If G is a Polish group acting continuously on a Polish space X, then*

$$E_1\not\leq_B E^X_G.$$

E_1 is also the subject of the *third dichotomy theorem* of [18]:

Theorem 4.5 *(Kechris-Louveau) If $E\leq_B E_1$, then exactly one of the following happens:*
(I) $E\leq_B E_0$;
(II) $E_1\leq_B E$.

A parallel result was obtained in [14] for the countable product of E_0.

Definition For $\vec{x}, \vec{y} \in (2^\omega)^\omega$ we set $\vec{x}(E_0)^\omega\vec{y}$

$$\forall n \in \omega(x_n E_0 y_n).$$

Theorem 4.6 *(Hjorth-Kechris) If $E \leq_B (E_0)^\omega$, then exactly one of the following happens:*
(I) $E \leq_B E_0$;
(II) $(E_0)^\omega \leq_B E$.

Among the Borel equivalence relations an especially important class are those arising in the form

$$E_H^X,$$

where H is a closed subgroup of the Polish group S_∞, consisting of all permutations of ω in the topology of pointwise convergence, acts continuously on the Polish space X. It is shown in [2] that these are up to Borel reducibility exactly the Borel isomorphism relations on classes of countable structures.

[14] further proved that whenever H a is sufficiently nice closed subgroup of S_∞ – for instance, abelian – acting continuously on a Polish space X with a Borel orbit equivalence relation, we have either

$$E_H^X \leq E$$

for some *countable* Borel equivalence relation E or

$$(E_0)^\omega \leq_B E_H^X.$$

Without any great conviction, and mainly out of a desire to be provocative, it was then conjectured that this result should hold for H an arbitrary closed subgroup of S_∞.

This paper proves that conjecture.

5 Model theoretic generalities

Definition For $\mathcal{L}$ a countable language, we let $\mathcal{L}_{\omega_1,\omega}$ be the countably infinitary logic obtained from the atomic formulas of $\mathcal{L}$ by closing under countable conjunction

$$\bigwedge$$

countable disjunction

$$\bigvee$$

as well as the first order operations of $\wedge$, $\vee$, $\neg$, $\exists x$ and $\forall x$, but allowing only formulas with finitely many variables.

$\mathcal{L}_{\omega,\omega}$ is the set of first order formulas generated by $\mathcal{L}$.

We will use the term "fragment" in a specific sense; our meaning is more narrow than the usual model theoretic definition, in as much as we require that F be countable and that it include all atomic formulas, and hence determine the language from which it was cleaved.

Definition A set of formulas $F \subset \mathcal{L}_{\omega_1,\omega}$ is a *fragment* if
(i) F is countable;
(ii) F is closed under the first order operations of $\wedge$, $\vee$, $\neg$, $\exists x$ and $\forall x$;
(iii) F is closed under all substitution instances;
(iv) F includes all atomic formulas;
(v) F is closed under subformulas.

Definition For $\mathcal{L}$ a countable language, we let $\mathrm{Mod}(\mathcal{L})$ be the space of all countable $\mathcal{L}$-structures whose underlying set is ω. We equip this with the topology generated by quantifier free formulas, so that the basic open sets have the form

$$\{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}) : \mathcal{M} \models \varphi(\vec{a})\}$$

for some $\varphi \in \mathcal{L}_{\omega,\omega}$ in which no quantifiers appear and some $\vec{a} \in \omega^{<\omega}$. We may naturally identify $\mathrm{Mod}(\mathcal{L})$ with

$$\prod_k (\prod_{\sigma_0(k)} \omega^{(\omega^k)} \prod_{\sigma_1(k)} 2^{(\omega^k)}),$$

where each $\sigma_0(k) \in \{0, 1, 2, ..., \aleph_0\}$ is the number of function symbols of arity k and $\sigma_1(k) \in \{0, 1, 2, ..., \aleph_0\}$ is the number of relation symbols of arity k.

Thus $\mathrm{Mod}(\mathcal{L})$ in the topology of quantifier free formulas is a recursive Polish space all of whose elements are subsets of the hereditarily finite sets. Thus as remarked at the end of section 2, there is no controversy in forming

$$L_\alpha[\mathcal{M}]$$

for α an ordinal and $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$.

It is also well known, and can be found at [10] 2.3, that the resulting space $\mathrm{Mod}(\mathcal{L})$ equipped with the topology of *all first order formulas* is again Polish. Thus for any theory T the set

$$\{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}) : \mathcal{M} \models T\}$$

is a closed subset and hence a Polish space in its own right. Both these topologies, first order and quantifier free, give rise to the same Borel structure; thus from our point of view they are largely interchangeable.

Notation From now on let us take part in the usual model theoretic convention regarding sequences from a structure, thus using

$$\vec{a} \in \mathcal{M}$$

(rather than $\vec{a} \in \mathcal{M}^{<\omega}$) to indicate that $\vec{a}$ is a sequence in $\mathcal{M}$.

In the case that

$$\rho : \mathcal{M} \to \mathcal{N}$$

is a mapping between two models and $\vec{a} = \langle a_0, a_1, ..., a_l \rangle$ is a sequence from $\mathcal{M}$ of length $l+1$ let us simply write

$$\rho(\vec{a})$$

for the sequence $\langle \rho(a_0), \rho(a_1), ..., \rho(a_l) \rangle \in \mathcal{N}$.

Definition For F a fragment and $\mathcal{M}$ and $\mathcal{N}$ models, let us say that a map

$$\rho : \mathcal{M} \to \mathcal{N}$$

is F*-elementary* if for all $\vec{a} \in \mathcal{M}$ and and $\varphi(\vec{x}) \in F$ we have

$$(\mathcal{M} \models \varphi(\vec{a})) \Leftrightarrow (\mathcal{N} \models \varphi(\rho(\vec{a}))).$$

Notation For each language $\mathcal{L}$ let us fix a corresponding infinite sequence $(\mathbf{c}_i)_{i \in \omega}$ of fresh constants. For $k \in \omega$ we let $\mathcal{L}^k$ be the extension of $\mathcal{L}$ obtained by adding the constant symbols $(\mathbf{c}_i)_{i<k}$.

For $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ and $\vec{a} \in \mathcal{M}$ we define

$$\langle \mathcal{M}; \vec{a} \rangle$$

to be the expansion of $\mathcal{M}$ to $\mathcal{L}^{lh(\vec{a})}$ obtained by insisting that at each $i < lh(\vec{a})$

$$\langle \mathcal{M}; \vec{a} \rangle \models \mathbf{c}_i = a_i.$$

Naturally we may also write

$$\langle \mathcal{M}; \vec{a}, \vec{b}, ... \rangle$$

in place of $\langle \mathcal{M}; \vec{a}^\frown \vec{b}^\frown ... \rangle$.

For F a fragment in $\mathcal{L}$ and $\mathcal{L}^k$ as in the previous definition, we may view F as giving rise to a canonical fragment for $\mathcal{L}^k$ by simply closing F under all substitution instances of the larger language. When there seems to be no threat of confusion it will be notationally convenient to identify F with this canonical extension

Definition For $\mathcal{L}$ a recursive language, F a fragment for $\mathcal{L}$, $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$, and $\vec{a} \in \mathcal{M}$ we let

$$F\text{−type}_{\mathcal{M}}(\vec{a})$$

be the F type of $\vec{a}$ from the point of view of $\mathcal{M}$. Explicitly:

$$\{\psi(\vec{x}) : \psi \in F, \mathcal{M} \models \psi(\vec{a})\}.$$

Notation For $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ and $\mathcal{L}' \subset \mathcal{L}$, we let

$$\mathcal{M}|_{\mathcal{L}'}$$

denote the reduction of $\mathcal{M}$ to the language $\mathcal{L}'$.

Definition For $\sigma \in \mathcal{L}_{\omega_1,\omega}$ we let

$$\mathrm{Mod}(\sigma)$$

be the set of $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ with

$$\mathcal{M} \models \sigma.$$

Of course the space $\mathrm{Mod}(\sigma)$ may no longer be a Polish space, and we have an understandable bias in favor of not stepping too far away from this nice class of spaces. One solution would be to take the topology indicated by the smallest fragment containing σ; another is provided by the next definition and lemma.

Definition A formula $\psi \in \mathcal{L}_{\omega_1,\omega}$ is Π^0_1 if it has the form

$$\bigwedge_{i\in\omega} \psi_j$$

where each ψ_j is a quantifier free first order formula. A formula is Σ^0_1 if it has the form

$$\bigvee_{i\in\omega} \psi_j$$

where each ψ_j is quantifier free and first order. After this we just iterate through the ordinals in the usual fashion, with formulas of the form

$$\bigvee_{j\in\omega} \exists\vec{x}\psi_j(\vec{y},\vec{x})$$

being Σ^0_α if each ψ_j is Π^0_β for some $\beta < \alpha$ and

$$\bigwedge_{j\in\omega} \forall \vec{y} \psi_j(\vec{y}, \vec{x})$$

being Π^0_α if each ψ_j is Σ^0_β for some $\beta < \alpha$.

The set of models satisfying a Π^0_α formula will be a $\underset{\sim}{\Pi}^0_\alpha$ subset of $\mathrm{Mod}(\mathcal{L})$.

Lemma 5.1 *Let $\mathcal{L}$ be a recursive language and let $\sigma \in \mathcal{L}_{\omega_1,\omega}$. Then there is a countable language $\hat{\mathcal{L}} \supset \mathcal{L}$ and $\hat{\sigma} \in \hat{\mathcal{L}}_{\omega_1,\omega}$ such that*

(i) $\hat{\sigma} \in \Pi^0_2$;

(ii) the reduction map

$$\mathrm{Red} : \mathrm{Mod}(\hat{\mathcal{L}}) \to \mathrm{Mod}(\mathcal{L})$$

$$\mathcal{M} \mapsto \mathcal{M}|_{\mathcal{L}}$$

restricted to $\mathrm{Mod}(\hat{\sigma})$ *witnesses*

$$\cong |_{\mathrm{Mod}(\hat{\sigma})} \leq_B \cong |_{\mathrm{Mod}(\sigma)} \leq_B \cong |_{\mathrm{Mod}(\hat{\sigma})}.$$

Proof Let F be the fragment generated by σ. For each $\psi(\vec{x}) \in F$ we introduce a new relation $R_{\psi(\vec{x})}$ into our language $\hat{\mathcal{L}}$. Then for $\mathcal{N} \in \mathrm{Mod}(\mathcal{L})$ we define

$$\mathrm{Exp}(\mathcal{N}) \in \mathrm{Mod}(\hat{\mathcal{L}})$$

to be the expansion of $\mathcal{N}$ given by

$$\mathrm{Exp}(\mathcal{N}) \models R_{\psi(\vec{x})}(\vec{a})$$

if and only if

$$\mathcal{N} \models \psi(\vec{a}).$$

We also obtain that

$$\mathrm{Red} \circ \mathrm{Exp} : \mathrm{Mod}(\mathcal{L}) \to \mathrm{Mod}(\mathcal{L})$$

is the identity map.

The set

$$\{\mathrm{Exp}(\mathcal{N}) : \mathcal{N} \in \mathrm{Mod}(\mathcal{L})\}$$

is easily seen to be defined by a Π^0_2 formula in $\mathrm{Mod}(\hat{\mathcal{L}})$ and

$$\{\mathrm{Exp}(\mathcal{N}) : \mathcal{N} \models \sigma\}$$

equals

$$\{\mathrm{Exp}(\mathcal{N}) : \mathrm{Exp}(\mathcal{N}) \models \forall a R_{\psi(x)}(a)\}$$

where $\psi(x)$ is the formula

$$(x = x) \wedge \sigma.$$

□

Thus by the statement at 3C [17] that $\underset{\sim}{\Pi}^0_2$ subspaces of Polish spaces are Polish, we may identify any space $\mathrm{Mod}(\sigma)$ with a Polish subspace of some $\mathrm{Mod}(\hat{\mathcal{L}})$

Definition We let S_∞ be the group of all permutations of ω equipped with the topology of pointwise convergence; we let it act in the natural manner on any space of the form $\mathrm{Mod}(\mathcal{L})$. (See [2] or [10] §2.1.)

Definition For $\sigma \in \mathcal{L}_{\omega_1,\omega}$ we let $\cong|_{\mathrm{Mod}(\sigma)}$ denote the isomorphism relation on models of σ in $\mathrm{Mod}(\mathcal{L})$.

Here of course the orbit equivalence relation arising from the action of S_∞ on $\mathrm{Mod}(\mathcal{L})$ is just the relation $\cong|_{\mathrm{Mod}(\sigma)}$ of isomorphism.

Theorem 5.2 *(Becker-Kechris; see [2]) Let H be a closed subgroup of S_∞ and X a Polish space on which H acts continuously with orbit equivalence relation E^X_H. Then there is a countable language $\mathcal{L}$ and $\sigma \in \mathcal{L}_{\omega_1,\omega}$ such that*

$$E^X_H \leq_B \cong|_{\mathrm{Mod}(\sigma)} \leq_B E^X_H.$$

It will be helpful at some point to make the assumption that all our models have something close to a pairing function, which enables us to encode the type of any finite tuple by the type of a single point.

Lemma 5.3 *Let $\mathcal{L}$ be a recursive language and $\sigma \in \mathcal{L}_{\omega_1,\omega}$. Then there is a richer recursive language $\hat{\mathcal{L}} \supset \mathcal{L}$ and $\hat{\sigma} \in \hat{\mathcal{L}}$ such that:*

(i) for each $\mathcal{M} \in \mathrm{Mod}(\hat{\sigma})$ and $\vec{a} \in \mathcal{M}$ there is some point $b_{\vec{a}} \in \mathcal{M}$ for any fragment F we have $F\text{-type}_{\mathcal{M}}(\vec{a})$ first order definable from $F\text{-type}_{\mathcal{M}}(b_{\vec{a}})$;

(ii) $\cong|_{\mathrm{Mod}(\hat{\sigma})} \leq_B \cong|_{\mathrm{Mod}(\sigma)} \leq_B \cong|_{\mathrm{Mod}(\hat{\sigma})}$.

Proof There are many ways to see this.

We can for instance pass from $\mathcal{M}$ to $\mathcal{M}^{eq}$ in the sense §4.3 [15]. The relevant processes of going back and forth are clearly Borel.

□

It will also be convenient at some stage to assume that each point in our model is repeatedly infinitely often.

Lemma 5.4 *Let $\mathcal{L}$ be a recursive language and $\sigma \in \mathcal{L}_{\omega_1,\omega}$. Then there is a richer recursive language $\hat{\mathcal{L}} \supset \mathcal{L}$ and $\hat{\sigma} \in \hat{\mathcal{L}}$ such that:*

(i) for each $\mathcal{M} \in \mathrm{Mod}(\hat{\sigma})$ and $\vec{a} \in \mathcal{M}$ and $b \in \mathcal{M}$ there are infinitely many $b' \in \mathcal{M}$ such that

$$\langle \mathcal{M}; \vec{a}^\frown b\rangle \cong \langle \mathcal{M}; \vec{a}^\frown b'\rangle;$$

(ii) $\cong|_{\mathrm{Mod}(\hat{\sigma})} \leq_B \cong|_{\mathrm{Mod}(\sigma)} \leq_B \cong|_{\mathrm{Mod}(\hat{\sigma})}$;
(iii) moreover if each $\mathcal{M} \in \mathrm{Mod}(\sigma)$ *satisfies 5.3(ii) then so too each* $\hat{\mathcal{M}} \in \mathrm{Mod}(\hat{\sigma})$ *satisfies 5.3(ii).*

Proof We introduce an equivalence relation E to the language $\mathcal{L}$ and let $\hat{\sigma}$ express that each E-equivalence class is infinite and that E is a congruence with respect to the language $\mathcal{L}$ and that

$$\mathcal{M}/E,$$

the result of quotienting out of the model $\mathcal{M}$ by the equivalence relation E, is a model of σ in the natural interpretation of $\mathcal{L}$.

$\square$

6 Aside: Something about Gandy-Harrington

We momentarily detrain for a kind of extended philosophical aside on Gandy-Harrington forcing viewed as a device to obtain "generic" members of a non-empty Σ^1_1 set; the main arguments of section 11 and beyond are always working with the combinatorics of Σ^1_1 sets and this naturally requires us to consider "typical" elements which are devoid of accidental properties. We present a sequence of little lemmas trying to suggest that the useful content of considering a *generic member* x of a Σ^1_1 set A is exhausted by the specification $\omega_1^{\mathrm{ck}(x)} = \omega_1^{\mathrm{ck}}$.

Most of this section is never quoted again; only 6.10 and 6.11 reappear. These are in any case well known and have direct proofs.

We state everything for the point class of lightfaced Σ^1_1. It should be clear that they all hold in a relativized form for the relativized class $\Sigma^1_1(z)$ whenever z is an element of a recursive Polish space.

Definition For X a recursive Polish space we let $\mathcal{P}_X$ denote the non-empty Σ^1_1 subsets of X ordered by inclusion. We write $A_1 \leq A_2$, read as "A_1 is stronger than A_2", if

$$A_1 \subset A_2.$$

In this way $\mathcal{P}_X$ becomes a partial order.

A subset $G \subset \mathcal{P}_X$ is a *filter* if

(i) for any $A_1, A_2 \in G$ there is some $B \in G$ with $B \leq A_1$, $B \leq A_2$;

(ii) if $A_1 \leq A_2$ and $A_1 \in G$ then $A_2 \in G$.

A set $D \subset \mathcal{P}_X$ is *dense* if for all $A_1 \in \mathcal{P}_X$ there is some $A_2 \leq A_1$ with $A_2 \in D$. For $\mathcal{D}$ a collection of dense sets, we say that a filter G is $\mathcal{D}$-generic if for all $D \in \mathcal{D}$ we have

$$G \cap D \neq \emptyset.$$

For the accuracy and clarity of later computations it is helpful to isolate the precise degree of genericity required of our filters.

Definition A filter $G \subset \mathcal{P}_X$ is *sufficiently generic* if whenever

$$C \subset X \times \omega^\omega$$

is a recursive closed set either

(i) there is $A \in G$ such that for all $x \in A$ and $f \in \omega^\omega$

$$(x, f) \notin C;$$

or

(ii) there is $A \in G$ and $n \in \omega$ such that for all $x \in A$ there is some $f \in \omega^\omega$ with

$$f(0) = n$$

and

$$(x, f) \in C.$$

Here are two easy facts about this definition, the second of which helps justify our choice of terminology. Neither of them should require a proof.

Fact 6.1 *G is sufficiently generic if and only if for $k \in \omega$ and*

$$C \subset X \times \omega^\omega$$

either:

(i) there is $A \in G$ such that for all $x \in A$ and $f \in \omega^\omega$

$$(x, f) \notin C;$$

or

(ii) there is $A \in G$ and $s \in \omega^k$ such that for all $x \in A$ there is some $f \in \omega^\omega$ with

$$f|_k = s$$

and

$$(x, f) \in C.$$

Fact 6.2 *There is a countable collection of dense sets, $\mathcal{D} = \{D_i : i \in \omega\}$, such that a filter G is sufficiently generic if and only if it is $\mathcal{D}$-generic.*

It turns out that a sufficiently generic filter gives rise to and can be encoded by a corresponding point in X. The next two lemmas are in some form standard and well known, and one can certainly find in [21], or in [8] working with a topological formulation, that there is a countable $\mathcal{D}$ which will enforce the conclusions of 6.3 and 6.4 for any $\mathcal{D}$-generic G.

We give fragmentary proofs only to dispel doubts whether the notion of "sufficiently generic" really is sufficient.

Lemma 6.3 *For $G \subset \mathcal{P}_X$ sufficiently generic there is a unique*

$$x(G) \in X$$

such that for all $\mathcal{O}$ basic open we have

$$x(G) \in \mathcal{O}$$

if and only

$$\mathcal{O} \in G.$$

Proof First to see that there is at most one such $x(G)$, note that for each rational number $q > 0$ we may let $(U_n)_{n\in\omega}$ be an enumeration of open sets such that

(i) $X = \bigcup_{n\in\omega} U_n$;

(ii) $d_X(U_n) < q$, where d_X is the complete metric for X and $d_X(U_n)$ is the diameter of U_n with respect to that metric;

(iii) $C = \{(x, f) : x \in \overline{U_{f(0)}}\}$ is a recursive closed subset of $X \times \omega^\omega$.

If $G \subset \mathcal{P}_X$ is sufficiently generic then there will be some k and $A \in G$ such that for all $x \in A$ one has that there is some $f \in \omega^\omega$ with $f(0) = k$ and $(x, f) \in C$. This amounts to saying that for all $x \in A$ we have

$$x \in \overline{U_k}.$$

Thus

$$A \subset \overline{U_k}$$

and thus

$$\overline{U_k} \in G$$

since G is a filter. But then for all

$$x_1, x_2 \in \bigcap \{\mathcal{O} : \mathcal{O} \text{ basic open} \in G\}$$

we have

$$d_X(x_1, x_2) < q.$$

Tending $q \to 0$ we have $x_1 = x_2$.

To see the converse we retrieve the argument from the first half of the proof to find an infinite sequence of basic open sets

$$(W_n)_{n \in \omega}$$

such that at each n

$$\overline{W_{n+1}} \subset \overline{W_n}$$

$$d(W_n) < \frac{1}{n}$$

$$\overline{W_n} \in G$$

and for any basic open set $\mathcal{O} \in G$ we have some n with

$$\overline{W_n} \subset \mathcal{O}.$$

Then we may find a point

$$x(G) \in \bigcap_{n \in \omega} \overline{W_n},$$

which must be as required.

$\square$

Notation In the case that

$$X = X_0 \times X_1 \times ... \times X_n$$

is the product of $n + 1$ many recursive Polish spaces, we define for each $i \leq n$ the projection map

$$p_i : \mathcal{P}_X \twoheadrightarrow \mathcal{P}_{X_i}$$

given by

$$A \mapsto \{x_i \in X_i : \exists x_0 \exists x_1 ... \exists x_{i-1} \exists x_{i+1} ...((x_0, x_1, ..., x_{i-1}, x_i, x_{i+1}, ...) \in A)\}.$$

This is parallel to the usual definition of the projection map

$$\pi_i : X \twoheadrightarrow X_i$$

$$(x_0, x_1, ..., x_i, x_{i+1}, ...) \mapsto x_i.$$

The three lemmas are essentially due to Gandy, just rephrased in the context of our technical notion of sufficient genericity.

Lemma 6.4 *If $G \subset \mathcal{P}_X$ is sufficiently generic then for all $A \subset X$ in Σ^1_1 we have*

$$A \in G$$

if and only if

$$x(G) \in A.$$

Proof For the forward direction with $A \in G$ we recall that any such Σ^1_1 set has the form

$$A = \{x : \exists f((x, f) \in C)\}$$

for some $C \subset X \times \omega^\omega$ recursively closed.

Thus we can apply 6.1 as in [21] to successively obtain

$$s_0 \subset s_1 \subset ... \subset s_i \subset s_{i+1} ...,$$

each $s_n \in \omega^n$, and

$$A_0 \supset A_1 \supset .... \supset A_i \supset A_{i+1} ...$$

with each $A_i \in G$ and

$$\forall x \in A_i \exists f \supset s_i((x, f) \in A_i).$$

Then for

$$f = \bigcup_{i \in \omega} s_i$$

we have that for all basic open $\mathcal{O} \in G$ and $i \in \omega$

$$A_i \cap \mathcal{O} \neq \emptyset$$

and hence there is some $y \in \mathcal{O}$ with

$$(y, f) \in C.$$

Since the collection of basic open sets appearing in G forms a neighbourhood basis for $x(G)$ we indeed have

$$(x(G), f) \in C.$$

Conversely if $A \notin G$ then there will be some $B \in G$ with

$$B \cap A = \emptyset.$$

Then since $x(G) \in B$ by the only if direction above, we have

$$x(G) \notin A,$$

as demanded.

$\square$

Lemma 6.5 *If $G \subset \mathcal{P}_X$ is sufficiently generic, then for all $A \in \Sigma^1_1$ with $A \subset X$ we have either:*

(i) $x(G) \in A$; or

(ii) there is $B \in G$ with $A \cap B = 0$.

Proof If we let $C \subset X \times \omega^\omega$ be a recursive closed set such that

$$A = \{x : \exists f((x, f) \in C)\}$$

then the lemma follows from 6.4 and the definition of G being sufficiently generic.

$\square$

Theorem 6.6 *(Gandy)*

If $G \subset \mathcal{P}_X$ is sufficiently generic, then

$$\omega_1^{\mathrm{ck}(x(G))} = \omega_1^{\mathrm{ck}}.$$

Proof This is *exactly* as in the usual proof, since if

$$\{e\}^{x(G)}$$

codes a recursive well ordering then by 6.5 there will be some $B \in G$ such that for all $y \in B$ one has that

$$\{e\}^y$$

codes a well ordering. Then by the boundedness theorem for Σ^1_1 there must be some $\gamma < \omega_1^{\mathrm{ck}}$ such that for all $y \in B$ we have $\{e\}^y$ as a well ordering of rank less than γ. Since $x(G) \in B$ by 6.4 we are done.

$\square$

Corollary 6.7 *(Gandy) Any non-empty $A \in \Sigma^1_1$ contains some x with*

$$\omega_1^{\mathrm{ck}(x)} = \omega_1^{\mathrm{ck}}.$$

Proof Since we can clearly diagonalize in a filter below A to meet the countably many dense sets of 6.2.

□

The combination of 6.4 and 6.6 allows a converse, which we present below along with two other characterizations of being sufficiently generic which have a much more set theoretical flavor.

Lemma 6.8 *The following are equivalent:*

(0) G is sufficiently generic;

(i) there is some $x(G)$ in the intersection of the basic open sets appearing in G and for this $x(G)$ we have that

$$G = \{A \in \Sigma^1_1 : x(G) \in A\}$$

and for all $A \in \Sigma^1_1$ either

(a) $x(G) \in A$, or

(b) there is some Σ^1_1 set B with $A \cap B = 0$ and $x(G) \in B$;

(ii) there is some $x(G)$ in the intersection of the basic open sets appearing in G and for this $x(G)$ we have

$$G = \{A \in \Sigma^1_1 : x(G) \in A\}$$

and

$$\omega_1^{\mathrm{ck}(x(G))} = \omega_1^{\mathrm{ck}};$$

(iii) there is some $x(G)$ with

$$G = \{A \in \Sigma^1_1 : x(G) \in A\}$$

and for this $x(G)$ there is an ω-model M of ZFC minus the power set axiom and an M-generic filter

$$H \subset (\mathcal{P}_X)^M$$

such that

$x(H) = x(G)$;

(iv) there is some $x(G)$ with

$$G = \{A \in \Sigma^1_1 : x(G) \in A\}$$

and for this $x(G)$ and any $B \in G$ there is an ω-model M of ZFC minus the power set axiom and an M-generic filter

$$H \subset (\mathcal{P}_X)^M$$

such that

(a) $x(H) = x(G)$;

(b) $M[H] \models x(G) \in B$.

Proof The equivalence of (0) and (i) is given by 6.4 and 6.5 and the definition of being sufficiently generic.

(ii)⇒(0): Let us fix $C \subset X \times \omega^\omega$ a recursive closed set as in the definition of being sufficiently generic.

If there exists f_0 with $(x(G), f_0) \in C$ then for $k = f_0(0)$ we have that $x(G)$ is an element of the Σ^1_1 set

$$A = \{x \in X : \exists f((x, f) \in C \wedge f(0) = k)\}$$

and we are done.

Instead then suppose that $x(G)$ is an element of the Π^1_1 set

$$P = \{x : \forall f((x, f) \notin C)\}.$$

Fix a recursive continuous function

$$\rho : X \rightarrow \mathrm{LO}$$

such that

$$\rho^{-1}[\mathrm{WO}] = P.$$

Since $\omega_1^{\mathrm{ck}(x(G))} = \omega_1^{\mathrm{ck}}$ we have some $\gamma < \omega_1^{\mathrm{ck}}$ with

$$||\rho(x(G))|| = \gamma.$$

Then the Δ^1_1 set

$$B = \{x \in X : ||\rho(x)|| = \gamma\}$$

will be in G by our assumptions and is exactly as required for the definition of sufficient genericity.

(i)⇒(iv): Given $x(G)$ arising from a sufficiently generic G, and given a specific $B \in G$, we have that the conclusion of (iv) refers to x being in a certain Σ^1_1 set. Thus it will fail the conclusion of (iv) if and only if there is some Σ^1_1 set $A_0 \in G$ necessitating that failure. But this is plainly impossible, since given $A_0, B \in G$ we may choose a countable elementary submodel of some rank initial segment V_δ of V and force below $A_0 \cap B$ meeting all the dense sets existing in our countable submodel.

(iv)⇒(iii): Obviously.

(iii)⇒(ii): Fix M our ω-model and $x = x(G) \in M$. There are two possibilities. The first being that the ordinal (or an isomorphic copy of the ordinal) ω_1^{ck} is in $M[H]$, whence $M[H]$ correctly calculates the structure $L_{\omega_1^{\mathrm{ck}}}[x]$, and thus by internalizing the proof of 6.6 to $M[H]$ we have that $L_{\omega_1^{\mathrm{ck}}}[x]$ is indeed admissible. Alternatively the ordinal does not appear in $M[H]$, which since $M[H]$ satisfies Kripke-Platek set theory gives that there is no well ordering of ω in M of length ω_1^{ck}, and in particular that there is no x-recursive well ordering of ω of length ω_1^{ck}.

□

Corollary 6.9 *The set of $x \in X$ for which there is some sufficiently generic G with $x(G) = x$ is Σ^1_1.*

Proof It is well known that

$$\{x \in X : \omega_1^{\mathrm{ck}(x)} = \omega_1^{\mathrm{ck}}\}$$

is Σ^1_1, and indeed it follows at once from the statement of (iii) from 6.8.

$\square$

Corollary 6.10 *For any non-empty $A \in \Sigma^1_1$ the set*

$$\{x \in A : \omega_1^{\mathrm{ck}(x)} = \omega_1^{\mathrm{ck}}\}$$

is a non-empty Σ^1_1 set.

Corollary 6.11 *If $\omega_1^{\mathrm{ck}(x)} = \omega_1^{\mathrm{ck}}$ then for all $A \in \Sigma^1_1$ we have either*

(i) $x \in A$; or

(ii) there is $B \in \Sigma^1_1$ with

$$x \in B$$

and

$$A \cap B = 0.$$

Corollary 6.12 *If $X = X_0 \times X_1$ is a recursive Polish space and*

$$G_0 \subset \mathcal{P}_{X_0}$$

is a sufficiently generic filter, with $A \in \Sigma^1_1$ such that

$$\{x_0 : \exists x_1 \in X_1((x_0, x_1) \in A)\} \in G,$$

then there is a sufficiently generic filter $G \subset \mathcal{P}_X$ with

$$A \in G$$

$$p_0[G] = G_0.$$

Proof Since

$$\omega_1^{\mathrm{ck}(x(G_0))} = \omega_1^{\mathrm{ck}}$$

we can apply the relativized version of 6.7 to find some x_1 with

$$(x(G_0), x_1) \in A$$

and

$$\omega_1^{\mathrm{ck}(x(G_0),x_1)} = \omega_1^{\mathrm{ck}(x(G_0))} = \omega_1^{\mathrm{ck}}.$$

Then by 6.8 we have that

$$G = \{B \in \mathcal{P}_X : (x(G_0), x_1) \in B\}$$

is as required.

$\square$

7 The central definition

We do everything below for Σ^1_1 sets, a recursive language $\mathcal{L}$, and formulas and fragments in $L_{\omega_1^{\mathrm{ck}}}$. It should be clear that the entire sequence of statements relativizes to $\Sigma^1_1(z)$ and recursive in z.

Notation For $\mathcal{L}$ a recursive language and $A \subset \mathrm{Mod}(\mathcal{L})$ a Σ^1_1 set, $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$, and

$$\vec{a} = \langle a_0, a_1, ..., a_n \rangle \in \mathcal{M},$$

let us write

$$\mathcal{M} \models A(\vec{a})$$

if there is some $\pi \in S_\infty$ with

$$\pi(a_l) = l$$

all $l \leq n$ and

$$\pi \cdot \mathcal{M} \in A.$$

Notation For each recursive language $\mathcal{L}$ fix a Σ^1_1 set

$$\mathcal{U}^{\mathcal{L}} \subset \omega \times \mathrm{Mod}(\mathcal{L})$$

that is universal in the sense that any Σ^1_1 set $A \subset \mathrm{Mod}(\mathcal{L})$ appears as

$$A = \mathcal{U}^{\mathcal{L}}_n = \{\mathcal{M} : (n, \mathcal{M}) \in \mathcal{U}^{\mathcal{L}}\}$$

for some suitable choice of n.

Fact 7.1 *For $\mathcal{L}$ a recursive language we may find a function*

$$\mathcal{R} : \omega \times \omega \times \omega_1 \to \mathcal{L}_{\omega_1,\omega}$$

$$(n, l, \alpha) \mapsto \psi^{\mathcal{L}}_{n,\alpha}$$

such that

(i) $\mathcal{R}|_{\omega\times\omega\times\delta}$ is uniformly $\Delta_1^{L_\delta}$ for any countable admissible ordinal δ;

(ii) for $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ and $\vec{a} \in \mathcal{M}$ with $lh(\vec{a}) = l$ we have

$$\mathcal{M} \models \mathcal{U}^{\mathcal{L}}_n(\vec{a})$$

if and only if for all $\alpha < \omega_1^{\mathrm{ck}(\mathcal{M})}$

$$\mathcal{M} \models \neg\psi^{\mathcal{L}}_{n,l,\alpha}(\vec{a}),$$

which in turn happens if and only if for all $\beta < \aleph_1$

$$\mathcal{M} \models \neg\psi^{\mathcal{L}}_{n,l,\beta}(\vec{a}).$$

Proof Fix $\mathcal{L}$, n, and l. Let A be the Σ^1_1 set

$$\{\mathcal{N} : \exists \pi \in S_\infty : \forall k < l(\pi(k) = k) \wedge \pi \cdot \mathcal{N} \in \mathcal{U}^{\mathcal{L}}_n\}$$

obtained by suitably saturating $\mathcal{U}^{\mathcal{L}}_n$. Fix a recursive continuous

$$\rho : \mathrm{Mod}(\mathcal{L}) \to \mathrm{LO}$$

with

$$\rho^{-1}[\mathrm{WO}] = \mathrm{Mod}(\mathcal{L}) \setminus A.$$

Following [23] or 16.C [17] we obtain $\psi^{\mathcal{L}}_{n,l,\alpha} \in \mathcal{L}_{\omega_1,\omega}$ for each α with

$$\mathcal{M} \models \psi^{\mathcal{L}}_{n,l,\alpha}(\vec{a})$$

if and only if for a relatively comeager set of $\pi \in S_\infty$ in the set of permutations sending each a_j to j we have

$$||\rho(n, \pi \cdot \mathcal{M})|| \leq \alpha.$$

The various calculations of these $\psi^{\mathcal{L}}_{n,l,\alpha}$ proceed in the effective category, and we have that

$$(n, l, \alpha) \mapsto \psi^{\mathcal{L}}_{n,l,\alpha}$$

is as required.

□

The notation of 7.1 is convoluted when given in explicit detail, so it seems wise to tolerate an abbreviation.

Notation For B a Σ^1_1 set in $\mathrm{Mod}(\mathcal{L})$ with $B = \mathcal{U}^{\mathcal{L}}_n$ let us write

$$\psi_{B,\alpha}$$

for

$$\psi^{\mathcal{L}}_{n,l,\alpha},$$

where l is given by context and the formula $\psi^{\mathcal{L}}_{n,l,\alpha}$ is given by 7.1.

Strictly speaking the formula $\psi^{\mathcal{L}}_{n,l,\alpha}$ depends not just on B, l, and α, but also the choice of the index n with $B = \mathcal{U}^{\mathcal{L}}_n$. Although the notation is therefore not completely correct, this will never lead to problems and we will always understand the expression

$$\mathcal{M} \models \psi_{B,\alpha}(\vec{a})$$

to indicate that

$$\mathcal{M} \models \psi^{\mathcal{L}}_{n,l,\alpha}(\vec{a})$$

where $\mathcal{L}$ is the language of $\mathcal{M}$, n is some fixed choice of an index for B, and l is the length of the sequence $\vec{a}$.

Definition For $\mathcal{L} \in L_{\omega_1^{\rm ck}}$ and $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(\mathcal{L})$ let us say that an embedding

$$i : \mathcal{M} \to \mathcal{N}$$

is $\omega_1^{\mathrm{ck}(\mathcal{M},\mathcal{N})}$-*elementary* if for all $\psi \in L_{\omega_1^{\mathrm{ck}(\mathcal{M},\mathcal{N})}} \cap \mathcal{L}_{\omega_1,\omega}$ and $\vec{a} \in \mathcal{M}$ we have

$$\mathcal{M} \models \psi(\vec{a})$$

if and only if

$$\mathcal{N} \models \psi(i(\vec{a})).$$

Note here that the we only consider constructible ψ. The collection of formulas depends only on the size of the least $(\mathcal{M}, \mathcal{N})$-admissible ordinal, and *not* on the exact composition of say the model $L_{\omega_1^{\mathrm{ck}(\mathcal{M},\mathcal{N})}}[\mathcal{M}, \mathcal{N}]$.

However even with this restriction to this seemingly modest set of formulas we are already able to prove:

Fact 7.2 *Let*

$$i : \mathcal{M} \to \mathcal{N}$$

be $\omega_1^{\mathrm{ck}(\mathcal{M},\mathcal{N})}$-*elementary. Then for all* $\vec{a} \in \mathcal{M}$ *and* $A \in \Sigma^1_1$ *we have*

$$\mathcal{M} \models A(\vec{a})$$

if and only if

$$\mathcal{N} \models A(i(\vec{a})).$$

Proof Tracing through the implications given by the definitions, and 7.1 we have

$$\mathcal{M} \models \neg A(\vec{a})$$

if and only if

$$\exists \alpha < \omega_1^{\mathrm{ck}(\mathcal{M},\mathcal{N})} (\mathcal{M} \models \neg \psi_{A,\alpha}(\vec{a}))$$

if and only if (by elementarity assumption on i)

$$\exists \alpha < \omega_1^{\mathrm{ck}(\mathcal{M},\mathcal{N})} (\mathcal{N} \models \neg \psi_{A,\alpha}(i(\vec{a})))$$

if and only if

$$\mathcal{N} \models \neg A(i(\vec{a})).$$

$\square$

Up until now this section has consisted in technicalities; the next definition is central to what will follow.

Definition Let $F \in L_{\omega_1^{\mathrm{ck}}}$ be a fragment for some recursive language $\mathcal{L}$. A Σ^1_1 set A is *F-barren* for l if for all $\mathcal{M}_0, \mathcal{M}_1 \in \mathrm{Mod}(\mathcal{L})$ and

$$i : \mathcal{M}_0 \to \mathcal{M}_1$$

$\omega_1^{\mathrm{ck}(\mathcal{M}_0,\mathcal{M}_1)}$-elementary and $\vec{b} \in \mathcal{M}_1$ with $lh(\vec{b}) = l$ we have if

$$\mathcal{M}_1 \models A(\vec{b})$$

then

$$F - \mathrm{type}_{\mathcal{M}_1}(\vec{b}) \in \{F - \mathrm{type}_{\mathcal{M}_0}(\vec{a}) : \vec{a} \in \mathcal{M}_0\}.$$

8 Consequences of the definition

We fix $\mathcal{L}$ a recursive language and prove various facts about the notion of being barren. They relativize smoothly to languages which are recursive in z.

First of all: The property of being F-barren is Π^1_1 in the indices for Σ^1_1 sets.

Lemma 8.1 *Let $\mathcal{U}^{\mathcal{L}}$ be a Σ^1_1 subset of $\omega \times \mathrm{Mod}(\mathcal{L})$ which is universal in the sense of 7.1. Let $F \in L_{\omega_1^{\mathrm{ck}}}$ be a fragment for $\mathcal{L}$.*

Then

$$\{n \in \omega : \mathcal{U}^{\mathcal{L}}_n \text{ is } F\text{−barren for } l\}$$

is a Π^1_1 subset of ω.

Proof The set of $(\mathcal{M}, \mathcal{N}, i)$ such that

$$i : \mathcal{M} \to \mathcal{N}$$

is $\omega_1^{\mathrm{ck}(\mathcal{M},\mathcal{N})}$-elementary is Σ^1_1 by Spector-Gandy. The set of all $(\mathcal{M}, \mathcal{N}, i, \vec{a})$ such that F-type$_{\mathcal{N}}(\vec{a})$ is not realized in $\mathcal{M}$ is Δ^1_1.

That said, the lemma follows by the usual kinds of manipulation and counting of quantifiers.

□

Clearly the proof of 8.1 is uniform in F and $\mathcal{L}$. If we let P be the set of

$$(n, m, k) \in \omega \times \omega \times \omega$$

such that k gives the index for a recursive language, call it $\mathcal{L}_k$, $\{m\}$ is a recursive well order of ω, and α_m its rank, the α_mth object in $L_{\omega_1^{\mathrm{ck}}}$ is a fragment F_m for $\mathcal{L}_k$, and $\mathcal{U}^{\mathcal{L}}_n$ is F_m-barren, then we obtain that P is Π^1_1 set.

Corollary 8.2 *Let $F \in L_{\omega_1^{\mathrm{ck}}}$ be a fragment for $\mathcal{L}$. Let $A \in \Sigma^1_1$ be F-barren for l. Then there is a Δ^1_1 set $D \supset A$ which is again F-barren.*

Proof There are a couple of ways to see this, the most brutal of which is to simply read it off from the statement of Π^1_1 reflection from [9] after quoting 8.1.

Alternatively we may apply the boundedness theorem in the following way. Consider the Σ^1_1 set B of $(\mathcal{M}, \mathcal{N}, i, \vec{a})$ with

$$i : \mathcal{M} \to \mathcal{N}$$

$\mathcal{L}_{\omega,\omega}$-elementary and the F-type$_{\mathcal{N}}(\vec{a})$ not realized in $\mathcal{M}$. We then let $(\psi_{A,\alpha})_{\alpha \in \aleph_1}$ be as in 7.1 and the notational remark it precedes. We have then that for all $(\mathcal{M}, \mathcal{N}, i, \vec{a}) \in B$ there is some $\alpha < \omega_1^{\mathrm{ck}(\mathcal{M},\mathcal{N})}$ such that

$$\exists \beta < \alpha (\mathcal{N} \models \psi_{A,\beta}(\vec{a})).$$

But then by 3.4 we may find a single $\gamma_0 < \omega_1^{\mathrm{ck}}$ such that for all $(\mathcal{M}, \mathcal{N}, i, \vec{a}) \in B$

$$\exists \beta < \gamma_0 (\mathcal{N} \models \psi_{A,\beta}(\vec{a})).$$

Thus we may take D to be

$$\{\mathcal{N} \in \mathrm{Mod}(\mathcal{L}) : \mathcal{N} \models \bigwedge_{\beta < \gamma_0} \neg \psi_{A,\beta}(0, 1, ..., l-1)\}.$$

□

Definition A formula $\psi(x_0, x_1, ..., x_{l-1})$ is *F-barren* for l if the corresponding set

$$D_{\psi(\vec{x})} = \{\mathcal{M} : \mathcal{M} \models \psi(0, 1, ..., l-1)\}$$

is F-barren in the above sense.

Thus in the proof of 8.2 we actually obtained:

Corollary 8.3 *If A is an F-barren for l Σ^1_1 set then there is an F-barren for l formula $\psi(x_0, x_1, ..., x_{l-1}) \in L_{\omega_1^{\mathrm{ck}}}$ with*

$$D_{\psi(\vec{x})} \supset A.$$

Corollary 8.4 *For $F \in L_{\omega_1^{\mathrm{ck}}}$ a fragment and $l \in \omega$ we have a set $A^l_F \in \Sigma^1_1$ such that*

$$\mathcal{M} \models A^l_F(\vec{a})$$

if and only if for no $A \in \Sigma^1_1$ that is F-barren for l do we have

$$\mathcal{M} \models A(\vec{a}).$$

Proof At once from 8.3 and Spector-Gandy.

□

Definition For each $l \in \omega$ and $F \in L_{\omega_1^{\mathrm{ck}}}$ let A^l_F be the set of m such that there is no F-barren for l set $A \in \Sigma^1_1$ with

$$\mathcal{M} \models A(0, 1, ..., l-1).$$

It is not so elegant to continually be writing a superscript l, so we just use A_F where context indicates l. Similarly we use the abbreviated term *F-barren* when there seems no danger of confusion.

Lemma 8.5 *Let ψ be F-barren. Then there is a fragment $H \in L_{\omega_1^{\mathrm{ck}}}$ such that whenever*

$$i : \mathcal{M} \to \mathcal{N}$$

is H-elementary and $\vec{b} \in \mathcal{N}$,

$$\mathcal{N} \models \psi(\vec{b})$$

implies that $F\text{-type}_{\mathcal{N}}(\vec{b})$ appears in $\{F\text{-type}_{\mathcal{M}}(\vec{a}) : \vec{a} \in \mathcal{M}\}$.

Proof The set of $(\mathcal{M}, \mathcal{N}, i, \vec{b})$ for which

$$\mathcal{M} \models \psi(\vec{b})$$

and i is elementary with respect to first order formulas but $F\text{-type}_{\mathcal{N}}(\vec{b})$ is not realized over $\mathcal{M}$ is Σ^1_1; the assumption on ψ then guarantees some $\alpha < \omega_1^{\mathrm{ck}(\mathcal{M},\mathcal{N})}$ and some $\varphi \in L_\alpha$ and some $\vec{a} \in \mathcal{M}$ with

$$(\mathcal{M} \models \varphi(\vec{a})) \Leftrightarrow (\mathcal{N} \models \neg\varphi(i(\vec{a}))).$$

Thus by the boundedness theorem for Σ^1_1 we have some single $\gamma_0 < \omega_1^{\mathrm{ck}}$ such that whenever

$$i : \mathcal{M} \to \mathcal{N}$$

$$F\text{−type}_{\mathcal{N}}(\vec{b}) \notin \{F\text{−type}_{\mathcal{M}}(\vec{a}) : \vec{a} \in \mathcal{M}\}$$

then there exists $\alpha < \gamma_0$ and $\varphi \in L_\alpha$ with

$$(\mathcal{M} \models \varphi(\vec{a})) \Leftrightarrow (\mathcal{N} \models \neg\varphi(\vec{a})).$$

We finish with $H = \mathcal{L}_{\omega_1,\omega} \cap L_{\gamma_0}$.

□

For future reference it may be helpful to note that the set of F-barren $\psi(\vec{x})$ is uniformly in F Π^1_1 in the codes as is the set of H as in 8.5. Thus by Spector-Gandy we have that the set of

$$(F, \psi) \in L_{\omega_1^{\mathrm{ck}}}$$

such that F is a fragment for $\mathcal{L}$ and ψ is F-barren is a $\Sigma_1^{L_{\omega_1^{\mathrm{ck}}}}$ set as is the set of triples

$$(F, H, \psi)$$

fulfilling the conclusion of 8.5.

In the course of section 11 we will need the following two lemmas.

Lemma 8.6 *Suppose $\mathcal{M} \models A_F(a)$ and $\omega_1^{\mathrm{ck}(\mathcal{M})} = \omega_1^{\mathrm{ck}}$. Then we may find $\mathcal{N} \in \mathrm{Mod}(\mathcal{L})$ and $\omega_1^{\mathrm{ck}(\mathcal{N},\mathcal{M})}$-elementary*

$$j : \mathcal{N} \to \mathcal{M}$$

with $F\text{-type}_{\mathcal{M}}(a)$ not realized in $\mathcal{N}$.

Proof By 6.11 we obtain that the failure of the lemma would give some Σ^1_1 set B with $\mathcal{M} \models B(a)$ and for all $\hat{\mathcal{M}}, \hat{a}$ with

$$\hat{\mathcal{M}} \models B(\hat{a})$$

we have that there is no $\hat{\mathcal{N}} \in \mathrm{Mod}(\mathcal{L})$ and $\omega_1^{\mathrm{ck}(\hat{\mathcal{N}},\hat{\mathcal{M}})}$-elementary

$$\hat{j} : \hat{\mathcal{N}} \to \hat{\mathcal{M}}$$

with F-type$_{\hat{\mathcal{M}}}(\hat{a})$ not realized in $\hat{\mathcal{N}}$. However this would be a direct contradiction to our assumption that $\mathcal{M} \models A_F(a)$.

$\square$

Lemma 8.7 *Suppose $\mathcal{M} \models A_F(a)$ and $\omega_1^{\mathrm{ck}(\mathcal{M})} = \omega_1^{\mathrm{ck}}$ and $B \in \Sigma^1_1$ with*

$$\mathcal{M} \models B(a).$$

Then we may find $\mathcal{N} \in \mathrm{Mod}(\mathcal{L})$ and $\omega_1^{\mathrm{ck}(\mathcal{M},\mathcal{N})}$-elementary

$$j : \mathcal{M} \to \mathcal{N}$$

and $b \in \mathcal{N}$ with

$$\mathcal{N} \models B(b)$$

and F-type$_{\mathcal{N}}(b)$ not realized in $\mathcal{M}$.

Proof Otherwise we may again apply 6.11 to find some single Σ^1_1 set C such that

$$\mathcal{M} \models C(a)$$

and for all $\hat{\mathcal{M}}, \hat{a}$ with

$$\hat{\mathcal{M}} \models C(\hat{a})$$

we have that there is no $\hat{\mathcal{N}} \in \mathrm{Mod}(\mathcal{L})$ and $\omega_1^{\mathrm{ck}(\hat{\mathcal{M}},\hat{\mathcal{N}})}$-elementary

$$\hat{j} : \hat{\mathcal{M}} \to \hat{\mathcal{N}}$$

with some $\hat{b} \in \hat{\mathcal{N}}$ having

$$\hat{\mathcal{N}} \models B(\hat{b})$$

but F-type$_{\hat{\mathcal{N}}}(\hat{b})$ not realized in $\hat{\mathcal{M}}$.

Claim: There are no $\hat{\mathcal{M}}, \hat{\mathcal{N}} \in \mathrm{Mod}(\mathcal{L})$ and $\omega_1^{\mathrm{ck}(\hat{\mathcal{M}},\hat{\mathcal{N}})}$-elementary

$$\hat{j} : \hat{\mathcal{M}} \to \hat{\mathcal{N}}$$

with some $\hat{b} \in \hat{\mathcal{N}}$ having

$$\hat{\mathcal{N}} \models B(\hat{b}) \wedge C(\hat{b})$$

but F-type$_{\hat{\mathcal{N}}}(\hat{b})$ not realized in $\hat{\mathcal{M}}$.

Proof of Claim: Otherwise fix $\hat{\mathcal{M}}, \hat{\mathcal{N}}$, $\hat{j}$, $\hat{b}$ as indicated. Then the elementarity of $\hat{j}$ along with 7.2 gives that there will exist some $\hat{a} \in \hat{\mathcal{M}}$ with

$$\hat{\mathcal{M}} \models B(\hat{a}) \wedge C(\hat{a}).$$

Then we have a contradiction to assumption on C. (Claim$\square$)

But then the claim implies that the set $B \cap C$ provides a direct contradiction to $\mathcal{M} \models A_F(a)$.

$\square$

9 Aside: Which sentences have uncountable models?

The definition just considered leads naturally to a characterization of when a sentence $\varphi \in \mathcal{L}_{\omega_1,\omega}$ admits an uncountable model. We carry a parameter $z \in 2^\omega$ through the next couple of lemmas.

Nothing in this section will be used again; it may indeed be safely ignored.

Definition Let $\mathcal{L}$ be a recursive in z language and $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(\mathcal{L})$. An embedding

$$i : \mathcal{M} \to \mathcal{N}$$

is $\Sigma^1_1(z)$-*elementary* if for all $A \in \Sigma^1_1(z)$ and $\vec{a} \in \mathcal{M}$

$$\mathcal{M} \models A(\vec{a})$$

implies

$$\mathcal{N} \models A(i(\vec{a})).$$

Lemma 9.1 *Let $(\mathcal{M}_n)_{n\in\omega}$ be a sequence in $\mathrm{Mod}(\mathcal{L})$ and suppose that at each n we have a $\Sigma^1_1(z)$ elementary map*

$$\rho_n : \mathcal{M}_n \to \mathcal{M}_{n+1}.$$

Then for the direct limit model

$$\mathrm{Dir\,Lim}_{n\in\omega}(\mathcal{M}_n) = \mathcal{M}_\infty$$

and for the various induced direct limit maps

$$\rho_{n,\infty} : \mathcal{M}_n \to \mathcal{M}_\infty$$

we have that each $\rho_{n,\infty}$ is $\Sigma^1_1(z)$-elementary.

Proof It suffices to prove that for any $\Sigma^1_1(z)$ set A we have that if

$$\mathcal{M}_0 \models A$$

then

$$\mathcal{M}_\infty \models A.$$

Fix at each n an enumeration $(a^n_i)_{i\in\omega}$ of $\mathcal{M}_n$. For each $n \leq m$ let

$$\rho_{n,m} : \mathcal{M}_n \to \mathcal{M}_m$$

be the composition obtained from our given embeddings. These are all clearly $\Sigma^1_1(z)$ elementary.

We may apply the definition of $\mathcal{M} \models A$ and our space $\mathrm{Mod}(\mathcal{L})$ to find a recursive in z array

$$(\varphi_{n,l}(v_0, v_1, ..., v_{n-1}))_{(n,l)\in\omega\times\omega}$$

of quantifier free formulas such that for all $\mathcal{N} \in \mathrm{Mod}(\mathcal{L})$ we have

$$\mathcal{N} \models A$$

if and only if there is a permutation $\pi \in S_\infty$ and $f : \omega \to \omega$ such that at each $k \in \omega$

$$\mathcal{N} \models \varphi_{k,f(k)}(\pi(0), \pi(1), ..., \pi(k-1)).$$

Claim: We may find a sequence $(m_n)_{n\in\omega}$ of natural numbers and a sequence of finite sets $F_n \subset \mathcal{M}_n$ and corresponding k_n equal to the size of F_n, with $\{b_i^n : i < k_n\}$ enumerating F_n such that the following all hold:

(i) for each n and $i < k_n$ we

$$\rho_n(b_i^n) = b_i^{n+1};$$

(ii) at each n there is a permutation π_n of ω and $f : \omega \to \omega$ such that
(a) $\pi_n(i) = b_i^n$ all $i < k_n$, and
(b) for all $l \in \omega$

$$\mathcal{M}_n \models \varphi_{l,f(l)}(\pi_n(0), \pi_n(1), ..., \pi_n(l-1));$$

(c) for all $l \leq n$ we have $f(l) = m_l$;
(iii) for all $i, l \leq n$ there is some $j(i,l) < k_n$ such that

$$b_n^{j(i,l)} = \rho_{l,n}(a_l^i).$$

Proof of Claim: We show the existence of these things by induction on n. Suppose we have done this up to $F_n = \{b_i^n : i < k_n\} \subset \mathcal{M}_n$ and we wish to continue one more step and find F_{n+1}.

Then obviously we start with $b_i^{n+1} = \rho_n(b_i^n)$ for each $i < k_n$. Since ρ_n is a $\Sigma_1^1(z)$-elementary map we obtain that there is some permutation π_{n+1} of ω and function $f : \omega \to \omega$ such that
(a) $\pi_{n+1}(i) = b_i^{n+1}$ all $i < k_n$, and
(b) for all $l \in \omega$

$$\mathcal{M}_{n+1} \models \varphi_{l,f(l)}(\pi_{n+1}(0), \pi_{n+1}(1), ..., \pi_{n+1}(l-1));$$

(c) for all $l \leq \mathcal{N}$ we have $f(l) = m_l$.

But then we can go to some large enough k_{n+1} so that

$$\{\rho_{l,n+1}(a_l^i) : i, l \leq n+1\} \subset \{\pi_{n+1}(j) : j < k_{n+1}\}.$$

Taking $b_{n+1}^j = \pi_{n+1}(j)$ and $m_{n+1} = f(n+1)$ we continue to the next round. (Claim$\square$)

Now we can define $b_i^\infty \in \mathcal{M}_\infty$ to be

$$\rho_{n,\infty}(b_i^n)$$

for any sufficiently large n; this is well defined by part (i) of the claim. The first order elementarity of the various ρ_n maps along with (ii) is sufficient to ensure that at every $k \in \omega$

$$\mathcal{M}_\infty \models \varphi_{k,m_k}(b_0^\infty, b_1^\infty, ..., b_{k-1}^\infty).$$

Clause (iii) of the claim gives that $\{b_i^\infty : i \in \omega\}$ enumerates $\mathcal{M}_\infty$.

□

Thus from 7.2 we have that $\omega_1^{\mathrm{ck}(\mathcal{M},\mathcal{N},z)}$-elementary embeddings are $\Sigma^1_1(z)$-elementary. With a bit of shoving and squeezing we could probably go through the next few steps only using this original notion of $\omega_1^{\mathrm{ck}(\mathcal{M},\mathcal{N},z)}$-elementarity, but the new definition is definitely more flexible.

Definition A $\Sigma^1_1(z)$ set A is said to be *unyielding* if whenever

$$i : \mathcal{M} \to \mathcal{N}$$

is $\Sigma^1_1(z)$-elementary and

$$\mathcal{N} \models A$$

then

$$\forall b \in \mathcal{N} \exists a \in \mathcal{M}(i(a) = b).$$

Lemma 9.2 *If A is not unyielding then there is some pair $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(\mathcal{L})$ with*

$$\omega_1^{\mathrm{ck}(\mathcal{M},\mathcal{N})} = \omega_1^{\mathrm{ck}},$$

$$\mathcal{N} \models A,$$

and

$$i : \mathcal{M} \to \mathcal{N}$$

$\Sigma^1_1(z)$*-elementary but not onto.*

Proof A simple consequence of A failing to be unyielding is that we may find $(\mathcal{A}, \mathcal{B}, j)$ with

$$\mathcal{B} \models A,$$

while the map

$$j : \mathcal{A} \to \mathcal{B}$$

is not onto and yet we still have for all $\psi \in L_{\omega_1^{\mathrm{ck}(z)}}[z] \cap \mathcal{L}_{\omega_1,\omega}$ and $\vec{c} \in \mathcal{A}$

$$\mathcal{A} \models \psi(\vec{c})$$

if and only if

$$\mathcal{B} \models \psi(j(\vec{c})).$$

Thus by 6.7 we can find $(\mathcal{M}, \mathcal{N}, i)$ with

$$\omega_1^{\mathrm{ck}(\mathcal{M},\mathcal{N})} = \omega_1^{\mathrm{ck}},$$

$$\mathcal{N} \models A,$$

$$i : \mathcal{M} \to \mathcal{N}$$

not onto, and for all $\psi \in L_{\omega_1^{\mathrm{ck}(z)}}[z] \cap \mathcal{L}_{\omega_1,\omega}$ and $\vec{a} \in \mathcal{M}$ we have

$$\mathcal{M} \models \psi(\vec{a})$$

if and only if

$$\mathcal{N} \models \psi(j(\vec{a})).$$

But then it follows by 7.1 that j is $\Sigma^1_1(z)$ elementary.

□

We then have a couple of lemmas in direct analogy with their counterparts in 8.

Lemma 9.3 *The property of being unyielding is $\Pi^1_1(z)$ in the indices for $\Sigma^1_1(z)$ sets.*

Proof Given 9.2, this has a proof like 8.1.

□

Corollary 9.4 *If $A \in \Sigma^1_1(z)$ is unyielding then there exists a $\Delta^1_1(z)$ set $D \supset A$ which is again unyielding.*

Proof This follows 9.3 as 8.2 followed 8.1.

□

Definition A sentence ψ is unyielding if the corresponding set

$$D_\psi = \{\mathcal{M} : \mathcal{M} \models \psi\}$$

is unyielding in the above sense.

Corollary 9.5 *If A is an unyielding $\Sigma^1_1(z)$ set then there is an unyielding sentence $\psi \in L_{\omega_1^{\mathrm{ck}(z)}}[z]$ with*

$$D_\psi \supset A.$$

Proof By the method of 8.3.

□

Corollary 9.6 *We have a set $A^{\mathrm{u},z} \in \Sigma^1_1(z)$ such that*

$$\mathcal{M} \models A^{\mathrm{u},z}$$

if and only if for no $A \in \Sigma^1_1(z)$ that is unyielding do we have

$$\mathcal{M} \models A.$$

Proof By 9.6 and Spector-Gandy.

□

With the definition given by 9.6 we can state the theorem for this section. This is parallel to a similar result due to [7] in the special case that σ is the Scott sentence of a countable model.

Theorem 9.7 *Let $\mathcal{L}$ be a recursive language and $\sigma \in L_{\omega_1^{\mathrm{ck}(z)}}[z]$. The following three things are equivalent:*

(1) $\sigma \in \mathcal{L}_{\omega_1,\omega}$ has an uncountable model.

(2) There are $\mathcal{M},\mathcal{N} \in \mathrm{Mod}(\sigma)$ and $\Sigma^1_1(z)$-elementary

$$i : \mathcal{M} \to \mathcal{N}$$

that is not onto.

(3) For some $\mathcal{N} \in \mathrm{Mod}(\sigma)$

$$\mathcal{N} \models A^{\mathrm{u},z}.$$

Proof First let us suppose in (1) that $\mathcal{A}$ is a model of σ of size $\aleph_1$. Let F be the fragment generated by σ Then a standard Skolem hull argument enables us to find a chain of F-elementary models

$$\mathcal{M}' \prec_F \mathcal{N}' \prec_F \mathcal{A}$$

such that

(i) $\mathcal{M}'$ is properly included in $\mathcal{N}'$;

(ii) $\mathcal{M}'$ and $\mathcal{N}'$ are countable;

(iii) for all $A \in \Sigma^1_1(z)$ and $\vec{a} \in \mathcal{N}_0$

$$\exists \alpha < \aleph_1 (\mathcal{N}' \models \psi_{A,\alpha}(\vec{a}))$$

if and only if

$$\exists \alpha < \aleph_1 (\mathcal{M}' \models \psi_{A,\alpha}(\vec{a})).$$

Thus if we choose $\mathcal{M},\mathcal{N} \in \mathrm{Mod}(\mathcal{L})$ isomorphic to $\mathcal{M}',\mathcal{N}'$ respectively then we can obtain a $\Sigma^1_1(z)$-elementary map between $\mathcal{M},\mathcal{N}$ that is not onto, and hence we have (2).

The implication from (2) to (3) should be obvious.

So this leaves us with the real work of showing that (3) implies (1).

Claim: We may build by induction on $\alpha < \aleph_1$ a sequence of pairs of models $(\mathcal{A}_\alpha, \mathcal{B}_\alpha)$ and such that:

(i) $\mathcal{A}_\alpha \in \mathrm{Mod}(\sigma)$ and $\omega_1^{\mathrm{ck}(\mathcal{A}_\alpha,z)} = \omega_1^{\mathrm{ck}(z)}$;

(ii) $\mathcal{B}_\alpha$ is isomorphic to $\mathcal{A}_\alpha$;

(iii) for F the fragment generated by σ and $\alpha < \beta < \aleph_1$ we have

$$\mathcal{B}_\alpha \prec_F \mathcal{B}_\beta$$

and

$$\mathcal{B}_\alpha \neq \mathcal{B}_\beta;$$

(iv) for $\alpha_0 < \alpha_1 < \aleph_1$ there are $\Sigma^1_1(z)$ elementary maps

$$\tau_{\alpha_0,\alpha_1} : \mathcal{A}_{\alpha_0} \to \mathcal{A}_{\alpha_1},$$

and these commute, in the sense that for $\alpha_0 < \alpha_1 < \alpha_2$ we have

$$\tau_{\alpha_0,\alpha_2} = \tau_{\alpha_1,\alpha_2} \circ \tau_{\alpha_0,\alpha_1}.$$

Proof of Claim; By induction on α.

Something does actually need to be said about the base case of this induction, so suppose that

$$\mathcal{N} \models A^{\mathrm{u},z}$$

and let us assume that $\omega_1^{\mathrm{ck}(\mathcal{N},z)} = \omega_1^{\mathrm{ck}(z)}$. But then it follows by 6.11 and the formulation of being unyielding that we may find some $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ with

$$\omega_1^{\mathrm{ck}(\mathcal{M},\mathcal{N},z)} = \omega_1^{\mathrm{ck}(z)}$$

and an $\omega_1^{\mathrm{ck}(\mathcal{M},\mathcal{N},z)}$-elementary map

$$i : \mathcal{M} \to \mathcal{N}$$

which is not onto. We then start the proof of the claim with $\mathcal{A}_0 = \mathcal{M}$ and $\mathcal{A}_1 = \mathcal{N} = \mathcal{B}_1$ and $\mathcal{B}_0 = i[\mathcal{M}]$.

For the successor steps of the induction we imagine that we have some α and $\beta < \alpha$

$$\mathcal{B}_\beta \neq \mathcal{B}_\alpha$$

and corresponding $\Sigma^1_1(z)$ elementary $\tau_{\beta,\alpha} : \mathcal{A}_\beta \to \mathcal{A}_\alpha$ which is not onto. Then since $\tau_{\beta,\alpha}$ is $\Sigma^1_1(z)$-elementary we may transfer any relevant $\Sigma^1_1(z)$ fact about $\mathcal{A}_\beta$ up to $\mathcal{A}_\alpha$. In particular we may find a model $\mathcal{A}$ with

$$\omega_1^{\mathrm{ck}(\mathcal{A}_\alpha,\mathcal{A},z)} = \omega_1^{\mathrm{ck}(z)}$$

and an $\omega_1^{\mathrm{ck}(z)}$-elementary map

$$\tau : \mathcal{A}_\alpha \to \mathcal{A}$$

which is not onto. Taking $\mathcal{A}_{\alpha+1} = \mathcal{A}$ and $\mathcal{B}_{\alpha+1}$ an appropriate isomorphic copy of $\mathcal{A}_{\alpha+1}$ we can continue to a further round.

For the limit stages of the construction we naturally wish to take

$$\mathcal{B}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{B}_\alpha$$

whenever λ is a limit, and then we are just left with trying to show that we can obtain an appropriate $\mathcal{A}_\lambda$. And this is where 9.1 comes to our rescue.

We find a model $\mathcal{A}_\infty$ from 9.1 to be the direct limit of the $\mathcal{A}_{\alpha(n)}$'s for some

$$\alpha(n) \to \lambda.$$

The direct limit construction gives

$$\tau_{\alpha(n),\infty} : \mathcal{A}_{\alpha(n)} \to \mathcal{A}_\infty$$

which commute and are $\Sigma^1_1(z)$ elementary; by composing with suitable $\tau_{\alpha,\alpha(n)}$ we may find

$$\tau_{\alpha,\infty} : \mathcal{A}_\alpha \to \mathcal{A}_\infty$$

all $\alpha < \lambda$.

The $\Sigma^1_1(z)$-elementarity of these maps ensures that we may find some

$$\rho : \mathcal{A}_\lambda \cong \mathcal{A}_\infty$$

with

$$\omega_1^{\mathrm{ck}(\mathcal{A}_\lambda, z)} = \omega_1^{\mathrm{ck}(z)}.$$

Choosing $\tau_{\alpha,\lambda} = \rho \circ \tau_{\alpha,\infty}$ we finish. (Claim$\square$)

But then the union

$$\mathcal{B}_{\aleph_1} = \bigcup_{\alpha < \aleph_1} \mathcal{B}_\alpha$$

is as required

$\square$

An immediate corollary of this theorem is a result previously proved by Keisler and Shelah using very different techniques. They showed the property of $\sigma \in \mathcal{L}_{\omega_1,\omega}$ allowing an uncountable model to be absolute between models of set theory, and moreover if we restrict ourselves to

$$\sigma \in \mathcal{L}_{\omega_1,\omega} \cap V_\alpha$$

some $\alpha < \omega_1$ then the set of such sentences is $\underset{\sim}{\Sigma}^1_1$.

However I should add to this that it would be excessive to claw through the several reflection lemmas above if our only goal was to reproduce Keisler-Shelah. Hugh Woodin has in any case shown that their theorem allows a short proof using iterations of ultrapowers of the Frechet filter at ω_1.

10 The split in cases

We wish to prove a dichotomy theorem for the isomorphism relation on countable models of some countably infinitary sentence σ. After relativizing to a real we may assume the existence of some $z \in 2^\omega$ and countable language $\mathcal{L}$ such that:

(i) $\mathcal{L}$ is recursive in z;
(ii) the Scott ranks of models of σ are bounded below $\omega_1^{\mathrm{ck}(z)}$;
(iii) $\sigma \in L_{\omega_1^{\mathrm{ck}(z)}}[z]$.

The arguments depend not at all on y, and thus there is no harm in lightening our notation and assuming:

(i) $\mathcal{L}$ is a recursive language;
(ii) the Scott ranks of models of σ are bounded below ω_1^{ck};
(iii) $\sigma \in L_{\omega_1^{\mathrm{ck}}}$;

and by 5.1 we may as well even assume

(iv) $\mathrm{Mod}(\sigma)$ is a Π_2^0 subset of $\mathrm{Mod}(\mathcal{L})$;

following 5.3 and 5.4 we may further insist that

(v) each tuple in $\mathcal{M} \in \mathrm{Mod}(\sigma)$ has its type first order definable from a single $a \in \mathcal{M}$;
(vi) for all $\vec{a}, b \in \mathcal{M}$, $\mathcal{M} \in \mathrm{Mod}(\sigma)$, there are infinitely many $c \in \mathcal{M}$ with

$$\langle \mathcal{M}; \vec{a}, b\rangle \cong \langle \mathcal{M}; \vec{a}, c\rangle.$$

And in the usual way we may assume that there are no function symbols to contend with:

(vii) $\mathcal{L}$ is relational.

CASE(I) There is a fragment $F \in L_{\omega_1^{\mathrm{ck}}}$ and there is $\mathcal{M} \in \mathrm{Mod}(\sigma)$ such that

$$\forall \vec{b} \in \mathcal{M} \exists \vec{a} \in \mathcal{M}(\langle \mathcal{M}; \vec{b}\rangle \models A_F(\vec{a})).$$

In other words, for all $\vec{b}$ in $\mathcal{M}$ there is some other $\vec{a}$ in $\mathcal{M}$ such that whenever

$$\langle \mathcal{M}; \vec{b}\rangle \models A(\vec{a})$$

we must have that A is not F-barren.

Definition Let X_F be the set of $\mathcal{M}$ as in case assumption.

Clearly $X_F \in \Sigma^1_1$.

CASE(II) Not case (I).

Here we need some observations and definitions before splitting into two further subcases.

By 7.1 we may find a transfinite sequence $\psi_{A_F,\alpha}$ such that

$$\alpha \mapsto \psi_{A_F,\alpha}$$

is uniformly Δ_1 over admissible sets and the following three things are equivalent for all $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ and $\vec{a}, \vec{b} \in \mathcal{M}$:

(1) $\langle \mathcal{M}; \vec{b} \rangle \models A_F(\vec{a})$;

(2) $\forall \alpha (\langle \mathcal{M}; \vec{b} \rangle \models \neg \psi_{A_F,\alpha}(\vec{a}))$;

(3) $\forall \alpha < \omega_1^{\mathrm{ck}(\mathcal{M})} (\langle \mathcal{M}; \vec{b} \rangle \models \neg \psi_{A_F,\alpha}(\vec{a}))$;

By case assumption and boundedness we can find a single $\gamma_0^F < \omega_1^{\mathrm{ck}}$ such for all $\mathcal{M} \in \mathrm{Mod}(\sigma)$ there is $\vec{b} \in \mathcal{M}$ such that

$$\langle \mathcal{M}; \vec{b} \rangle \models \forall \vec{a} \bigvee_{\alpha < \gamma_0^F} \psi_{A_F,\alpha}(\vec{a}).$$

Lemma 10.1 *Under the assumptions of case(II) we may find for each fragment $F \in L_{\omega_1^{\mathrm{ck}}}$ a corresponding fragment $H \in L_{\omega_1^{\mathrm{ck}}}$ such that for all $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(\sigma)$ and*

$$i : \mathcal{M} \to \mathcal{N}$$

H-elementary map, we have every

$$F\text{−type}_{\mathcal{N}}(\vec{a})$$

is realized over $\mathcal{M}$.

Proof Note that if $H_{\forall \vec{x} \bigvee_{\alpha < \gamma_0^F} \psi_{A_F,\alpha}(\vec{x})}$ is the fragment generated by the formula $\forall \vec{x} \bigvee_{\alpha < \gamma_0^F} \psi_{A_F,\alpha}(\vec{x})$ then we must have that for any $H_{\forall \vec{x} \bigvee_{\alpha < \gamma_0^F} \psi_{A_F,\alpha}(\vec{x})}$-elementary map

$$i : \mathcal{M} \to \mathcal{N}$$

between countable models of σ there will be some $\vec{b} \in \mathcal{M}$ such that

$$\langle \mathcal{N}; i(\vec{b}) \rangle \models \forall \vec{x} \bigvee_{\alpha < \gamma_0^F} \psi_{A_F,\alpha}(\vec{x});$$

and thus for every $\vec{a} \in \mathcal{N}$ we would by 8.3 have some barren ψ such that

$$\langle \mathcal{N}; i(\vec{b}) \rangle \models \psi(\vec{a}).$$

Now as in the proof of 8.5 we may apply 3.4 to find a single fragment $H \supset H_{\forall \vec{x} \bigvee_{\alpha < \gamma_0^F} \psi_{A_F,\alpha}(\vec{x})}$ such that for all $\mathcal{M}, \mathcal{N} \models \sigma$, all $\vec{a} \in \mathcal{N}$, and all

$$i : \mathcal{M} \to \mathcal{N}$$

which is H-elementary and for all $\vec{b} \in \mathcal{M}$ with

$$\langle \mathcal{N}; i(\vec{b})\rangle \models A_F(\vec{a})$$

we obtain the F-type$_{\langle \mathcal{N}; i(\vec{b})\rangle}(\vec{a})$ appearing in $\langle \mathcal{M}; \vec{b}\rangle$, and in particular F-type$_{\mathcal{N}}(\vec{a})$ realized over $\mathcal{M}$. This H suffices for the lemma.

□

Note that for any fragment F the set of H satisfying the conclusion of 10.1 is uniformly Π^1_1 in the codes. Thus by Spector-Gandy:

Corollary 10.2 *Under assumptions of case(II), we may find a* $\Delta_1^{L_{\omega_1^{\mathrm{ck}}}}$ *function*

$$F \mapsto H_F$$

such that for all $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(\sigma)$ *and*

$$i : \mathcal{M} \to \mathcal{N}$$

H-elementary map, we have every

$$F\text{−type}_{\mathcal{N}}(\vec{a})$$

is realized over $\mathcal{M}$.

Now we may assign to each $\mathcal{M} \in \mathrm{Mod}(\sigma)$ and fragment $F \in L_{\omega_1^{\mathrm{ck}}}$ a corresponding tree

$$T^F_{\mathcal{M}} \subset \mathcal{M}^{<\omega}$$

such that:

(i) there is in $L_{\omega_1^{\mathrm{ck}}}$ a sequence of formulas $(\varphi^F_{i\in\omega})_{i\in\omega}$ in H_F, depending on F but not on $\mathcal{M}$, such that for each $n \in \omega$ and $\vec{a} \in \mathcal{M}^n$ we have

$$\mathcal{M} \models \varphi^F_n(\vec{a})$$

if and only if

$$\vec{a} \in T^F_{\mathcal{M}};$$

(ii) a substructure $\mathcal{N}$ of $\mathcal{M}$ is an H_F-elementary substructure if and only if there is some infinite branch

$$f \in [T^F_{\mathcal{M}}]$$

with

$$\mathcal{N} = \{f(n) : n \in \omega\}.$$

This is just a general fact; it does not depend at all on the specifics of our case assumption. We can tear apart all the sentences of interest and insist that the $(\varphi^F_{i\in\omega})_{i\in\omega}$ be chosen so as to appropriately Skolemize the formulas appearing in H_F.

However in the setting of our case assumption we must moreover obtain:

(iii) for each $\vec{a} \in \mathcal{M}$ and $f \in [T^F_{\mathcal{M}}]$ there will exist some $\vec{c} \in \{f(n) : n \in \omega\}$ such that

$$F\text{−type}_{\mathcal{M}}(\vec{a}) = F\text{−type}_{\mathcal{M}}(\vec{c});$$

this in particular implies the weaker but more useful conclusion that

(iv) for each $\vec{a} \in \mathcal{M}$ and $f \in [T^F_{\mathcal{M}}]$ there will exist some $\vec{c} \in \{f(n) : n \in \omega\}$ such that $F\text{−type}_{\mathcal{M}}(\vec{a})$ is H-definable from $\vec{c}$. That is to say, there is some formula $\psi \in H$ such that

(a) $\mathcal{M} \models \psi(\vec{c}, \vec{a})$;
(b) $\mathcal{M} \models \forall \vec{b}^0, \vec{b}^1 (\bigwedge_{\varphi \in F} (\psi(\vec{c}, \vec{b}^0) \wedge \psi(\vec{c}, \vec{b}^1) \Rightarrow (\varphi(\vec{b}^0) \Leftrightarrow \varphi(\vec{b}^1)))$.

Thus we may define for any $F \in L_{\omega_1^{\mathrm{ck}}}$, $H \supset H_F$, $\mathcal{M} \in \mathrm{Mod}(\sigma)$ a kind of ranking function

$$\rho^H_{F,\mathcal{M}} : T^F_{\mathcal{M}} \times \mathcal{M}^{<\omega} \rightarrow \aleph_1$$

defined by the specification that:

$\rho^H_{F,\mathcal{M}}(\vec{a}, \vec{b}) = 0$ if $F\text{−type}_{\mathcal{M}}(\vec{b})$ is H-definable from $\vec{a}$;
$\rho^H_{F,\mathcal{M}}(\vec{a}, \vec{b}) = \sup\{\rho^H_{F,\mathcal{M}}(\vec{a}^\frown c, \vec{b}) + 1 : \vec{a}^\frown c \in T^F_{\mathcal{M}}\}$ whenever $F\text{−type}_{\mathcal{M}}(\vec{b})$ is not H-definable from $\vec{a}$.

CASE(IIa): For all fragments $F \in L_{\omega_1^{\mathrm{ck}}}$ there is some fragment $H \supset H_F$ in $L_{\omega_1^{\mathrm{ck}}}$ such that for all $\mathcal{M} \in \mathrm{Mod}(\sigma)$ there is some $\vec{a} \in \mathcal{M}$ with

$$\sup\{\rho^H_{F,\mathcal{M}}(\vec{a}, \vec{b}) : \vec{b} \in \mathcal{M}\} < \omega.$$

CASE(IIb): There is some fragment $F \in L_{\omega_1^{\mathrm{ck}}}$ such that for all $H \supset H_F$ in $L_{\omega_1^{\mathrm{ck}}}$ there is $\mathcal{M} \in \mathrm{Mod}(\sigma)$ with

$$\sup\{\rho^H_{F,\mathcal{M}}(\vec{a}, \vec{b}) : \vec{b} \in \mathcal{M}\}$$

infinite for all $\vec{a} \in \mathcal{M}$.

11 Proof for case(I)

Fix the fragment F as in case hypothesis. It will be convenient to work with models of σ whose underlying set is something other than ω.

Notation Let Ω be the disjoint union of $(2^{<\omega})^{<\omega}$ and $(2^{<\omega})^{<\omega} \times \omega$. Let $\mathrm{Mod}_\Omega(\mathcal{L})$ be the space of $\mathcal{L}$-structures on Ω equipped with the topology of quantifier free formulas. Let $\mathrm{Mod}_\Omega(\sigma)$ be the subspace of models of σ. We again have that $\mathrm{Sym}(\Omega)$, the collection of permutations of Ω, is a Polish group in the topology of pointwise convergence and gives rise to the isomorphism relation on $\mathrm{Mod}_\Omega(\mathcal{L})$.

We can continue then by analogy with section 7.

Definition For $A \subset \mathrm{Mod}_\Omega(\mathcal{L})$ a Σ^1_1 set and $f \subset \Omega$ finite, $\mathcal{M} \in \mathrm{Mod}_\Omega(\mathcal{L})$, $(a_u)_{u\in f}$ a collection elements of $\mathcal{M}$ indexed by the elements of f, let us write

$$\mathcal{M} \models A((a_u)_{u\in f})$$

if there is $\pi \in \mathrm{Sym}(\Omega)$ with

$$\pi(a_u) = u$$

all $u \in f$ and

$$\pi \cdot \mathcal{M} \in A.$$

For $(a_u)_{u\in f}$ in $\mathcal{M}$ we let $\langle \mathcal{M}; (a_u)_{u\in f}\rangle$ be the expansion of $\mathcal{M}$ by fresh constants $(\mathbf{c}_u)_{u\in f}$ with

$$\langle \mathcal{M}; (a_u)_{u\in f}\rangle \models \mathbf{c}_u = a_u$$

all $u \in f$. Given $f_0, f_1, ..., f_n$ finite disjoint subsets of Ω we may naturally write

$$\langle \mathcal{M}; (a_u)_{u\in f_0}, (a_u)_{u\in f_1}, ...\rangle$$

in place of

$$\langle \mathcal{M}; (a_u)_{u\in f_0\cup f_1\cup ...}\rangle$$

and

$$\mathcal{M} \models A((a_u)_{u\in f_0}, (a_u)_{u\in f_1}, ..)$$

in place of

$$\mathcal{M} \models A((a_u)_{u\in f_0\cup f_1\cup ...}).$$

In general it is convenient to extend the model theoretic conventions for sequences to our present situation. We may just write

$$(a_u)_{u\in f} \in \mathcal{M}$$

to indicate that $(a_u)_{u\in f}$ is an f-indexed collection of points in $\mathcal{M}$.

Definition We write

$$\langle \mathcal{M}; (a_u)_{u\in f}\rangle \models A_F(b)$$

if for all Σ^1_1 sets A with

$$\langle \mathcal{M}; (a_u)_{u\in f}\rangle \models A(b),$$

we can find

$$\mathcal{N}_0, \mathcal{N}_1 \in \mathrm{Mod}_\Omega(\sigma),$$

$$(b_u)_{u\in f} \in \mathcal{N}_0,$$

$$d \in \mathcal{N}_1,$$

$$\tau : \mathcal{N}_0 \to \mathcal{N}_1$$

$\omega_1^{\mathrm{ck}(\mathcal{N}_0,\mathcal{N}_1)}$-elementary, with

$$\langle \mathcal{N}_1; (\tau(b_u))_{u\in f}\rangle \models A(d),$$

but

$$F\text{−type}_{\langle \mathcal{N}_1;(\tau(b_u))_{u\in f}\rangle}(d) \notin \{F\text{−type}_{\langle \mathcal{N}_0;(b_u)_{u\in f}\rangle}(e) : e \in \mathcal{N}_0\}.$$

Strictly speaking we should introduce a different $A_{F,f}$ for each finite subset f of Ω. We do not go to this trouble.

We should also distinguish these Σ^1_1 sets from their counterparts in the space $\mathrm{Mod}(\sigma)$. Again we do not go to this trouble.

Definition We let $X_F \subset \mathrm{Mod}_\Omega(\sigma)$ be the set of $\mathcal{M}$ such that for all $\vec{b} \in \mathcal{M}$ there is $a \in \mathcal{M}$ with

$$\langle \mathcal{M}; \vec{b}\rangle \models A_F(a).$$

Our case assumptions appropriately translated across to the space $\mathrm{Mod}_\Omega(\sigma)$ give that X_F is a non-empty set; here we have used that the conclusion of 5.3 is assumed to hold and so we can restrict ourselves to a a single point, instead of having to range over all sequences $\vec{c} \in \mathcal{M}$. 8.4 gives that X_F is Σ^1_1.

Lemma 11.1 *Let $\mathcal{N}_0, \mathcal{N}_1 \in \mathrm{Mod}_\Omega(\sigma)$,*

$$\tau : \mathcal{N}_0 \to \mathcal{N}_1$$

$\omega_1^{\mathrm{ck}(\mathcal{N}_0,\mathcal{N}_1)}$*-elementary, and let $b \in \mathcal{N}_1$ with*

$$F\text{−type}_{\mathcal{N}_1}(b) \notin \{F\text{−type}_{\mathcal{N}_0}(c) : c \in \mathcal{N}_0\}.$$

Then for all finite $f \subset \Omega$ and $(a_u)_{u\in f}$ in $\mathcal{N}_0$ we have

$$\langle \mathcal{N}_1; (\tau(a_u))_{u\in f}\rangle \models A_F(b).$$

Proof Fix $(a_u)_{u\in f}$ in $\mathcal{N}_0$. Since $F-\text{type}_{\mathcal{N}_1}(b) \notin \{F-\text{type}_{\mathcal{N}_0}(c) : e \in \mathcal{N}_0\}$ we have in particular that

$$F-\text{type}_{\langle \mathcal{N}_1;(\tau(a_u))_{u\in f}\rangle}(b) \notin \{F-\text{type}_{\langle \mathcal{N}_0;(a_u)_{u\in f}\rangle}(c) : c \in \mathcal{N}_0\}.$$

Thus we necessarily have that for all $A \in \Sigma^1_1$ with

$$\langle \mathcal{N}_1; (\tau(a_u))_{u\in f}\rangle \models A(b)$$

we can indeed find a pair of models and an elementary embedding (namely: $\langle \mathcal{N}_0; (a_u)_{u\in f}\rangle$, $\langle \mathcal{N}_1; (\tau(a_u))_{u\in f}\rangle$, $\tau!$) as in the definition of A_F as above.

$\square$

Definition We let $\mathbb{Q}_0$ the set of sequences $\langle A, (a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, (a_{\vec{s}})_{\vec{s}\in f_1}, ...(b_v)_{v\in g_n}\rangle$ such that:-

(i) $n \in \omega$; each of the $a_{\vec{s}}$ and b_v are in Ω;

(ii) there are $l_0, l_1, ..., l_n \in \omega$ with each $f_i = 2^{l_0} \times 2^{l_1} \times ... \times 2^{l_n}$ and each g_i a finite subset of $f_i \times \omega$;

(iii) $A \in \Sigma^1_1$;

(iv) there is some $\mathcal{M} \in \text{Mod}_\Omega(\sigma)$ with

$$\mathcal{M} \models A((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (b_v)_{v\in g_n});$$

for all $\mathcal{M} \in \text{Mod}_\Omega(\sigma)$ with

$$\mathcal{M} \models A((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (b_v)_{v\in g_n})$$

we have

$$\mathcal{M} \models X_F;$$

(v) for each $m \leq n$ there is a linear ordering $<^m$ on $f_m = 2^{l_0} \times 2^{l_1} \times ... \times 2^{l_m}$ such that for all $\vec{t} \in f_m$ and $\mathcal{M} \in \text{Mod}_\Omega(\sigma)$ with

$$\mathcal{M} \models A((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (b_v)_{v\in g_n})$$

we have

$$\langle \mathcal{M}; (a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (a_{\vec{s}})_{\vec{s}\in f_{m-1}}, (b_v)_{v\in g_{m-1}}, (a_{\vec{s}})_{\vec{s}<^m\vec{t}}\rangle \models A_F(a_{\vec{t}});$$

(vi) for each $\vec{s} \neq \vec{t} \in \bigcup_{m\leq n} f_m$ there is some $\psi_{\vec{s},\vec{t}} \in F$ such that whenever

$$\mathcal{M} \models A((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (b_v)_{v\in g_n})$$

we have must have

$$\mathcal{M} \models \psi_{\vec{s},\vec{t}}(a_{\vec{s}}) \wedge \neg\psi_{\vec{s},\vec{t}}(a_{\vec{t}}).$$

In fact the ordering $<^m$ in (v) can be taken to be the lexicographic ordering under some appropriate isomorphism of $2^{l_0} \times 2^{l_1} \times ... \times 2^{l_m}$ with $2^{l_0+l_1+...+l_m}$. There is no need for us to keep count of this specific detail and so we ignore it.

Note that it has been hard wired into our definitions that whenever $\vec{s} \neq \vec{t}$ and

$$\langle \mathcal{M}; (a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (b_v)_{v\in g_n}\rangle \models A$$

for any Σ^1_1 set A then it must be the case that

$$a_{\vec{s}} \neq a_{\vec{t}},$$

$$a_{\vec{s}} \neq b_{\vec{t}},$$

$$a_{\vec{s}} \neq b_{\vec{s}},$$

$$b_{\vec{s}} \neq b_{\vec{t}}.$$

Part (vi) of the definition of $\mathbb{Q}_0$ represents a minor redundancy. If we were to define $\mathbb{Q}_0^*$ to be the set obtained by sequences only satisfying (i)-(v), then we would obtain that in some sense $\mathbb{Q}_0$ is dense in $\mathbb{Q}_0^*$.

Lemma 11.2 *Suppose $\langle A, (a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, (a_{\vec{s}})_{\vec{s}\in f_1}, ...(b_v)_{v\in g_n}\rangle$ satisfies (i)-(v) from the definition of $\mathbb{Q}_0$ above. Then we may find a Σ^1_1 set $\hat{A} \subset A$ such that*

$$\langle \hat{A}, (a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, (a_{\vec{s}})_{\vec{s}\in f_1}, ...(b_v)_{v\in g_n}\rangle \in \mathbb{Q}_0.$$

Proof For each $m \leq n$ let $<^m$ be the indicated ordering on f_m as in (v) of the definition of $\mathbb{Q}_0$. Let $\mathcal{M} \in \mathrm{Mod}_\Omega(\sigma)$ be such that

$$\mathcal{M} \models A((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (b_v)_{v\in g_n}).$$

Then for each $m \leq n$ and $\vec{t} \in f_m$ we have

$$\langle \mathcal{M}; (a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (a_{\vec{s}})_{\vec{s}<^m\vec{t}}\rangle \models A_F(a_{\vec{t}}),$$

and thus in particular it is an outright consequence of the definition of A_F that for any $\vec{s} <^m \vec{t}$

$$\mathcal{M} \models \neg \bigwedge_{\psi\in F} (\psi(a_{\vec{s}}) \Leftrightarrow \psi(a_{\vec{t}})).$$

Thus we may choose some $\psi_{\vec{s},\vec{t}} \in F$ with

$$\mathcal{M} \models \psi_{\vec{s},\vec{t}}(a_{\vec{s}}) \wedge \neg\psi_{\vec{s},\vec{t}}(a_{\vec{t}}).$$

Then we can define $\hat{A} \subset A$ by the specification that for all $\mathcal{N} \in \mathrm{Mod}_\Omega(\sigma)$

$$\mathcal{N} \models \hat{A}((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (b_v)_{v\in g_n})$$

if and only if

$$\mathcal{N} \models A((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (b_v)_{v\in g_n})$$

and for all $m \leq n$ and $\vec{t} \in f_m$

$$\mathcal{N} \models \bigwedge_{\vec{s}<^m\vec{t}} \neg\psi_{\vec{s},\vec{t}}(a_{\vec{t}}) \wedge \bigwedge_{\vec{s}>^m\vec{t}} \psi_{\vec{t},\vec{s}}(a_{\vec{t}}).$$

$\square$

Notation Suppose $l_0, l_1, ..., l_n \in \omega$ and $f_i = 2^{l_0} \times 2^{l_1} \times ... \times 2^{l_i}$. Then for each $m \leq n$ we let

$$f_i^{(m)} = 2^{l_0} \times 2^{l_1} \times ... \times 2^{l_{m-1}} \times 2^{l_m+1} \times 2^{l_{m+1}} \times ... \times 2^{l_i}.$$

(Naturally when $m > i$ we simply have $f_i^{(m)} = f_i$.)

For $\vec{s} \in f_i$ we define $\vec{s}^{(m,0)} \in f_i^{(m)}$ by

$$(\vec{s}^{(m,0)})_j = (\vec{s})_j$$

for $j \leq i$, $j \neq m$, and

$$(\vec{s}^{(m,0)})_m = (\vec{s})_j^\frown 0;$$

similarly

$$(\vec{s}^{(m,1)})_j = (\vec{s})_j$$

for $j \leq i$, $j \neq m$,

$$(\vec{s}^{(m,1)})_m = (\vec{s})_j^\frown 1.$$

Thus in particular if $\vec{s} \in f_i$ and $m > i$ then

$$\vec{s}^{(m,0)} = \vec{s}^{(m,1)} = \vec{s}.$$

We let

$$f_i^{(m,0)} = \{\vec{s}^{(m,0)} : \vec{s} \in f_i\},$$
$$f_i^{(m,1)} = \{\vec{s}^{(m,1)} : \vec{s} \in f_i\}.$$

In a similar fashion we extend this to $g_i \subset f_i \times \omega$. So for any $v = (\vec{s}, k) \in g_i$ we let

$$v^{(m,0)} = (\vec{s}^{(m,0)}, k),$$

$$v^{(m,1)} = (\vec{s}^{(m,1)}, k).$$

Then

$$g_i^{(m,0)} = \{v^{(m,0)} : v \in g_i\},$$
$$g_i^{(m,1)} = \{v^{(m,1)} : v \in g_i\},$$
$$g_i^{(m)} = g_i^{(m,0)} \cup g_i^{(m,1)}.$$

The definitions are set up so that given any $g_i \subset \Omega$ finite and $m \leq i$ we have

$$g_i^{(m,0)} \cap g = g_i^{(m,1)} \cap g = 0.$$

This along with our assumption that $\mathrm{Mod}_\Omega(\sigma)$ satisfies the conclusion of 5.4 is being used in the next lemma; together they keep a leash on the sets of the form $(b_v)_{v \in g_i}$. Without these assumptions we can move forward only by painfully enumerating when $b_v = b_w$.

Lemma 11.3 *Let*

$$\langle A, (a_{\vec{s}})_{\vec{s} \in f_0}, (b_v)_{v \in g_0}, (a_{\vec{s}})_{\vec{s} \in f_1}, \ldots (b_v)_{v \in g_n} \rangle$$

be in $\mathbb{Q}_0$*, where each* $f_i = 2^{l_0} \times 2^{l_1} \times \ldots \times 2^{l_i}$ *and let* $m \leq n$.

Then there is some

$$\langle B, (a_{\vec{s}})_{\vec{s} \in f_0}, (b_v)_{v \in g_0}, (a_{\vec{s}})_{\vec{s} \in f_1}, \ldots, (b_v)_{v \in g_{m-1}}, (a_{\vec{s}})_{\vec{s} \in f_m^{(m)}}, (b_v)_{v \in g_m^{(m)}}, \ldots (b_v)_{v \in g_n^{(m)}} \rangle$$

in $\mathbb{Q}_0$ *such that for all*

$$\mathcal{M} \models B((a_{\vec{s}})_{\vec{s} \in f_0}, (b_v)_{v \in g_0}, (a_{\vec{s}})_{\vec{s} \in f_1}, \ldots, (b_v)_{v \in g_{m-1}}, (a_{\vec{s}})_{\vec{s} \in f_m^{(m)}}, (b_v)_{v \in g_m^{(m)}}, (b_v)_{v \in g_n^{(m)}})$$

we have

$$\mathcal{M} \models A((a_{\vec{s}})_{\vec{s} \in f_0}, (b_v)_{v \in g_0}, (a_{\vec{s}})_{\vec{s} \in f_1}, \ldots, (b_v)_{v \in g_{m-1}}, (a_{\vec{s}^{(m,0)}})_{\vec{s} \in f_m}, (b_{v^{(m,0)}})_{v \in g_m}, \ldots (b_{v^{(m,0)}})_{v \in g_n})$$

and

$$\mathcal{M} \models A((a_{\vec{s}})_{\vec{s} \in f_0}, (b_v)_{v \in g_0}, (a_{\vec{s}})_{\vec{s} \in f_1}, \ldots, (b_v)_{v \in g_{m-1}}, (a_{\vec{s}^{(m,1)}})_{\vec{s} \in f_m}, (b_{v^{(m,1)}})_{v \in g_m}, \ldots (b_{v^{(m,1)}})_{v \in g_n})$$

Proof For notational simplicity let us take the case that $n = 1$, $l_0 = 0$, $l_1 = 1$, $m = 0$, $g_0 = 2^0 \times \{5, 6\} = \{0\} \times \{5, 6\}$, $g_1 = 0$. Thus f_0 contains only the empty set and f_1 is the set of size two whose only elements are $\langle 0, \langle 0 \rangle \rangle$ and $\langle 0, \langle 1 \rangle \rangle$; $f_0^{(0,0)}$ contains $\langle 0 \rangle$ and $f_1^{(0,0)}$ is the set whose elements are $\langle \langle 0 \rangle, \langle 0 \rangle \rangle$ and $\langle \langle 0 \rangle, \langle 1 \rangle \rangle$; $f_0^{(0,1)}$ contains $\langle 0 \rangle$ and $f_1^{(0,1)}$ contains $\langle \langle 1 \rangle, \langle 0 \rangle \rangle$ and $\langle \langle 1 \rangle, \langle 1 \rangle \rangle$.

Let us assume that

$$\langle 0, \langle 0 \rangle \rangle <^1 \langle 0, \langle 1 \rangle \rangle$$

describes the linear ordering $<^1$ on $2^0 \times 2^1 = f_1$.

Now from clause (v) of the definition of $\mathbb{Q}_0$ we may find $\mathcal{N}_0, \mathcal{N}_1 \in \mathrm{Mod}_\Omega(\sigma)$ and $\omega_1^{\mathrm{ck}(\mathcal{N}_0, \mathcal{N}_1)}$-elementary

$$\tau_0 : \mathcal{N}_0 \to \mathcal{N}_1$$

and

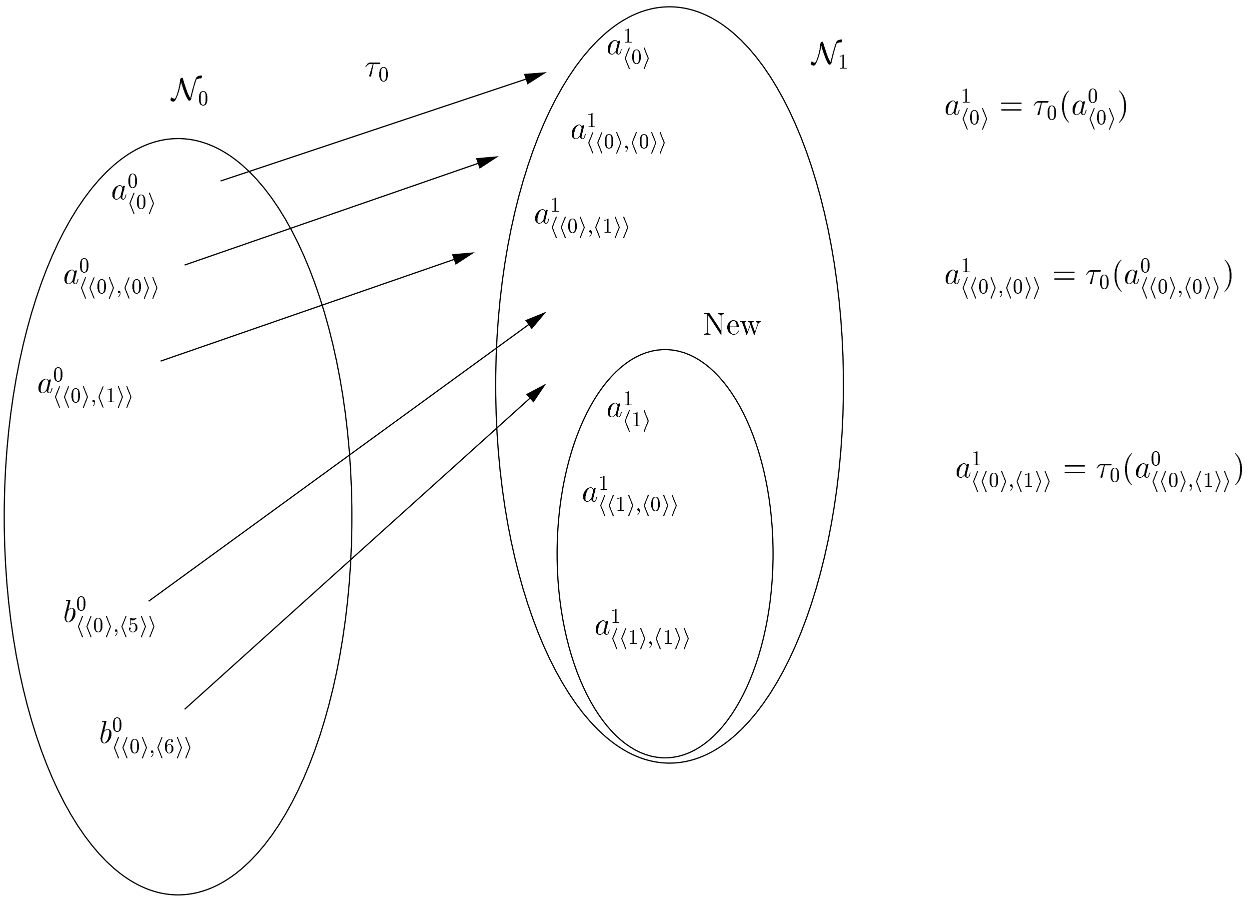
$$a^0_{\langle 0 \rangle}, b^0_{\langle \langle 0 \rangle, \langle 5 \rangle \rangle}, b^0_{\langle \langle 0 \rangle, \langle 6 \rangle \rangle}, a^0_{\langle \langle 0 \rangle, \langle 0 \rangle \rangle}, a^0_{\langle \langle 0 \rangle, \langle 1 \rangle \rangle} \in \mathcal{N}_0,$$
$$a^1_{\langle 1 \rangle}, b^1_{\langle \langle 1 \rangle, \langle 5 \rangle \rangle}, b^1_{\langle \langle 1 \rangle, \langle 6 \rangle \rangle}, a^1_{\langle \langle 1 \rangle, \langle 0 \rangle \rangle}, a^1_{\langle \langle 1 \rangle, \langle 1 \rangle \rangle} \in \mathcal{N}_1,$$
$$\mathcal{N}_0 \models A((a^0_{\vec{s}^{(0,0)}})_{\vec{s} \in f_0}, (b^0_{v^{(0,0)}})_{v \in g_0}, (a^0_{\vec{s}^{(0,0)}})_{\vec{s} \in f_1}),$$

$$\mathcal{N}_1 \models A((a^1_{\vec{s}(0,1)})_{\vec{s}\in f_0}, (b^1_{v(0,1)})_{v\in g_0}, (a^1_{\vec{s}(0,1)})_{\vec{s}\in f_1}),$$
$$F\text{−type}_{\mathcal{N}_1}(a^1_{\langle 1\rangle}) \notin \{F\text{−type}_{\mathcal{N}_0}(c) : c \in \mathcal{N}_0\}$$

and therefore by 11.1

$$\langle \mathcal{N}_1; \tau_0(a^0_{\langle 0\rangle})\rangle \models A_F(a^1_{\langle 1\rangle}).$$

We use the assumption that the conclusion of 5.4 holds to obtain $b^1_{\langle\langle 1\rangle,\langle 5\rangle\rangle}, b^1_{\langle\langle 1\rangle,\langle 6\rangle\rangle}$ distinct to $\tau_0(b^0_{\langle\langle 0\rangle,\langle 5\rangle\rangle}), \tau_0(b^0_{\langle\langle 0\rangle,\langle 6\rangle\rangle})$.



So far the only properties mentioned for the pair $(\mathcal{N}_0, \mathcal{N}_1)$ is that it falls inside finitely many Σ^1_1 sets. Thus we may assume from 6.11

$$\omega_1^{\mathrm{ck}(\mathcal{N}_0,\mathcal{N}_1)} = \omega_1^{\mathrm{ck}}.$$

The elementarity of the map τ_0 is sufficient by 7.2 to give

$$\mathcal{N}_1 \models A((\tau_0(a^0_{\vec{s}(0,0)}))_{\vec{s}\in f_0}, (\tau_0(b^0_{v(0,0)}))_{v\in g_0}, (\tau_0(a^0_{\vec{s}(0,0)}))_{\vec{s}\in f_1}),$$

and hence

$$\langle \mathcal{N}_1; (\tau_0(a^0_{\vec{s}(0,0)}))_{\vec{s}\in f_0}, (\tau_0(b^0_{v(0,0)})_{v\in g_0})\rangle \models A_F(\tau_0(a^0_{\langle\langle 0\rangle,\langle 0\rangle\rangle}))$$

by clause (v) of the definition of $\mathbb{Q}_0$.

By 6.11 and 8.7 we may find $\mathcal{N}_2 \in \mathrm{Mod}_\Omega(\sigma)$ with

$$\omega_1^{\mathrm{ck}(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2)} = \omega_1^{\mathrm{ck}(\mathcal{N}_0,\mathcal{N}_1)} = \omega_1^{\mathrm{ck}}$$

and ω_1^{ck}-elementary

$$\tau_1 : \mathcal{N}_1 \to \mathcal{N}_2$$

and

$$a^2_{\langle\langle 0\rangle,\langle 0\rangle\rangle}, a^2_{\langle\langle 0\rangle,\langle 1\rangle\rangle} \in \mathcal{N}_2,$$

with

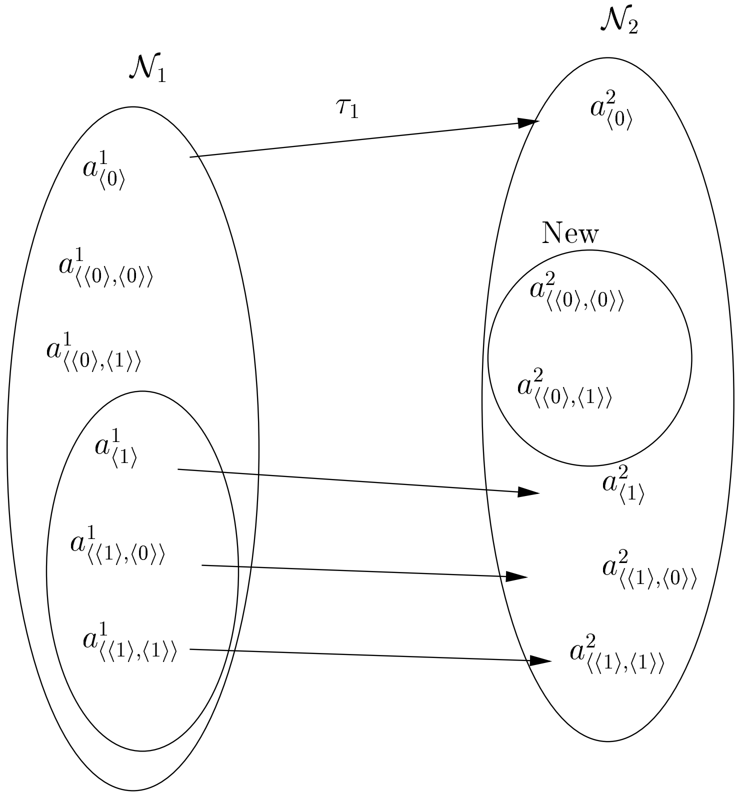
$$\mathcal{N}_2 \models A((\tau_1 \circ \tau_0(a^0_{\vec{s}(0,0)}))_{\vec{s}\in f_0}, (\tau_1 \circ \tau_0(b^0_{v(0,0)}))_{v\in g_0}, (a^2_{\vec{s}(0,0)})_{\vec{s}\in f_1}),$$

and

$$F\mathrm{-type}_{\mathcal{N}_2}(a^2_{\langle\langle 0\rangle,\langle 0\rangle\rangle}) \notin \{F\mathrm{-type}_{\mathcal{N}_1}(c) : c \in \mathcal{N}_1\},$$

and therefore in particular

$$\langle \mathcal{N}_2; (\tau_1\circ\tau_0(a^0_{\langle 0\rangle}))_{\{\langle 0\rangle\}}, (\tau_1(a^1_{\langle 1\rangle}))_{\{\langle 1\rangle\}}, (\tau_1\circ\tau_0(b^0_{v(0,0)}))_{v(0,0)\in g_0^{(0,0)}}, (\tau_1(b^1_{v(0,1)}))_{v(0,1)\in g_0^{(0,1)}}\rangle \models A_F(a^2_{\langle\langle 0\rangle,\langle 0\rangle\rangle}).$$



We can repeat and find $\mathcal{N}_3$, again with

$$\omega_1^{\mathrm{ck}(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2,\mathcal{N}_3)} = \omega_1^{\mathrm{ck}},$$

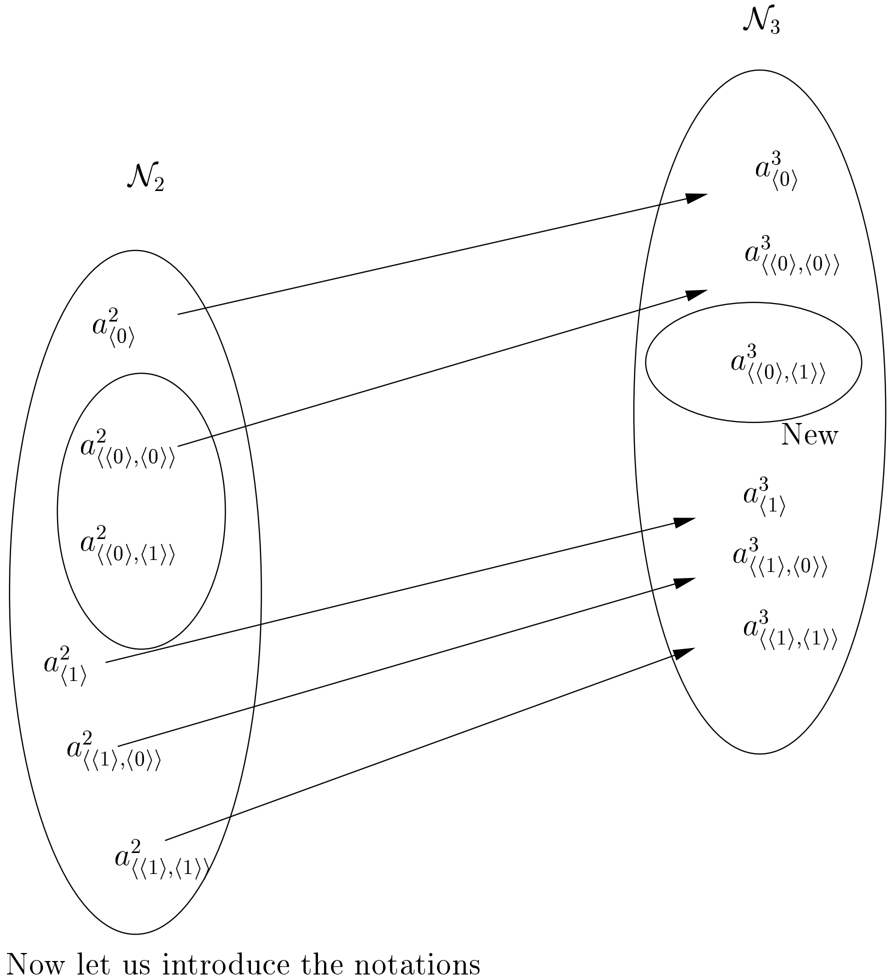
and some $a^3_{\langle\langle 0\rangle,\langle 1\rangle\rangle} \in \mathcal{N}_3$, with

$$\mathcal{N}_3 \models A((\tau_2\circ\tau_1\circ\tau_0(a^0_{\vec{s}(0,0)}))_{\vec{s}\in f_0}, (\tau_2\circ\tau_1\circ\tau_0(b^0_{v(0,0)}))_{v\in g_0}, (\tau_2(a^2_{\langle\langle 0\rangle,\langle 0\rangle\rangle}))_{\{\langle\langle 0\rangle,\langle 0\rangle\rangle\}}, (a^3_{\langle\langle 0\rangle,\langle 1\rangle\rangle})_{\{\langle\langle 0\rangle,\langle 1\rangle\rangle\}}),$$

$$F\mathrm{-type}_{\mathcal{N}_3}(a^3_{\langle\langle 0\rangle,\langle 1\rangle\rangle}) \notin \{F\mathrm{-type}_{\mathcal{N}_2}(c) : c \in \mathcal{N}_0\}$$

and thus

$$\langle \mathcal{N}_3; (\tau_2\circ\tau_1\circ\tau_0(a^0_{\langle 0\rangle}))_{\{\langle 0\rangle\}}, \cdots, (\tau_2(a^2_{\langle\langle 0\rangle,\langle 0\rangle\rangle}))_{\{\langle\langle 0\rangle,\langle 0\rangle\rangle\}}\rangle \models A_F(a^3_{\langle\langle 0\rangle,\langle 1\rangle\rangle}).$$



Now let us introduce the notations

$$\tau_{0,3} = \tau_2\circ\tau_1\circ\tau_0 : \mathcal{N}_0 \to \mathcal{N}_3,$$

$$\tau_{1,3} = \tau_2\circ\tau_1 : \mathcal{N}_1 \to \mathcal{N}_3,$$

$$\tau_{2,3} = \tau_2 : \mathcal{N}_2 \to \mathcal{N}_3.$$

We can then go on and let

$$b^3_{v(0,0)} = \tau_2\circ\tau_1\circ\tau_0(b^0_{v(0,0)})$$

for $v^{(0,0)} \in g_0^{(0,0)}$,

$$b^3_{v^{(0,1)}} = \tau_2 \circ \tau_1(b^1_{v^{(0,1)}})$$

for $v^{(0,1)} \in g_0^{(0,1)}$,

$$a^3_{\langle 0\rangle} = \tau_{0,3}(a^0_{\langle 0\rangle}).$$
$$a^3_{\langle\langle 0\rangle,\langle 0\rangle\rangle} = \tau_{2,3}(a^2_{\langle\langle 0\rangle,\langle 0\rangle\rangle}),$$
$$a^3_{\langle\langle 1\rangle,\langle 0\rangle\rangle} = \tau_{1,3}(a^1_{\langle\langle 1\rangle,\langle 0\rangle\rangle}),$$
$$a^3_{\langle\langle 1\rangle,\langle 1\rangle\rangle} = \tau_{1,3}(a^1_{\langle\langle 1\rangle,\langle 1\rangle\rangle}),$$
$$a^3_{\langle 1\rangle} = \tau_{1,3}(a^1_{\langle 1\rangle}).$$

We have that each $\tau_{i,j}$ is $\omega_1^{\mathrm{ck}(\mathcal{N}_i,\mathcal{N}_j)}$-elementary, and hence respects Σ^1_1 truths in the sense indicated by 7.2. Thus we have all the following in $\mathcal{N}_3$:-

(a)

$$\mathcal{N}_3 \models A((a^3_{\vec{s}^{(0,0)}})_{\vec{s}\in f_0}, (b^3_{v^{(0,0)}})_{v\in g_0}, (a^3_{\vec{s}^{(0,0)}})_{\vec{s}\in f_1});$$

(b)

$$\langle \mathcal{N}_3; a^3_{\langle 0\rangle}\rangle \models A_F(a^3_{\langle 1\rangle});$$

(c)

$$\langle \mathcal{N}_3; (a^3_{\langle 0\rangle})_{\{\langle 0\rangle\}}, (a^3_{\langle 1\rangle})_{\{\langle 1\rangle\}}, (b^3_v)_{v\in g_0^{(0)}}\rangle \models A_F(a^3_{\langle\langle 0\rangle,\langle 0\rangle\rangle});$$

(e) and by one last application of 11.1,

$$\langle \mathcal{N}_3; (a^3_{\langle 0\rangle})_{\{\langle 0\rangle\}}, (a^3_{\langle 1\rangle})_{\{\langle 1\rangle\}}, (b^3_v)_{v\in g_0^{(0)}}, (a^3_{\langle\langle 0\rangle,\langle 0\rangle\rangle})_{\{\langle\langle 0\rangle,\langle 0\rangle\rangle\}}\rangle \models A_F(a^3_{\langle\langle 0\rangle,\langle 1\rangle\rangle}).$$

Now holding onto $a^3_{\langle 0\rangle}$, $a^3_{\langle 1\rangle}$, $(b^3_{v^{(0,0)}})_{v^{(0,0)}\in g_0^{(0,0)}}$, $(b^3_{v^{(0,1)}}))_{v^{(0,1)}\in g_0^{(0,1)}}$, $a^3_{\langle\langle 0\rangle}$, $a^3_{\langle\langle 0\rangle,\langle 1\rangle\rangle}$, we can work on $a^3_{\langle 1\rangle}$ and obtain in perfect analogy to the construction of $\mathcal{N}_1$ given $\mathcal{N}_0$ above some ω_1^{ck}-elementary map

$$\tau_3 : \mathcal{N}_3 \to \mathcal{N}_4$$

and some $a^4_{\langle\langle 1\rangle,\langle 0\rangle\rangle}, a^4_{\langle\langle 1\rangle,\langle 1\rangle\rangle} \in \mathcal{N}_4$ with

$$\mathcal{N}_4 \models A((\tau_3(a^4_{\vec{s}^{(0,1)}}))_{\vec{s}\in f_0}, (\tau_3(b^4_{v^{(0,1)}}))_{v\in g_0}, (a^4_{\vec{s}^{(0,1)}})_{\vec{s}\in f_1}),$$

$$F\text{−type}_{\mathcal{N}_4}(a^4_{\langle\langle 1\rangle,\langle 0\rangle\rangle}) \notin \{F\text{−type}_{\mathcal{N}_3}(c) : c \in \mathcal{N}_3\}.$$

Then we go again to obtain $\mathcal{N}_5$,

$$\tau_4 : \mathcal{N}_4 \to \mathcal{N}_5,$$

$$\mathcal{N}_2 \models A((\tau_4\circ\tau_3(a^0_{\vec{s}^{(0,1)}}))_{\vec{s}\in f_0}, (\tau_4\circ\tau_3(b^3_{v^{(0,1)}}))_{v\in g_0}, (\tau_4(a^4_{\langle\langle 1\rangle,\langle 0\rangle\rangle}))_{\{\langle\langle 1\rangle,\langle 0\rangle\rangle\}}), (a^5_{\langle\langle 1\rangle,\langle 1\rangle\rangle})_{\{\langle\langle 1\rangle,\langle 1\rangle\rangle\}}),$$

$$F\text{−type}_{\mathcal{N}_5}(a^5_{\langle\langle 1\rangle,\langle 1\rangle\rangle}) \notin \{F\text{−type}_{\mathcal{N}_4}(c) : c \in \mathcal{N}_4\}.$$

We may again obtain that τ_4 and τ_5 preserve the satisfaction with respect to Σ^1_1 formulas.

Let

$$\tau_{3,5} = \tau_4 \circ \tau_3 : \mathcal{N}_3 \to \mathcal{N}_5.$$

Again

$$b^5_v = \tau_{3,5}(b^3_v),$$

for $v \in g_0^{(0)}$. And again

$$a^5_{\langle\langle 1\rangle,\langle 0\rangle\rangle} = \tau_4(a^5_{\langle\langle 1\rangle,\langle 0\rangle\rangle}),$$
$$a^5_{\langle 0\rangle} = \tau_{3,5}(a^3_0),$$
$$a^5_{\langle\langle 0\rangle,\langle 0\rangle\rangle} = \tau_{3,5}(a^3_{\langle\langle 0\rangle,\langle 0\rangle\rangle}),$$
$$a^5_{\langle\langle 1\rangle,\langle 0\rangle\rangle} = \tau_{3,5}(a^3_{\langle\langle 1\rangle,\langle 0\rangle\rangle}),$$
$$a^5_{\langle\langle 1\rangle,\langle 1\rangle\rangle} = \tau_{3,5}(a^3_{\langle\langle 1\rangle,\langle 1\rangle\rangle}),$$
$$a^5_{\langle 1\rangle} = \tau_{3,5}(a^3_{\langle 1\rangle}).$$

By the preservation of the relevant Σ^1_1 formulas we have

(a')

$$\mathcal{N}_5 \models A((a^5_{\vec{s}(0,0)})_{\vec{s}\in f_0}, (b^5_{v(0,0)})_{v\in g_0}, (a^5_{\vec{s}(0,0)})_{\vec{s}\in f_1});$$

(b')

$$\langle \mathcal{N}_5; a^5_{\langle 0\rangle}\rangle \models A_F(a^5_{\langle 1\rangle});$$

(c')

$$\langle \mathcal{N}_5; (a^5_{\langle 0\rangle})_{\{\langle 0\rangle\}}, (a^5_{\langle 1\rangle})_{\{\langle 1\rangle\}}, (b^5_v)_{v\in g_0^{(0)}}\rangle \models A_F(a^5_{\langle\langle 0\rangle,\langle 0\rangle\rangle});$$

(d')

$$\langle \mathcal{N}_5; (a^5_{\langle 0\rangle})_{\{\langle 0\rangle\}}, (a^5_{\langle 1\rangle})_{\{\langle 1\rangle\}}, (b^5_v)_{v\in g_0^{(0)}}, (a^5_{\langle\langle 0\rangle,\langle 0\rangle\rangle})_{\{\langle\langle 0\rangle,\langle 0\rangle\rangle\}}\rangle \models A_F(a^5_{\langle\langle 0\rangle,\langle 1\rangle\rangle});$$

(e')

$$\langle \mathcal{N}_5; (a^5_{\langle 0\rangle})_{\{\langle 0\rangle\}}, (a^5_{\langle 1\rangle})_{\{\langle 1\rangle\}}, (b^5_v)_{v\in g_0^{(0)}}, ..., (a^5_{\langle\langle 0\rangle,\langle 1\rangle\rangle})_{\{\langle\langle 0\rangle,\langle 1\rangle\rangle\}}\rangle \models A_F(a^5_{\langle\langle 1\rangle,\langle 0\rangle\rangle});$$

(g')

$$\langle \mathcal{N}_5; (a^5_{\langle 0\rangle})_{\{\langle 0\rangle\}}, (a^5_{\langle 1\rangle})_{\{\langle 1\rangle\}}, ...(a^5_{\langle\langle 1\rangle,\langle 0\rangle\rangle})_{\{\langle\langle 1\rangle,\langle 0\rangle\rangle\}}\rangle \models A_F(a^5_{\langle\langle 1\rangle,\langle 1\rangle\rangle}).$$

Finally with $\mathcal{N}_5$ and the various points indicated above in place, we can return to the task of "splitting" the indicated sequence in $\mathbb{Q}_0$. We can let

$$\langle 0\rangle <^0 \langle 1\rangle$$

define the ordering on 2^1 and

$$\langle\langle 0\rangle,\langle 0\rangle\rangle <^1 \langle\langle 0\rangle,\langle 1\rangle\rangle <^1 \langle\langle 1\rangle,\langle 0\rangle\rangle <^1 \langle\langle 1\rangle,\langle 1\rangle\rangle$$

define the ordering on $2^1 \times 2^1$. Taking B to be any Σ^1_1 set describing enough about $\mathcal{N}_5$ and applying 11.2 we finish.

$\square$

Definition For $\vec{s} \in f_i = 2^{l_0} \times 2^{l_1} \times ... \times 2^{l_i}$ and $\vec{t} \in 2^{k_0} \times 2^{k_1} \times ... \times 2^{k_j}$ some $j \geq i$, we define

$$\vec{s} * \vec{t} \in 2^{l_0+k_0} \times 2^{l_1+k_1} \times ... \times 2^{l_i+k_i}$$

by the specification that at each $m \leq i$

$$(\vec{s} * \vec{t})_m = (\vec{s})_m^\frown (\vec{t})_m,$$

the concatenation of the sequences $(\vec{s})_m$ and $(\vec{t})_m$. For

$$v = (\vec{s}, k) \in f_i \times \omega,$$

we let

$$v * \vec{t} = (\vec{s} * \vec{t}, k).$$

Definition For elements in $\mathbb{Q}_0$

$$p = \langle A, (a_{\vec{s}})_{\vec{s} \in f_0}, (b_v)_{v \in g_0}, (a_{\vec{s}})_{\vec{s} \in f_1}, ...(b_v)_{v \in g_n} \rangle$$

and

$$\bar{p} = \langle \bar{A}, (\bar{a}_{\vec{s}})_{\vec{s} \in \bar{f}_0}, (\bar{b}_v)_{v \in \bar{g}_0}, (\bar{a}_{\vec{s}})_{\vec{s} \in \bar{f}_1}, ...(\bar{b}_v)_{v \in \bar{g}_{\bar{n}}} \rangle$$

we set $\bar{p} \leq p$ if the following all take place:-

(i) $n \leq \bar{n}$;

(ii) there are $l_0, l_1, ..., l_n$, $k_0, ..., k_n$, such that for each $i \leq n$ we have $f_i = 2^{l_0} \times 2^{l_1} \times ... \times 2^{l_i}$ and $\bar{f}_i = 2^{l_0+k_0} \times 2^{l_1+k_1} \times ... \times 2^{l_i+k_i}$;

(iii) for $l_0, l_1, ..., l_n$, $k_0, ..., k_n$ as in (ii) we have that each

$$\bar{g}_i \supset \{v * \vec{t} : \vec{t} \in 2^{k_0} \times ... \times 2^{k_i}, v \in g_i\};$$

(iv) for any

$$\mathcal{M} \models \bar{A}((\bar{a}_{\vec{s}})_{\vec{s} \in \bar{f}_0}, (\bar{b}_v)_{v \in \bar{g}_0}, (\bar{a}_{\vec{s}})_{\vec{s} \in \bar{f}_1}, ...(\bar{b}_v)_{v \in \bar{g}_{\bar{n}}})$$

and $\vec{t} \in 2^{k_0} \times ...2^{k_n}$ we have

$$\mathcal{M} \models A((a_{\vec{s} * \vec{t}})_{\vec{s} \in f_0}, (b_{v * \vec{t}})_{v \in g_0}, (a_{\vec{s} * \vec{t}})_{\vec{s} \in f_1}, ...(b_{v * \vec{t}})_{v \in g_n}).$$

Lemma 11.4 *Fix*

$$p = \langle A, (a_{\vec{s}})_{\vec{s} \in f_0}, (b_v)_{v \in g_0}, ..., (a_{\vec{s}})_{\vec{s} \in f_n}, (b_v)_{v \in g_n}) \in \mathbb{Q}_0,$$

each $f_i = 2^{l_0} \times 2^{l_1} \times ...2^{l_i}$. *Suppose* $\vec{k} = \langle k_0, k_1, ..., k_n$ *and at each* $i \leq n$ *we have*

$$\bar{f}_i = \{\vec{s} * \vec{t} : \vec{s} \in f_i, \vec{t} \in 2^{k_0} \times 2^{k_1} \times ..2^{k_i}\},$$

$$\bar{g}_i = \{v * \vec{t} : v \in g_i, \vec{t} \in 2^{k_0} \times 2^{k_1} \times ..2^{k_i}\}.$$

Then we may find $\bar{p} \leq p$ *in* $\mathbb{Q}_0$ *of the form*

$$\bar{p} = \langle \bar{A}, (a_{\vec{s}})_{\vec{s} \in \bar{f}_0}, (b_v)_{v \in \bar{g}_0}, ..., (a_{\vec{s}})_{\vec{s} \in \bar{f}_n}, (b_v)_{v \in \bar{g}_n} \rangle.$$

Proof By repeated applications of 11.3.

□

Lemma 11.5 *If*

$$p = \langle A, (a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (a_{\vec{s}})_{\vec{s}\in f_n}, (b_v)_{v\in g_n}\rangle \in \mathbb{Q}_0,$$

then there is some

$$\bar{p} = \langle \bar{A}, ((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (a_{\vec{s}})_{\vec{s}\in f_n}, (b_v)_{v\in g_n}, (a_{\vec{s}})_{\vec{s}\in f_{n+1}}, (b_v)_{v\in g_{n+1}}\rangle \leq p$$

again in $\mathbb{Q}_0$.

Proof Let $l_0, l_1, ..., l_n$ be such that each $f_i = 2^{l_0} \times 2^{l_1} \times ... \times 2^{l_i}$. We then take $l_{n+1} = 0$, $f_{n+1} = 2^{l_0} \times 2^{l_1} \times ... \times 2^{l_n} \times 2^{l_{n+1}} = f_n \times \{0\}$, and $g_{n+1} = 0$.

Fix some $\mathcal{M} \models X_F$ in $\mathrm{Mod}_\Omega(\sigma)$ with

$$\mathcal{M} \models A((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (a_{\vec{s}})_{\vec{s}\in f_n}, (b_v)_{v\in g_n})).$$

By the definition of X_F we have that for all $\vec{b} \in \mathcal{M}$ there is some $a \in \mathcal{M}$ with

$$\langle \mathcal{M}; \vec{b}\rangle \models A_F(a).$$

Applying this repeatedly we may find $(a_{\vec{s}})_{\vec{s}\in f_{n+1}}$ and a linear ordering $<^{n+1}$ such that at each $\vec{t} \in f_{n+1}$ we have

$$\langle \mathcal{M}; (a_{\vec{s}})_{\vec{s}\in f_0}, ..., (b_v)_{v\in g_n}, (a_{\vec{s}})_{\vec{s}<^{n+1}\vec{t}}\rangle \models A_F(a_{\vec{t}}).$$

Thus we may let A' just be the Σ^1_1 given by

$$\mathcal{N} \models A'((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (a_{\vec{s}})_{\vec{s}\in f_n}, (b_v)_{v\in g_n}, (a_{\vec{s}})_{\vec{s}\in f_{n+1}})$$

if and only if

$$\mathcal{N} \models A((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (a_{\vec{s}})_{\vec{s}\in f_n}, (b_v)_{v\in g_n}))$$

and for each $\vec{t} \in f_{n+1}$ we have

$$\langle \mathcal{N}; (a_{\vec{s}})_{\vec{s}\in f_0}, ..., (b_v)_{v\in g_n}, (a_{\vec{s}})_{\vec{s}<^{n+1}\vec{t}}\rangle \models A_F(a_{\vec{t}}).$$

Applying 11.2 we can find $\bar{A} \subset A'$ as required.

□

Lemma 11.6 *Let*

$$p = \langle A, (a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (a_{\vec{s}})_{\vec{s}\in f_n}, (b_v)_{v\in g_n}\rangle \in \mathbb{Q}_0$$

and

$$\psi(x_0, x_1, ..., x_{m_0}, z_0, z_1, ..., z_{m_1}, y) \in F.$$

Then there is

$$\bar{p} = \langle \bar{A}, (a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in \bar{g}_0}, ..., (a_{\vec{s}})_{\vec{s}\in f_n}, (b_v)_{v\in \bar{g}_n} \rangle \in \mathbb{Q}_0$$

such that:

(i) $\bar{p} \leq p$;

(ii) for each $i \leq n$ *and* $\vec{c} = \langle c_0, c_1, ..., c_{m_0} \rangle$ *in*

$$\bigcup_{j\leq i} \{a_{\vec{s}} | \vec{s} \in f_j\},$$

$\vec{d} = \langle d_0, d_1, ..., d_{m_1} \rangle$ *in*

$$\bigcup_{j\leq i} \{b_v | v \in g_j\}$$

there is some $v \in \bar{g}_i$ *such that either*

(a) for all $\mathcal{M} \models \bar{A}((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in \bar{g}_0}, ..., (a_{\vec{s}})_{\vec{s}\in f_n}, (b_v)_{v\in \bar{g}_n})$ *we have*

$$\mathcal{M} \models \forall y \neg \psi(\vec{c}, \vec{d}, y),$$

or

(b) for all $\mathcal{M} \models \bar{A}((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in \bar{g}_0}, ..., (a_{\vec{s}})_{\vec{s}\in f_n}, (b_v)_{v\in \bar{g}_n})$ *we have*

$$\mathcal{M} \models \psi(\vec{c}, \vec{d}, b_v).$$

Proof We go to $\mathcal{M} \in \mathrm{Mod}_\Omega(\sigma)$ with $\omega_1^{\mathrm{ck}(\mathcal{M})} = \omega_1^{\mathrm{ck}}$ and $\mathcal{M} \models A((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (a_{\vec{s}})_{\vec{s}\in f_n}, (b_v)_{v\in g_n})$. By repeated applications of 8.6 and 6.10 we can find $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2, ..., \mathcal{M}_{2^{l_0+...l_n}} = \mathcal{M}$ such that:

(a) each

$$\omega_1^{\mathrm{ck}(\mathcal{M}_i, \mathcal{M}_j)} = \omega_1^{\mathrm{ck}};$$

(b) there are ω_1^{ck}-elementary embeddings

$$\tau_{i,j} : \mathcal{M}_i \to \mathcal{N}_j$$

for $i \leq j$ which commute in the sense that if $i \leq j \leq k$ then

$$\tau_{i,k} = \tau_{j,k} \circ \tau_{i,j};$$

(c) if $\vec{t}$ occupies the jth position in f_i under the ordering $<^i$ then

(c.i) $\tau_{2^{l_0+...l_i}+j, 2^{l_0+...l_n}}[\mathcal{M}_i]$ includes $(a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (a_{\vec{s}})_{\vec{s}\in f_i}, (b_v)_{v\in g_i}, (a_{\vec{s}})_{\vec{s}<^i \vec{t}}$;

(c.ii) but F-type$_{\mathcal{M}}(a_{\vec{t}})$ is not realized in $\mathcal{M}_j$.

Now we just fold into A all the information about which $\vec{c}, \vec{d}$ as above satisfy

$$\mathcal{M} \models \exists y \psi(\vec{c}, \vec{d}, y).$$

If

$$\vec{c} = \langle c_0, c_1, ..., c_{m_0} \rangle \in \bigcup_{j \leq i} \{a_{\vec{s}} | \vec{s} \in f_j\},$$

$$\vec{d} = \langle d_0, d_1, ..., d_{m_1} \rangle \in \bigcup_{j \leq i} \{b_v | v \in g_j\}$$

and $\mathcal{M} \models \exists y \psi(\vec{c}, \vec{d}, y)$ then we can apply the elementarity of the map $\tau_{2^{l_0+...l_i}, 2^{l_0+...l_n}}$ to find some $b_v \in \tau_{2^{l_0+...l_i}, 2^{l_0+...+l_n}}[\mathcal{M}_{2^{l_0+...l_i}}]$ with

$$\mathcal{M} \models \psi(\vec{c}, \vec{d}, b_v).$$

Thus we can choose to add information about b_v to our description of $\mathcal{M}$, at last obtaining a set $\bar{A}$ as indicated. 11.1 ensures that this process does not lead to a violation of (v) in the definition of $\mathbb{Q}_0$.

□

We need to decide on a filter for $\mathbb{Q}_0$ meeting enough dense sets. For the sake of concreteness and using that $\mathbb{Q}_0 \in L_{\omega_1^{\mathrm{ck}}+2\cdot\omega}$ I do so exactly in the next paragraph. It should be clear that any reasonably generic filter on $\mathbb{Q}_0$ will do.

Notation From now until the end of the section fix $G_0 \subset \mathbb{Q}_0$ an $L_{\omega_2^{\mathrm{ck}}}$-generic filter. That is to say:

(i) [G_0 is a filter]
  (a) if $p \in G_0$ and $p \leq q$, then $q \in G_0$;
  (b) if $p_1, p_2 \in G_0$ then there is some $p \in G_0$ with $p \leq p_1, p_2$;

(ii) [$L_{\omega_2^{\mathrm{ck}}}$-generic] if $D \subset \mathbb{Q}_0$ is in $L_{\omega_2^{\mathrm{ck}}}$ and is dense in the sense that

$$\forall q \in \mathbb{Q}_0 \exists p \in D(p \leq q)$$

then we have

$$D \cap G_0 \neq 0.$$

We may find such a filter for the usual reason: Letting $(D_n)_{n\in\omega}$ enumerate the dense sets in $L_{\omega_2^{\mathrm{ck}}}$ we find $p_0 \in D_0$ and then successively produce $p_{n+1} \in D_{n+1}$ with $p_{n+1} \leq p_n$ and finally let $G_0 = \{q : \exists n(p_n \leq q)\}$. The rationale for doing so is slightly more subtle. Basically we wish to use G_0 to associate to each $\vec{x} \in (2^\omega)^\omega$ a corresponding model $\mathcal{M}_{\vec{x}}$ of σ. In order for this to work out well we need that G_0 has taken certain decisions correctly; all these correct choices will be seen by 11.3, 11.5, 11.4, and 11.6 above to correspond to G_0 meeting various dense sets; all these dense sets are easily definable and obviously present already in $L_{\omega_2^{\mathrm{ck}}}$.

Of course we could build G_0 sufficient to our needs by hand, laboriously meeting each specific requirement in some kind of explicit procedure. It is however less fuss notationally to simply summon G_0 into existence as above.

Definition For $\vec{x} = (x_n)_{n\in\omega} \in (2^\omega)^\omega$ we define a structure $\mathcal{M}_{\vec{x}}$. We first just describe its underlying set and then wait until after lemma 11.7 to describe how it interprets the relations of $\mathcal{L}$.

$|\mathcal{M}_{\vec{x}}|$, the universe of $\mathcal{M}_{\vec{x}}$, consists in

$$\{a_{\langle y_0,y_1,..,y_m\rangle} | m \in \omega, \forall i \leq m(y_i E_0 x_i)\}$$

$$\dot\cup$$

$$\{b_w | w = (\langle y_0, y_1, ..., y_m\rangle, k) \text{ some } k, m, \vec{y} \text{ with } \forall i \leq m(y_i E_0 x_i)\}.$$

For the purposes of this definition one should think of $a_{\vec{y}} \neq a_{\vec{z}}$ any $\vec{y} \neq \vec{z}$, $b_v \neq b_w$ any $w \neq v$, and $a_{\vec{y}} \neq b_w$ any y, w.

Definition For $\vec{y} = \langle y_0, y_1, ..., y_m\rangle$, each $y_i \in 2^\omega$, and for $n \geq m$ and $\vec{l} = (l_0, l_1, ..., l_n) \in \omega^{n+1}$, we let

$$\pi_{\vec{l}}(\vec{y}) = \langle y_0|_{l_0}, y_1|_{l_1}, ..., y_m|_{l_m}\rangle.$$

Similarly if $w = (\vec{y}, k)$ then

$$\pi_{\vec{l}}(w) = (\langle y_0|_{l_0}, y_1|_{l_1}, ..., y_m|_{l_m}\rangle, k).$$

Definition A subset $\mathcal{F} \subset G_0$ of our filter is *cofinal* if there is some $q_0 \in G_0$ such that

$$\forall p \leq q_0(p \in G_0 \Rightarrow p \in \mathcal{F}).$$

We can say then that "such and such" happens cofinally often in G_0 if the set $\{p \in G_0 :$ "such and such happens for" $p\}$ is a cofinal subset of our filter.

Lemma 11.7 *For $R \in \mathcal{L}$, $\vec{x} \in (2^\omega)^\omega$, and*

$$a_{\vec{y}^0}, a_{\vec{y}^1}, ..., a_{\vec{y}^{k(0)}}, b_{w_0}, b_{w_1}, ..., b_{w_{k(1)}}$$

in $|\mathcal{M}_{\vec{x}}|$ we have one of the following two possibilities:

(i) for cofinally many

$$p = \langle A, (a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (a_{\vec{s}})_{\vec{s}\in f_n}, (b_v)_{v\in g_n}\rangle \in G_0,$$

with each $f_j = 2^{l_0} \times 2^{l_1} \times ... \times 2^{l_j}$, and $\vec{l} = \langle l_0, l_1, ..., l_n\rangle$, we have that for all

$$\mathcal{M} \models A((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (b_v)_{v\in g_n})$$

it is the case that

$$\mathcal{M} \models R(a_{\pi_{\vec{l}}(\vec{y}^0)}, ..., a_{\pi_{\vec{l}}(\vec{y}^{k(0)})}, b_{\pi_{\vec{l}}(w_0)}, ..., b_{\pi_{\vec{l}}(w_{k(1)})});$$

or

(ii) for cofinally many

$$p = \langle A, (a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (a_{\vec{s}})_{\vec{s}\in f_n}, (b_v)_{v\in g_n})\rangle \in G_0,$$

with each $f_j = 2^{l_0} \times 2^{l_1} \times ... \times 2^{l_j}$, *and* $\vec{l} = \langle l_0, l_1, ..., l_n\rangle$, *for all*

$$\mathcal{M} \models A((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (b_v)_{v\in g_n})$$

we have

$$\mathcal{M} \models \neg R(a_{\pi_{\vec{l}}(\vec{y}^0)}, ..., a_{\pi_{\vec{l}}(\vec{y}^{k(0)})}, b_{\pi_{\vec{l}}(w_0)}, ..., b_{\pi_{\vec{l}}(w_{k(1)})}).$$

Proof First let us go to $\vec{l} = \langle l_0, l_1, ..., l_n\rangle$ large enough to absorb all the relevant disagreements: That is to say:-

(a) for each $\vec{y}^i = \langle y^i_0, y^i_1, ..., y^i_{m(i)}\rangle$ we have

$$x_j(l) = y^i_j(l)$$

all $l \geq l_i$ and $j \leq m(i)$;

(b) for each $w_i = (\langle \hat{y}^i_0, \hat{y}^i_1, ..., \hat{y}^i_{\hat{m}(i)}\rangle, \hat{k}_i)$ we have

$$x_j(l) = \hat{y}^i_j(l)$$

all $l \geq l_i$ and $j \leq \hat{m}(i)$.

After possibly increasing $\vec{l}$ we may assume that there is a condition

$$q_0 = \langle B, (a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (a_{\vec{s}})_{\vec{s}\in f_n}, (b_v)_{v\in g_n}\rangle \in G_0,$$

with each $f_i = 2^{l_0} \times ... 2^{l_i}$, each $g_i \supset f_i \times \{\hat{k}_0, \hat{k}_1, ...\}$, deciding all the relevant questions about R: That is to say, for all

$$\vec{c} \in \{a_{\vec{s}}, b_v : \vec{s} \in f_0 \cup f_1 \cup ... \cup f_n, v \in g_0 \cup g_1 \cup ... \cup g_n\},$$

$$\mathcal{M}_0 \models B((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (b_v)_{v\in g_n}),$$

$$\mathcal{M}_1 \models B((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (b_v)_{v\in g_n}),$$

we have

$$\mathcal{M}_0 \models R(\vec{c})$$

if and only if

$$\mathcal{M}_1 \models R(\vec{c}).$$

Now for any $\bar{n} \geq n$ and

$$\vec{k} = \langle k_0, k_1, ..., k_{\bar{n}}\rangle$$

with $k_i \geq l_i$ all $i \leq n$, and any

$$p = \langle A, (a_{\vec{s}})_{\vec{s} \in \bar{f}_0}, (b_v)_{v \in \bar{g}_0}, (a_{\vec{s}})_{\vec{s} \in \bar{f}_1}, ...(b_v)_{v \in \bar{g}_{\bar{n}}} \rangle$$

with $p \leq q_0$ and each $\bar{f}_i = 2^{k_0} \times ...2^{k_i}$ we may discharge the assumptions (a) and (b) above to find some $\vec{t} \in 2^{k_0 - l_0} \times 2^{k_1 - l_1} \times ... \times 2^{k_n - l_n}$ such that each

$$\pi_{\vec{l}}(\vec{y}^i) * \vec{t} = \pi_{\vec{k}}(\vec{y}^i),$$

$$\pi_{\vec{l}}(w^i) * \vec{t} = \pi_{\vec{k}}(w_i).$$

Then for any

$$\mathcal{M}_B \models B((a_{\vec{s}})_{\vec{s} \in f_0}, (b_v)_{v \in g_0}, ..., (b_v)_{v \in g_n}),$$

$$\mathcal{M}_A \models A(a_{\vec{s}})_{\vec{s} \in \bar{f}_0}, (b_v)_{v \in \bar{g}_0}, (a_{\vec{s}})_{\vec{s} \in \bar{f}_1}, ...(b_v)_{v \in \bar{g}_{\bar{n}}}),$$

we have by $p \leq q_0$ that

$$\mathcal{M}_B \models R(a_{\pi_{\vec{l}}(\vec{y}^0)}, ..., a_{\pi_{\vec{l}}(\vec{y}^{k(0)})}, a_{\pi_{\vec{l}}(w_0)}, ..., a_{\pi_{\vec{l}}(w_{k(1)})})$$

if and only if

$$\mathcal{M}_A \models R(a_{\pi_{\vec{k}}(\vec{y}^0)}, ..., a_{\pi_{\vec{k}}(\vec{y}^{k(0)})}, b_{\pi_{\vec{k}}(w_0)}, ..., b_{\pi_{\vec{k}}(w_{k(1)})}).$$

$\square$

Exactly the same proof extends to arbitrary formulas in $L_{\omega_1^{\mathrm{ck}}}$.

Lemma 11.8 *For* $\psi \in \mathcal{L}_{\omega_1, \omega} \cap L_{\omega_1^{\mathrm{ck}}}$, $\vec{x} \in (2^\omega)^\omega$, *and*

$$a_{\vec{y}^0}, a_{\vec{y}^1}, ..., a_{\vec{y}^{k(0)}}, b_{w_0}, b_{w_1}, ..., b_{w_{k(1)}}$$

in $|\mathcal{M}_{\vec{x}}|$ *we have one of the following two possibilities:*

(i) for cofinally many

$$p = \langle A, (a_{\vec{s}})_{\vec{s} \in f_0}, (b_v)_{v \in g_0}, ..., (a_{\vec{s}})_{\vec{s} \in f_n}, (b_v)_{v \in g_n} \rangle \in G_0,$$

with each $f_j = 2^{l_0} \times 2^{l_1} \times ... \times 2^{l_j}$, *and* $\vec{l} = \langle l_0, l_1, ..., l_n \rangle$, *we have that for all*

$$\mathcal{M} \models A((a_{\vec{s}})_{\vec{s} \in f_0}, (b_v)_{v \in g_0}, ..., (b_v)_{v \in g_n})$$

it is the case that

$$\mathcal{M} \models \psi(a_{\pi_{\vec{l}}(\vec{y}^0)}, ..., a_{\pi_{\vec{l}}(\vec{y}^{k(0)})}, b_{\pi_{\vec{l}}(w_0)}, ..., b_{\pi_{\vec{l}}(w_{k(1)})});$$

or

(ii) for cofinally many

$$p = \langle A, (a_{\vec{s}})_{\vec{s} \in f_0}, (b_v)_{v \in g_0}, ..., (a_{\vec{s}})_{\vec{s} \in f_n}, (b_v)_{v \in g_n} \rangle \in G_0,$$

with each $f_j = 2^{l_0} \times 2^{l_1} \times ... \times 2^{l_j}$, *and* $\vec{l} = \langle l_0, l_1, ..., l_n \rangle$, *for all*

$$\mathcal{M} \models A((a_{\vec{s}})_{\vec{s} \in f_0}, (b_v)_{v \in g_0}, ..., (b_v)_{v \in g_n})$$

we have

$$\mathcal{M} \models \neg\psi(a_{\pi_{\vec{l}}(\vec{y}^0)}, ..., a_{\pi_{\vec{l}}(\vec{y}^{k(0)})}, b_{\pi_{\vec{l}}(w_0)}, ..., b_{\pi_{\vec{l}}(w_{k(1)})}).$$

Definition With 11.7 in hand we can turn $\mathcal{M}_{\vec{x}}$ into an $\mathcal{L}$-structure as follows. For R a relation symbol in the language,

$$\mathcal{M}_{\vec{x}} \models R(a_{\vec{y}^0}, a_{\vec{y}^1}, ..., b_{w_0}, b_{w_1}, ...)$$

if and only if the following two things happen:

(a) $a_{\vec{y}^0}, a_{\vec{y}^1}, ..., b_{w_0}, b_{w_1}, ...$ are in $|\mathcal{M}_{\vec{x}}|$ (as given in the definition prior to 11.7);

(b) for all sufficiently large $\vec{l}$ and for cofinally many for cofinally many

$$p = \langle A, (a_{\vec{s}})_{\vec{s} \in f_0}, (b_v)_{v \in g_0}, ..., (a_{\vec{s}})_{\vec{s} \in f_n}, (b_v)_{v \in g_n} \rangle \in G_0,$$

with each $f_j = 2^{l_0} \times 2^{l_1} \times ... \times 2^{l_j}$, and $\vec{l} = \langle l_0, l_1, ..., l_n \rangle$, we have that for all

$$\mathcal{M} \models A((a_{\vec{s}})_{\vec{s} \in f_0}, (b_v)_{v \in g_0}, ..., (b_v)_{v \in g_n})$$

it is the case that

$$\mathcal{M} \models R(a_{\pi_{\vec{l}}(\vec{y}^0)}, ..., a_{\pi_{\vec{l}}(\vec{y}^{k(0)})}, b_{\pi_{\vec{l}}(w_0)}, ..., b_{\pi_{\vec{l}}(w_{k(1)})}).$$

We *are* assuming $\mathcal{L}$ to have no function symbols, so this does suffice to define an $\mathcal{L}$-structure.

Lemma 11.9 *For* $\psi \in \mathcal{L}_{\omega_1,\omega} \cap L_{\omega_1^{\mathrm{ck}}}$ *and* $a_{\vec{y}^0}, a_{\vec{y}^1}, ..., b_{w_0}, b_{w_1}, ...$ *in* $|\mathcal{M}_{\vec{x}}|$, *we have*

$$\mathcal{M}_{\vec{x}} \models \psi(a_{\vec{y}^0}, a_{\vec{y}^1}, ..., b_{w_0}, b_{w_1}, ...)$$

if and only for all sufficiently large $\vec{l}$ *and cofinally many*

$$p = \langle A, (a_{\vec{s}})_{\vec{s} \in f_0}, (b_v)_{v \in g_0}, ..., (a_{\vec{s}})_{\vec{s} \in f_n}, (b_v)_{v \in g_n} \rangle \in G_0,$$

with each $f_j = 2^{l_0} \times 2^{l_1} \times ... \times 2^{l_j}$, *and* $\vec{l} = \langle l_0, l_1, ..., l_n \rangle$, *we have that:*
for all $\mathcal{M} \in \mathrm{Mod}_\Omega(\sigma)$, *if*

$$\mathcal{M} \models A((a_{\vec{s}})_{\vec{s} \in f_0}, (b_v)_{v \in g_0}, ..., (b_v)_{v \in g_n}),$$

then

$$\mathcal{M} \models \psi(a_{\pi_{\vec{l}}(\vec{y}^0)}, ..., a_{\pi_{\vec{l}}(\vec{y}^{k(0)})}, b_{\pi_{\vec{l}}(w_0)}, ..., b_{\pi_{\vec{l}}(w_{k(1)})}).$$

Proof This follows in the usual way by induction on the logical complexity of ψ and 11.8. In this induction on complexity, the passage from ψ to $\exists z\psi$ uses 11.6.

$\Box$

Lemma 11.10 *If*

$$\vec{x}(E_0)^\omega\vec{y},$$

then

$$\mathcal{M}_{\vec{x}} = \mathcal{M}_{\vec{y}}.$$

Proof Immediate from the definitions.

$\Box$

Lemma 11.11 *If*

$$\mathcal{M}_{\vec{x}} \cong \mathcal{M}_{\vec{y}},$$

then

$$\vec{x}(E_0)^\omega\vec{y}.$$

Proof Suppose that $\vec{x} = \langle x_0, x_1, ...\rangle$ and $\vec{z} = \langle z_0, z_1, ...\rangle$ are not $(E_0)^\omega$ equivalent. We will show that the resulting models are not isomorphic. In fact we will even show that the sets of F-types realized in the two models are distinct.

Fix k with x_k not E_0 equivalent to z_k. Let $\vec{y} = \langle x_0, x_1, ..., x_k\rangle$. I wish to claim that $F\text{-type}_{\mathcal{M}_{\vec{x}}}(a_{\vec{y}})$ does not appear in $\{F\text{-type}_{\mathcal{M}_{\vec{z}}}(c) : c \in \mathcal{M}_{\vec{z}}\}$. This splits into two cases.

Claim(1) For all $a_{\vec{r}} \in |\mathcal{M}_{\vec{z}}|$ we have $F\text{-type}_{\mathcal{M}_{\vec{x}}}(a_{\vec{y}}) \neq F\text{-type}_{\mathcal{M}_{\vec{z}}}(a_{\vec{r}})$.

Proof of claim: We may choose $\vec{l} = \langle l_0, l_1, ..., l_n\rangle$, $n \geq k$, $f_0 = 2^{l_0}$, $f_1 = 2^{l_0} \times 2^{l_1}$, ..., $f_n = 2^{l_0} \times ...2^{l_n}$, and

$$p = \langle A, (a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (a_{\vec{s}})_{\vec{s}\in f_n}, (b_v)_{v\in g_n}\rangle \in G_0,$$

with

$$\pi_{\vec{l}}(\vec{y}) \neq \pi_{\vec{l}}(\vec{r}).$$

Then for some $\psi_{\pi_{\vec{l}}(\vec{y}),\pi_{\vec{l}}(\vec{r})}$ it follows from the definition of the ordering on $\mathbb{Q}_0$ and 11.9 that

$$\mathcal{M}_{\vec{x}} \models \psi_{\pi_{\vec{l}}(\vec{y}),\pi_{\vec{l}}(\vec{r})}(a_{\vec{y}})$$

but

$$\mathcal{M}_{\vec{z}} \models \neg\psi_{\pi_{\vec{l}}(\vec{y}),\pi_{\vec{l}}(\vec{r})}(a_{\vec{r}}).$$

($\Box$Claim(1))

Claim(2) For all $b_w \in |\mathcal{M}_{\vec{z}}|$ we have $F\text{-type}_{\mathcal{M}_{\vec{x}}}(a_{\vec{y}}) \neq F\text{-type}_{\mathcal{M}_{\vec{z}}}(b_w)$.

Proof of claim: First of all we may assume that there are cofinally

$$p = \langle A, (a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (a_{\vec{s}})_{\vec{s}\in f_n}, (b_v)_{v\in g_n}\rangle \in G_0,$$

such that for some assignment
$$\vec{s} \mapsto \varphi_{\vec{s}}$$
of formulas in F to sequences in $\vec{s} \in \bigcup_{i \leq n} f_i$ such that for all
$$\mathcal{M} \models A((a_{\vec{s}})_{\vec{s} \in f_0}, (b_v)_{v \in g_0}, ..., (b_v)_{v \in g_n})$$
we have
$$\mathcal{M} \models \varphi_{\vec{s}}(a_{\vec{s}}) \wedge \neg\varphi_{\vec{s}}(b_{\pi_{\vec{l}}(w)});$$
the point is that if this failed then we can assume by genericity that we have
$$p = \langle A, (a_{\vec{s}})_{\vec{s} \in f_0}, (b_v)_{v \in g_0}, ..., (a_{\vec{s}})_{\vec{s} \in f_n}, (b_v)_{v \in g_n} \rangle \in G_0$$
and some fixed $\vec{t} \in \bigcup_{i \leq n} f_i$ such that
$$\mathcal{M} \models A((a_{\vec{s}})_{\vec{s} \in f_0}, (b_v)_{v \in g_0}, ..., (b_v)_{v \in g_n})$$
we have
$$\mathcal{M} \models \bigwedge_{\varphi \in F} (\varphi(a_{\vec{t}}) \Leftrightarrow \varphi(b_{\pi_{\vec{l}}(w)})),$$
and this puts us back exactly into claim(1).

But now for $\vec{t} = \pi_{\vec{l}}(\vec{y})$ we are again able to apply 11.9 and the definition of the ordering on $\mathbb{Q}_0$ to get
$$\mathcal{M}_{\vec{x}} \models \varphi_{\vec{t}}(a_{\vec{y}})$$
but
$$\mathcal{M}_{\vec{x}} \models \neg\varphi_{\vec{t}}(b_w).$$

(□Claim(2))

□

Lemma 11.12 *There is a Borel function*
$$\theta : (2^\omega)^\omega \to \mathrm{Mod}(\sigma)$$
such that for any $\vec{x} \in (2^\omega)^\omega$ we have
$$\theta(\vec{x}) \cong \mathcal{M}_{\vec{x}}.$$

Proof This follows by the uniformization theorem for Borel sets with countable sections, as found in 18C of [17], since $|\mathcal{M}_{\vec{x}}|$ is a countable and uniformly $\Delta^1_1(x)$ set and all the relations on $\mathcal{M}_{\vec{x}}$ are again uniformly $\Delta^1_1(x)$.

□

Corollary 11.13 *Under the assumptions of case(I), we have*
$$(E_0)^\omega \leq_B \cong|_{\mathrm{Mod}(\sigma)}.$$

Proof By 11.10, 11.11, and 11.12.

□

12 Proof for case(IIb)

We leave case(IIa) to last, and instead fix F and H_F as in the assumptions of case(IIb). We let Y_F be the set of $\mathcal{M} \in \mathrm{Mod}(\sigma)$ such that for all $\vec{a} \in T^F_{\mathcal{M}}$ and fragments $H \in L_{\omega_1^{\mathrm{ck}}}$ we have

$$\sup_{\vec{b}\in\mathcal{M}} \rho^H_{F,\mathcal{M}}(\vec{a},\vec{b}) \geq \omega.$$

Our case assumption is

$$Y_F \neq 0.$$

Lemma 12.1 *Given $\mathcal{M} \in \mathrm{Mod}(\sigma)$, the set of $(H, \alpha, \vec{a}, \vec{b})$ such that*
(i) $\alpha \in \omega_1^{\mathrm{ck}}$,
(ii) H is a fragment including H_F in $L_{\omega_1^{\mathrm{ck}}}$,
(iii) $\vec{a} \in T^F_{\mathcal{M}}$,
(iv) $\vec{b} \in \mathcal{M}$, and
(v) $\rho^H_{F,\mathcal{M}}(\vec{a},\vec{b}) = \alpha$
is uniformly $\Delta_1(\mathcal{M})$ over $L_{\omega_1^{\mathrm{ck}(\mathcal{M})}}[\mathcal{M}]$.

Proof It is a routine exercise in transfinite recursion to see that (v) is Δ_1 over any admissible set containing the relevant parameters; the other clauses are trivially Δ_1.

□

Corollary 12.2 *Given $\mathcal{M} \in \mathrm{Mod}(\sigma)$, the set of $(H, \alpha, \vec{a}, \vec{b})$ such that*
(i) $\alpha \in \omega_1^{\mathrm{ck}}$,
(ii) H is a fragment including H_F in $L_{\omega_1^{\mathrm{ck}}}$,
(iii) $\vec{a} \in T^F_{\mathcal{M}}$,
(iv) $\vec{b} \in \mathcal{M}$, and
(v) for all fragments $H' \supset H$ in $L_{\omega_1^{\mathrm{ck}}}$ we have $\rho^{H'}_{F,\mathcal{M}}(\vec{a},\vec{b}) \geq \alpha$
is uniformly $\Pi_1(\mathcal{M})$ over $L_{\omega_1^{\mathrm{ck}(\mathcal{M})}}[\mathcal{M}]$.

Corollary 12.3 *Given a fragment $H \in L_{\omega_1^{\mathrm{ck}}}$ and $\alpha \in \omega_1^{\mathrm{ck}}$ we have that the set of $(\mathcal{M}, \vec{a}, \vec{b})$ such that for all fragments $H' \supset H$ in $L_{\omega_1^{\mathrm{ck}}}$ we have $\rho^{H'}_{F,\mathcal{M}}(\vec{a},\vec{b}) \geq \alpha$ is Σ^1_1*

Proof From 12.2 by Spector-Gandy.

□

Lemma 12.4 *Given fragments H_0, H_1 with $H_0 \supset H_1 \supset H_F$, and given $\mathcal{M} \in \mathrm{Mod}(\sigma)$, we have for all $\vec{a} \in T^F_{\mathcal{M}}$, $\vec{b} \in \mathcal{M}$*

$$\rho^{H_0}_{F,\mathcal{M}}(\vec{a},\vec{b}) \leq \rho^{H_1}_{F,\mathcal{M}}(\vec{a},\vec{b}).$$

Proof Since H_0 has more formulas, there are more $\vec{c}$ with

$$\rho^{H_0}_{F,\mathcal{M}}(\vec{a}^\frown\vec{c},\vec{b})=0.$$

$\square$

Corollary 12.5 *For $\mathcal{M}\in\mathrm{Mod}(\sigma)$, $\vec{a}\in T^F_{\mathcal{M}}$, $\vec{b}\in\mathcal{M}$ there is $H_0\supset H_F$ in $L_{\omega_1^{\mathrm{ck}}}$ such that for all fragments $H^*\supset H_0$ in $L_{\omega_1^{\mathrm{ck}}}$ we have*

$$\rho^{H_0}_{F,\mathcal{M}}(\vec{a},\vec{b})=\rho^{H^*}_{F,\mathcal{M}}(\vec{a},\vec{b}).$$

Definition Thus given $\mathcal{M}\in\mathrm{Mod}(\sigma)$, $\vec{a}\in T^F_{\mathcal{M}}$, and $\vec{b}\in\mathrm{Mod}(\sigma)$ we may naturally define

$$\rho^{\infty}_{F,\mathcal{M}}(\vec{a},\vec{b})$$

to be the unique γ such that for all sufficiently large fragments $H\in L_{\omega_1^{\mathrm{ck}}}$

$$\rho^{H}_{F,\mathcal{M}}(\vec{a},\vec{b})=\gamma.$$

The next lemma provides the whole reason why the $\rho^H_{F,\mathcal{M}}(\cdot,\cdot)$ rank function is useful. It states that if $\vec{b}\in\mathcal{M}$, $c\in\mathcal{M}$, and each b_i (from the finite sequence $\vec{b}=\langle b_0,b_1,...\rangle$) has lower rank than c, then the F-type of c is not definable from $\vec{b}$.

Lemma 12.6 *If*

(i) $\mathcal{M}\in\mathrm{Mod}(\sigma)$,
(ii) $\vec{a}\in T^F_{\mathcal{M}}$,
(iii) $c\in\mathcal{M}$,
(iv) $\vec{b}=\langle b_0,b_1,...,b_n\rangle\in\mathcal{M}$,
(iv) $H\supset H_F$ is a fragment in $L_{\omega_1^{\mathrm{ck}}}$, and
(vi) $\forall i\le n(\rho^H_{F,\mathcal{M}}(\vec{a},b_i)<\rho^H_{F,\mathcal{M}}(\vec{a},c))$,

then there is no $\psi\in H$ such that

$$\mathcal{M}\models\psi(\vec{a},\vec{b},c)$$

and

$$\mathcal{M}\models\forall c_0,c_1((\psi(\vec{a},\vec{b},c_0)\wedge\psi(\vec{a},\vec{b},c_1))\Rightarrow\bigwedge_{\varphi\in F}(\varphi(c_0)\Leftrightarrow\varphi(c_1))).$$

Proof We successively find $d_0,d_1,...,d_k,...$ in $\mathcal{M}$ such that for all $k\in\omega$

$$\vec{a}^\frown d_0^\frown d_1^\frown...^\frown d_k\in T^F_{\mathcal{M}}$$

and for all $i\le n$

$$\rho^H_{F,\mathcal{M}}(\vec{a}^\frown d_0^\frown d_1^\frown...^\frown d_k,b_i)<\rho^H_{F,\mathcal{M}}(\vec{a}^\frown d_0^\frown...^\frown d_k,c).$$

Suppose we have succeeded so far in finding $d_0, d_1, ..., d_k$ and we wish to produce d_{k+1}.

First if we are at a successor ordinal, with

$$\rho^H_{F,\mathcal{M}}(\vec{a}^\frown d_0^\frown ...^\frown d_k, c) = \delta + 1$$

we can by the definition of this rank function choose d_{k+1} with

$$\rho^H_{F,\mathcal{M}}(\vec{a}^\frown d_0^\frown ...^\frown d_k^\frown d_{k+1}, c) = \delta.$$

Since at each $i \leq n$ we necessarily have

$$\rho^H_{F,\mathcal{M}}(\vec{a}^\frown d_0^\frown ...^\frown d_k, b_i) > \rho^H_{F,\mathcal{M}}(\vec{a}^\frown d_0^\frown ...^\frown d_k^\frown d_{k+1}, b_i),$$

we do indeed obtain

$$\rho^H_{F,\mathcal{M}}(\vec{a}^\frown d_0^\frown d_1^\frown ...^\frown d_k^\frown d_{k+1}, b_i) < \rho^H_{F,\mathcal{M}}(\vec{a}^\frown d_0^\frown ...^\frown d_k^\frown d_{k+1}, c).$$

The other possibility is that we are at some limit ordinal with

$$\rho^H_{F,\mathcal{M}}(\vec{a}^\frown d_0^\frown ...^\frown d_k, c) = \lambda.$$

Then since $\vec{b}$ is finite we may apply the inductive assumption on $\langle d_0, d_1, ..., d_k\rangle$ and find some $\gamma < \lambda$ with each

$$\rho^H_{F,\mathcal{M}}(\vec{a}^\frown d_0^\frown ...^\frown d_k, b_i) \leq \gamma.$$

Then we can choose d_{k+1} with

$$\rho^H_{F,\mathcal{M}}(\vec{a}^\frown d_0^\frown ...^\frown d_k^\frown d_{k+1}, c) \geq \gamma.$$

This process must eventually terminate, and it can only do so with some $\vec{d}$ such that at each $i \leq n$

$$0 = \rho^H_{F,\mathcal{M}}(\vec{a}^\frown \vec{d}, b_i) < \rho^H_{F,\mathcal{M}}(\vec{a}^\frown \vec{d}, c).$$

But now whatever F-types we can define over H from $\vec{a}^\frown \vec{d}^\frown \vec{b}$ we can already define from $\vec{a}^\frown \vec{d}$.

$\square$

Corollary 12.7 *Fix $\mathcal{M} \in \mathrm{Mod}(\sigma)$, $c \in \mathcal{M}$, $\vec{b} = \langle b_0, b_1, ..., b_n\rangle \in \mathcal{M}$, $\vec{a} \in \mathcal{M}$.*

If

$$\forall i \leq n(\rho^\infty_{F,\mathcal{M}}(\vec{a}, b_i) < \rho^\infty_{F,\mathcal{M}}(\vec{a}, c))$$

then for all fragments $H \in L_{\omega_1^{\mathrm{ck}}}$ we have that F-type$_{\mathcal{M}}(c)$ is not H-definable from $\vec{a}^\frown \vec{b}$; that is to say, there is no $\psi \in H$ such that

$$\mathcal{M} \models \psi(\vec{a}, \vec{b}, c)$$

and

$$\mathcal{M} \models \forall c_0, c_1((\psi(\vec{a}, \vec{b}, c_0) \wedge \psi(\vec{a}, \vec{b}, c_1)) \Rightarrow \bigwedge_{\varphi \in F} (\varphi(c_0) \Leftrightarrow \varphi(c_1))).$$

Lemma 12.8 *$\mathcal{M}, c, \vec{b}, \vec{a}$ as in 12.7.*

For any Σ^1_1 set A with

$$\mathcal{M} \models A(\vec{a}, \vec{b}, c)$$

there exists $\mathcal{N} \in \mathrm{Mod}(\sigma)$ and $\vec{a}^*, \vec{b}^*, c_0^*, c_1^* \in \mathcal{N}$ such that

$$\mathcal{N} \models A(\vec{a}^*, \vec{b}^*, c_0^*),$$

$$\mathcal{N} \models A(\vec{a}^*, \vec{b}^*, c_1^*),$$

but F-$\mathrm{type}_{\mathcal{N}}(c_0^*) \neq F$-$\mathrm{type}_{\mathcal{N}}(c_1^*)$.

Proof Otherwise applying boundedness and 7.1, in a manner similar but easier than the proof of 8.2, we obtain some $\psi \in \mathcal{L}_{\omega_1,\omega} \cap L_{\omega_1^{\mathrm{ck}}}$ such that

(i) for all $\mathcal{N} \in \mathrm{Mod}(\sigma)$ and $\vec{a}^*, \vec{b}^*, c^* \in \mathcal{N}$,

$$\mathcal{N} \models A(\vec{a}^*, \vec{b}^*, c^*) \Rightarrow \mathcal{N} \models \psi(\vec{a}^*, \vec{b}^*, c^*);$$

(ii) for all $\mathcal{N} \in \mathrm{Mod}(\sigma)$ and $\vec{a}^*, \vec{b}^*, c_0^*, c_1^* \in \mathcal{N}$ with

$$\mathcal{N} \models \psi(\vec{a}^*, \vec{b}^*, c_0^*),$$

$$\mathcal{N} \models \psi(\vec{a}^*, \vec{b}^*, c_1^*),$$

we have F-$\mathrm{type}_{\mathcal{N}}(c_0^*) = F$-$\mathrm{type}_{\mathcal{N}}(c_1^*)$.

In particular then we have $\mathcal{M} \models A(\vec{a}, \vec{b}, c)$ and for any $d \in \mathcal{M}$

$$\mathcal{M} \models \psi(\vec{a}, \vec{b}, d) \Rightarrow \bigwedge_{\varphi \in F} (\varphi(c) \Leftrightarrow \varphi(d)).$$

Thus F-$\mathrm{type}_{\mathcal{M}}(c)$ is H-definable from $\vec{a}^\frown\vec{b}$ for any fragment $H \supset H_F$ which contains ψ; which in turn is in violation of the assumptions on $\vec{a}, \vec{b}, c$, $\mathcal{M}$, and the statement of 12.7.

$\square$

Corollary 12.9 *Suppose $\mathcal{M} \in \mathrm{Mod}(\sigma)$, $\vec{a} \in \mathcal{M}$, $\vec{b} = \langle b_0, b_1, ..., b_n\rangle \in \mathcal{M}$, $\vec{c} = \langle c_0, c_1, ..., c_k\rangle \in \mathcal{M}$, and*

$$\forall i \leq n \forall j \leq k (\rho^{\infty}_{F,\mathcal{M}}(\vec{a}, b_i) < \rho^{\infty}_{F,\mathcal{M}}(\vec{a}, c_j)).$$

Then for any Σ^1_1 set A with

$$\mathcal{M} \models A(\vec{a}, \vec{b}, \vec{c})$$

we may find some $\mathcal{N} \in \mathrm{Mod}(\sigma)$, $\vec{a}^*, \vec{b}^*, \vec{c}^0 = \langle c_0^0, c_1^0, ..., c_k^0\rangle, \vec{c}^1 = \langle c_0^1, c_1^1, ..., c_k^1\rangle \in \mathcal{M}$ such that

$$\mathcal{N} \models A(\vec{a}^*, \vec{b}^*, \vec{c}^*),$$

$$\mathcal{N} \models A(\vec{a}^*, \vec{b}^*, \vec{c}^{**}),$$

for all $i, j \leq k$

$$F{-}\mathrm{type}_{\mathcal{N}}(c_i^*) \neq F{-}\mathrm{type}_{\mathcal{N}}(c_j^{**}).$$

Proof First let us fix H, $\vec{\beta}, \vec{\gamma}$, such that for all $i \leq n, j \leq k$

$$\rho^{\infty}_{F,\mathcal{M}}(\vec{a}, b_i) = \rho^{H}_{F,\mathcal{M}}(\vec{a}, b_i) = \beta_i,$$

$$\rho^{\infty}_{F,\mathcal{M}}(\vec{a}, c_j) = \rho^{H}_{F,\mathcal{M}}(\vec{a}, c_j) = \gamma_i.$$

By 12.3 we have that these various statements about the ranks are Σ^1_1. Thus appealing to 12.8 we can find $\mathcal{N}_0 \in \mathrm{Mod}(\sigma)$ and $\vec{a}^{*0}, \vec{b}^{*0}, \vec{c}^{*0} = \langle c^{*0}_0, c^{*0}_1, ..., c^{*0}_k\rangle, \vec{c}^{**0} = \langle c^{**0}_0, c^{**0}_1, ..., c^{**0}_k\rangle \in \mathcal{N}_0$ such that

$$\mathcal{N}_0 \models A(\vec{a}^{*0}, \vec{b}^{*0}, \vec{c}^{*0}),$$

$$\mathcal{N}_0 \models A(\vec{a}^{*0}, \vec{b}^{*0}, \vec{c}^{**0}),$$

$$\rho^{\infty}_{F,\mathcal{N}_0}(\vec{a}^{*0}, b^{*0}_i) = \rho^{H}_{F,\mathcal{N}_0}(\vec{a}^{*0}, b^{*0}_i) = \beta_i,$$

$$\rho^{\infty}_{F,\mathcal{N}_0}(\vec{a}^{*0}, c^{*0}_j) = \rho^{H}_{F,\mathcal{N}_0}(\vec{a}^{*0}, c^{*0}_j) = \gamma_i,$$

$$\rho^{\infty}_{F,\mathcal{N}_0}(\vec{a}^{*0}, c^{**0}_j) = \rho^{H}_{F,\mathcal{N}_0}(\vec{a}^{*0}, c^{**0}_j) = \gamma_i,$$

and for some $\varphi_{*0} \in F$

$$\mathcal{N}_0 \models \varphi_{*0}(c^{*0}_0) \wedge \neg\varphi_{*0}(c^{**0}_0).$$

But now we can apply the same reasoning to $\mathcal{N}_0$, $\vec{c}^{0**}$ and all the Σ^1_1 sets implicit in the same description and find $\mathcal{N}_1 \in \mathrm{Mod}(\sigma)$, $\vec{a}^{*1}, \vec{b}^{*1}, \vec{c}^{*1} = \langle c^{*1}_0, c^{*1}_1, ..., c^{*1}_k\rangle, \vec{d}^{\,0} = \langle d^0_0, d^0_1, ..., d^0_k\rangle, \vec{d}^{\,1} = \langle d^1_0, d^1_1, ..., d^1_k\rangle \in \mathcal{N}_0$ and $\varphi \in F$ such that

$$\mathcal{N}_1 \models A(\vec{a}^{*1}, \vec{b}^{*1}, \vec{c}^{*1}),$$

$$\mathcal{N}_1 \models A(\vec{a}^{*1}, \vec{b}^{*1}, \vec{d}^{0}),$$

$$\mathcal{N}_1 \models A(\vec{a}^{*1}, \vec{b}^{*1}, \vec{d}^{1}),$$

$$\mathcal{N}_1 \models \varphi_{*0}(c^{*0}_0) \wedge \neg\varphi_{*0}(d^0_0) \wedge \neg\varphi_{*0}(d^1_0),$$

$$\mathcal{N}_1 \models \varphi(d^0_1) \wedge \neg\varphi(d^1_1),$$

$$\rho^{\infty}_{F,\mathcal{N}_1}(\vec{a}^{*1}, b^{*1}_i) = \rho^{H}_{F,\mathcal{N}_1}(\vec{a}^{*1}, b^{*1}_i) = \beta_i,$$

$$\rho^{\infty}_{F,\mathcal{N}_1}(\vec{a}^{*1}, c^{*1}_j) = \rho^{H}_{F,\mathcal{N}_1}(\vec{a}^{*1}, c^{*1}_j) = \gamma_i,$$

$$\rho^{\infty}_{F,\mathcal{N}_1}(\vec{a}^{*1}, d^{0}_j) = \rho^{H}_{F,\mathcal{N}_1}(\vec{a}^{*1}, d^{0}_j) = \gamma_i,$$

$$\rho^{\infty}_{F,\mathcal{N}_1}(\vec{a}^{*1}, d^{1}_j) = \rho^{\infty}_{F,\mathcal{N}_1}(\vec{a}^{*1}, d^{1}_j) = \gamma_i.$$

Then depending on whether $\varphi(c^{*1}_1)$ holds in $\mathcal{N}_1$ we either let $\vec{c}^{**1} = \vec{d}^1$ and $\varphi_{*1} = \neg\varphi$ or $\vec{c}^{**1} = \vec{d}^0$ and $\varphi_{*1} = \varphi$.

Continuing in this fashion we at last arrive at $\mathcal{N}_k$, $\varphi_{*0}, \varphi_{*1}, ..., \varphi_{*k}$ in F, $\vec{a}^{*k}, \vec{b}^{*k}, \vec{c}^{*k} = \langle c^{*k}_0, c^{*k}_1, ..., c^{*k}_k\rangle, \vec{c}^{**k} = \langle c^{**k}_0, c^{**k}_1, ..., c^{**k}_k\rangle \in \mathcal{N}_0$ and $\varphi_{*0}, \varphi_{*1}, ..., \varphi_{*k} \in F$ such that

$$\mathcal{N}_k \models A(\vec{a}^{*k}, \vec{b}^{*k}, \vec{c}^{*k}),$$

$$\mathcal{N}_k \models A(\vec{a}^{*k}, \vec{b}^{*k}, \vec{c}^{**k}),$$

and at each $i \leq k$

$$\mathcal{N}_k \models \varphi_{*i}(c_i^{*k}) \wedge \neg\varphi_{*i}(c_i^{**k}).$$

$\mathcal{N} = \mathcal{N}_k^*$, $\vec{a}^* = \vec{a}^{*k}$, $\vec{b}^* = \vec{b}^{*k}$, $\vec{c}^0 = \vec{c}^{*k}$, $\vec{c}^1 = \vec{c}^{**k}$ is then as required.

□

Lemma 12.10 *There is some $\gamma < \omega_1^{\mathrm{ck}}$ such that for all $\mathcal{M} \in \mathrm{Mod}(\sigma)$, $\vec{a} \in T^F_{\mathcal{M}}$, $\vec{b} \in \mathcal{M}$ we have*

$$\rho^{H_F}_{F,\mathcal{M}}(\vec{a}, \vec{b}) < \gamma.$$

Proof Consider the Σ^1_1 relation

$$(\mathcal{M}, \vec{a}, \vec{b}) R(\mathcal{N}, \vec{c}, \vec{d})$$

if

$$\mathcal{M} = \mathcal{N},$$
$$\vec{c} = \vec{d},$$

$\vec{c}$ extends $\vec{a}$,

$$\vec{a}, \vec{c} \in T^F_{\mathcal{M}},$$

and F-type$_{\mathcal{N}}(\vec{d})$ is not H_F definable from $\vec{c}$ over $\mathcal{N}$. This relation is well founded by case assumption. Thus by 3.1 it has rank some $\gamma < \omega_1^{\mathrm{ck}}$, which, after unwinding the definitions, can be seen to be as required.

□

So far nothing mentioned is specific to case(IIb) as against case(IIa). We have simply used the existence of the ranking function and not that its supremum over the $\vec{b}$ is infinite; the next lemma however specifically uses our case assumptions.

Note there is a natural tendency for the supremums of the ranking functions to cluster around a limit ordinal. If $\sup_{\vec{b} \in \mathcal{M}} \rho^H_{F,\mathcal{M}}(\vec{a}, \vec{b}) = \beta + 1$, then this implies directly from the definitions that whenever $\vec{a}^\frown c \in T^F_{\mathcal{M}}$ then $\sup_{\vec{b} \in \mathcal{M}} \rho^H_{F,\mathcal{M}}(\vec{a}^\frown c, \vec{b}) \leq \beta$. This dropping can happen finitely many times, since there is no infinite descending chain of ordinals; thus we can eventually produce $\vec{a}^* \supset \vec{a}$ in $T^F_{\mathcal{M}}$ such that for all $(\vec{a}^*)^\frown c$ we have

$$\sup_{\vec{b} \in \mathcal{M}} \rho^H_{F,\mathcal{M}}(\vec{a}^*, \vec{b}) = \sup_{\vec{b} \in \mathcal{M}} \rho^H_{F,\mathcal{M}}((\vec{a}^*)^\frown c, \vec{b});$$

but then this $\sup_{\vec{b} \in \mathcal{M}} \rho^H_{F,\mathcal{M}}(\vec{a}^*, \vec{b})$ must be a limit.

Lemma 12.11 *There is an infinite limit ordinal $\lambda_0 < \omega_1^{\mathrm{ck}}$, $\mathcal{M} \in Y_F$, $\vec{a}^0 \in T^F_{\mathcal{M}}$, $H_0 \in L_{\omega_1^{\mathrm{ck}}}$ a fragment including H_F, such that*

(i) $\sup_{\vec{b} \in \mathcal{M}} \rho^{H_0}_{F,\mathcal{M}}(\vec{a}^0, \vec{b}) = \lambda_0$

(ii) $\sup_{\vec{b} \in \mathcal{M}} \rho^{\infty}_{F,\mathcal{M}}(\vec{a}^0, \vec{b}) = \lambda_0$.

Proof Suppose instead that for every such fragment $H_0 \in L_{\omega_1^{\rm ck}}$, infinite limit λ_0, $\mathcal{M} \in Y_F$, $\vec{a}^0 \in T^F_M$,

$$\sup_{\vec{b}\in\mathcal{M}} \rho^{H_0}_{F,\mathcal{M}}(\vec{a}^0, \vec{b}) = \lambda_0$$

implies the existence of some $\gamma_0 < \lambda_0$ such that for all $\vec{b} \in \mathcal{M}$ there is $H_{\vec{b}} \supset H_0$ in $L_{\omega_1^{\rm ck}}$ with

$$\rho^{H_{\vec{b}}}_{F,\mathcal{M}}(\vec{a}^0, \vec{b}) \leq \gamma_0.$$

Note that the indicated property of γ_0 is Σ_1 over $L_{\omega_1^{\rm ck(\mathcal{M})}}[\mathcal{M}]$ since the complexity class Σ_1 is closed under bounded quantification inside any admissible set.

Then taking H_0 as a parameter and considering the Σ^1_1 set of such $(\mathcal{M}, \vec{a}^0, \vec{b})$ and applying 3.4 we can find a single $\delta_0 < \omega_1^{\rm ck}$ such that we can always find the fragment $H_{\vec{b}} \in L_{\delta_0}$, and thus we have that $H_1 = \mathcal{L}_{\omega_1,\omega} \cap L_{\delta_0+\omega}$ suffices to obtain that for *every* $\mathcal{M} \in Y_F$, $\vec{a}^0, \vec{b}$ as above

$$\sup_{\vec{b}\in\mathcal{M}} \rho^{H_1}_{F,\mathcal{M}}(\vec{a}^0, \vec{b}) < \lambda_0.$$

But then for every such $\mathcal{M}, \vec{a}^0$ we can find some limit ordinal $\lambda_1 < \lambda_0$ and some $(\vec{a}^0)^\frown(\vec{a}^1) \in T^F_{\mathcal{M}}$ with

$$\sup_{\vec{b}\in\mathcal{M}} \rho^{H_1}_{F,\mathcal{M}}((\vec{a}^0)^\frown(\vec{a}^1), \vec{b}) = \lambda_1.$$

Then we can run up and find H_2 (depending on only on H_1 and λ_1) which again drops the supremums of the ranks below λ_1.

This process must eventually terminate for each $\mathcal{M} \in Y_F$ and $\vec{a}^0 \in T^F_{\mathcal{M}}$. It can only come to rest by us having eventually obtained some $(\vec{a}^0)^\frown(\vec{a}^1)^\frown ...^\frown(\vec{a}^n) \in T^F_{\mathcal{M}}$ and $H_n \in L_{\omega_1^{\rm ck}}$ with

$$\sup_{\vec{b}\in\mathcal{M}} \rho^{H_n}_{F,\mathcal{M}}((\vec{a}^0)^\frown(\vec{a}^1)...^\frown(\vec{a}^n), \vec{b}) = 0.$$

But then we have shown that for *every* $\mathcal{M} \in Y_F$ there is $\vec{d} \in T^F_{\mathcal{M}}$ and $H_n \in L_{\omega_1^{\rm ck}}$ with

$$\sup_{\vec{b}\in\mathcal{M}} \rho^{H_n}_{F,\mathcal{M}}(\vec{d}, \vec{b}) = 0.$$

Thus by 3.4 we can in fact find a single $\alpha < \omega_1^{\rm ck}$ such that for all $\mathcal{M} \in Y_F$ there is $\vec{d} \in T^F_{\mathcal{M}}$ and $H_n \in L_\alpha$ with

$$\sup_{\vec{b}\in\mathcal{M}} \rho^{H_n}_{F,\mathcal{M}}(\vec{d}, \vec{b}) = 0.$$

Taking $H = L_{\alpha+\omega} \cap \mathcal{L}_{\omega_1,\omega}$ we finish with a contradiction to the case assumption.

□

Notation From now until the end of the proof for case(IIb) fix an infinite limit ordinal $\lambda_0 < \omega_1^{\rm ck}$ and fragment $H_0 \supset H_F$ such that $H_0 \in L_{\omega_1^{\rm ck}}$ and there exists $\mathcal{M} \in Y_F$, $\vec{a} \in T^F_{\mathcal{M}}$ such that

(i) $\sup_{\vec{b}\in\mathcal{M}} \rho^{H_0}_{F,\mathcal{M}}(\vec{a}, \vec{b}) = \lambda_0$;

(ii) $\sup_{\vec{b}\in\mathcal{M}}\rho^{\infty}_{F,\mathcal{M}}(\vec{a},\vec{b})=\lambda_0$.

In what follows let us assume for notational simplicity that we can actually take $\vec{a}$ to be the empty sequence. Thus:

(i) $\sup_{\vec{b}\in\mathcal{M}}\rho^{H_0}_{F,\mathcal{M}}(0,\vec{b})=\lambda_0$;

(ii) $\sup_{\vec{b}\in\mathcal{M}}\rho^{\infty}_{F,\mathcal{M}}(0,\vec{b})=\lambda_0$.

Again it will be convenient to pass to the space $\mathrm{Mod}_\Omega(\sigma)$, of models of σ with underlying set $\Omega=(2^{<\omega})^{<\omega}$. We then let Z_F be the set of $\mathcal{M}\in\mathrm{Mod}_\Omega(\sigma)$ such that:

(i) there is an isomorphic copy, $\mathcal{N}\cong\mathcal{M}$, with $\mathcal{N}\in Y_F$;

(ii) $\sup_{\vec{b}\in\mathcal{M}}\rho^{H_0}_{F,\mathcal{M}}(0,\vec{b})=\lambda_0$;

(iii) $\sup_{\vec{b}\in\mathcal{M}}\rho^{\infty}_{F,\mathcal{M}}(0,\vec{b})=\lambda_0$.

Note that Z_F is a Σ^1_1 set.

Definition Let $\mathbb{Q}_1$ be the set of sequences of the form

$$\langle A,H,\vec{\gamma},(a_{\vec{s}})_{\vec{s}\in f_0},(b_v)_{v\in g_0},...,(a_{\vec{s}})_{\vec{s}\in f_n},(b_v)_{v\in g_n}\rangle$$

such that

(i) $n\in\omega$; $\vec{\gamma}\in\lambda_0^{<\omega}$; $\gamma_0=0$, $\gamma_1>1$, each $\gamma_{i+1}>\gamma_i$;

(ii) for some $\vec{l}=\langle l_0,l_1,...,l_n\rangle$ we have each $f_i=2^{l_0}\times 2^{l_1}\times ...2^{l_i}$ and each g_i a finite subset of $f_i\times\omega$;

(iii) A is a non-empty Σ^1_1 subset of $\mathrm{Mod}_\Omega(\sigma)$;

(iv) every $\mathcal{M}\in\mathrm{Mod}_\Omega(\sigma)$ with

$$\mathcal{M}\models A((a_{\vec{s}})_{\vec{s}\in f_0},(b_v)_{v\in g_0},...,(a_{\vec{s}})_{\vec{s}\in f_n},(b_v)_{v\in g_n})$$

is in Z_F;

(v) H is a fragment in $L_{\omega_1^{\mathrm{ck}}}$ extending H_0; for any

$$\mathcal{M}\models A((a_{\vec{s}})_{\vec{s}\in f_0},(b_v)_{v\in g_0},...,(a_{\vec{s}})_{\vec{s}\in f_n},(b_v)_{v\in g_n})$$

one has for each $i\le n$ and $\vec{s}\in f_i$, $v\in g_i$ that

$$0<\rho^{H}_{F,\mathcal{M}}(0,a_{\vec{s}})=\rho^{\infty}_{F,\mathcal{M}}(0,a_{\vec{s}})\in[\gamma_i,\gamma_{i+1}),$$

$$0\le\rho^{H}_{F,\mathcal{M}}(0,b_v)=\rho^{\infty}_{F,\mathcal{M}}(0,b_v)\in[\gamma_i,\gamma_{i+1});$$

(vi) for each $\vec{u},\vec{t}\in\bigcup_{i\le n}f_i$ with $\vec{u}\neq\vec{t}$ there exists $\psi_{\vec{u},\vec{t}}\in F$ such that for all

$$\mathcal{M}\models A((a_{\vec{s}})_{\vec{s}\in f_0},(b_v)_{v\in g_0},...,(a_{\vec{s}})_{\vec{s}\in f_n},(b_v)_{v\in g_n})$$

one has

$$\mathcal{M}\models\psi_{\vec{u},\vec{t}}(a_{\vec{u}})\wedge\neg\psi_{\vec{u},\vec{t}}(a_{\vec{t}}).$$

We can think of elements in $\mathbb{Q}_1$ as partial descriptions of models of σ. This is very much parallel to the definition of $\mathbb{Q}_0$, but with the difference that at (v) we do not ask that differing $a_{\vec{s}}$'s have the potential to come from the images of a nested array of elementary submodels, but rather that they occupy strata with respect to the rank functions; for $i < j \leq n$ and $\vec{s} \in f_i$, $\vec{u} \in f_j$, we are requiring that

$$\gamma_i \leq \rho^{\infty}_{F,\mathcal{M}}(0, a_{\vec{s}}) < \gamma_{i+1} \leq \gamma_j < \rho^{\infty}_{F,\mathcal{M}}(0, a_{\vec{t}}) < \gamma_{j+1}.$$

We then define an ordering on $\mathbb{Q}_1$ parallel to the ordering on $\mathbb{Q}_0$.

Definition For

$$p = \langle A, H, \vec{\gamma}, (a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, (a_{\vec{s}})_{\vec{s}\in f_1}, ...(b_v)_{v\in g_n}\rangle$$

and

$$\bar{p} = \langle \bar{A}, \bar{H}, \vec{\delta}, (a_{\vec{s}})_{\vec{s}\in \bar{f}_0}, (b_v)_{v\in \bar{g}_0}, (a_{\vec{s}})_{\vec{s}\in \bar{f}_1}, ...(b_v)_{v\in \bar{g}_{\bar{n}}}\rangle$$

both in $\mathbb{Q}_1$ we set $\bar{p} \leq p$ if the following all take place:-

(i) $n \leq \bar{n}$; $\vec{\delta}$ extends the sequence $\vec{\gamma}$; $\bar{H} \supset H$;

(ii) there are $l_0, l_1, ..., l_n$, $k_0, ..., k_n$, such that for each $i \leq n$ we have $f_i = 2^{l_0} \times 2^{l_1} \times ... \times 2^{l_i}$ and $\bar{f}_i = 2^{l_0+k_0} \times 2^{l_1+k_1} \times ... \times 2^{l_i+k_i}$;

(iii) for $l_0, l_1, ..., l_n$, $k_0, ..., k_n$ as in (ii) we have that each

$$\bar{g}_i \supset \{v * \vec{t} : \vec{t} \in 2^{k_0} \times ... \times 2^{k_i}, v \in g_i\};$$

(iv) for any

$$\mathcal{M} \models \bar{A}((a_{\vec{s}})_{\vec{s}\in \bar{f}_0}, (b_v)_{v\in \bar{g}_0}, (\bar{a}_{\vec{s}})_{\vec{s}\in \bar{f}_1}, ...(b_v)_{v\in \bar{g}_{\bar{n}}})$$

and $\vec{t} \in 2^{k_0} \times ...2^{k_n}$ we have

$$\mathcal{M} \models A((a_{\vec{s}*\vec{t}})_{\vec{s}\in f_0}, (b_{v*\vec{t}})_{v\in g_0}, (a_{\vec{s}*\vec{t}})_{\vec{s}\in f_1}, ...(b_{v*\vec{t}})_{v\in g_n}).$$

Lemma 12.12 *For*

$$p = \langle A, H, \vec{\gamma}, (a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, (a_{\vec{s}})_{\vec{s}\in f_1}, ...(b_v)_{v\in g_n}\rangle$$

in $\mathbb{Q}_1$ *and* $m \leq n$, *there is*

$$\bar{p} = \langle \bar{A}, H, \vec{\gamma}, (a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, (a_{\vec{s}})_{\vec{s}\in f_1}, ...(b_v)_{v\in g_{m-1}}, (a_{\vec{s}})_{\vec{s}\in f_m^{(m)}}, (b_v)_{v\in g_m^{(m)}}, ..., (b_v)_{v\in g_n^{(m)}}\rangle \in \mathbb{Q}_1$$

with

$$\bar{p} \leq p.$$

Proof The lemma asserts that we may fix $(a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, (a_{\vec{s}})_{\vec{s}\in f_1}, ...(b_v)_{v\in g_{m-1}}$ as a parameter and essentially split the sequence $(a_{\vec{s}})_{\vec{s}\in f_m^{(m)}}, (b_v)_{v\in g_m^{(m)}}, ..., (b_v)_{v\in g_n^{(m)}}$ into two sequences with different F-types. This is exactly as stated at 12.9.

$\square$

Lemma 12.13 *For all*

$$p = \langle A, H, \vec{\gamma}, (a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, (a_{\vec{s}})_{\vec{s}\in f_1}, ...(b_v)_{v\in g_n}\rangle$$

in $\mathbb{Q}_1$ *and* $\lambda < \lambda_0$ *there is*

$$\bar{p} = \langle \bar{A}, \bar{H}, \vec{\delta}, (a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, (a_{\vec{s}})_{\vec{s}\in f_1}, ...(b_v)_{v\in g_n}, (a_{\vec{s}})_{\vec{s}\in f_{n+1}}, (b_v)_{v\in g_{n+1}}\rangle$$

in $\mathbb{Q}_1$ *with*

(i) $\bar{p} \leq p$*;*

(ii) $\vec{\delta} = \langle \gamma_0, \gamma_1, ..., \gamma_n, \gamma_{n+1}, \gamma_{n+2}\rangle$ *for some* $\gamma_{n+2} > \lambda$*.*

Proof Let $\vec{\gamma} = \langle \gamma_0, \gamma_1, ...\gamma_{n+1}\rangle$. Fix some $\mathcal{M} \in Z_F$ with

$$\mathcal{M} \models A((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, (a_{\vec{s}})_{\vec{s}\in f_1}, ...(b_v)_{v\in g_n}).$$

Appealing to the definition of Z_F we may find some $\bar{H} \supset H$ and $\gamma > \lambda, \gamma_{n+1}$ and $c \in \mathcal{M}$ with

$$\rho^{\infty}_{F,\mathcal{M}}(0, c) = \rho^{\bar{H}}_{F,\mathcal{M}}(0, c) = \gamma.$$

Following 12.9 we may find some $\mathcal{N}$ with

$$\mathcal{N} \models A((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, (a_{\vec{s}})_{\vec{s}\in f_1}, ...(b_v)_{v\in g_n})$$

inside which we can obtain $|f_n| = |2^{l_0+l_1+...+l_n}|$ many such c, and so there will be $(c_j)_{j<|2^{l_0+l_1+...+l_n}|}$ such that each

$$\rho^{\infty}_{F,\mathcal{N}}(0, c_j) = \rho^{\bar{H}}_{F,\mathcal{N}}(0, c_j) = \gamma$$

but

$$F\text{−type}_{\mathcal{N}}(c_{j(0)}) \neq F\text{−type}_{\mathcal{N}}(c_{j(1)})$$

for $j(0) \neq j(1)$.

Then we may let $l_{n+1} = 0$ and $f_{n+1} = f_n \times \{0\}$, $g_{n+1} = 0$, reindex the set $(c_j)_{j<|2^{l_0+l_1+...+l_n}|}$ as $(a_{\vec{s}})_{\vec{s}\in f_{n+1}}$, let $\vec{\delta} = \langle \gamma_0, \gamma_1, ..., \gamma_n, \gamma_{n+1}, \gamma + 1\rangle$ and choose $\bar{A} \in \Sigma^1_1$ to describing enough about

$$\langle \mathcal{N}, (a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, (a_{\vec{s}})_{\vec{s}\in f_1}, ...(b_v)_{v\in g_n}), (a_{\vec{s}})_{\vec{s}\in f_{n+1}}, (b_v)_{v\in g_{n+1}}\rangle$$

to ensure that for

$$\bar{p} = \langle \bar{A}, \bar{H}, \vec{\delta}, (a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, (a_{\vec{s}})_{\vec{s}\in f_1}, ...(b_v)_{v\in g_n}, (a_{\vec{s}})_{\vec{s}\in f_{n+1}}, (b_v)_{v\in g_{n+1}}\rangle \in \mathbb{Q}_1$$

and for any $\mathcal{N}^* \in \mathrm{Mod}_{\Omega}(\sigma)$ with

$$\mathcal{N}^* \models \bar{A}((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, (a_{\vec{s}})_{\vec{s}\in f_1}, ..., (a_{\vec{s}})_{\vec{s}\in f_{n+1}}, (b_v)_{v\in g_{n+1}})$$

we have

$$\mathcal{N}^* \models A((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, (a_{\vec{s}})_{\vec{s}\in f_1}, ...(b_v)_{v\in g_n}).$$

$\square$

Lemma 12.14 *For all*

$$p = \langle A, H, \vec{\gamma}, (a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, (a_{\vec{s}})_{\vec{s}\in f_1}, ...(b_v)_{v\in g_n}\rangle \in \mathbb{Q}_1$$

there is

$$\bar{p} = \langle \bar{A}, \bar{H}, \vec{\gamma}, (a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in \bar{g}_0}, (a_{\vec{s}})_{\vec{s}\in f_1}, ...(a_{\vec{s}})_{\vec{s}\in f_{\bar{n}}}(b_v)_{v\in \bar{g}_{\bar{n}}}\rangle \in \mathbb{Q}_1$$

such that for all

$$\vec{c} \in \{a_{\vec{s}}, b_v : \vec{s} \in \bigcup_{j\leq n} f_j, v \in \bigcup_{j\leq n} g_j\}$$

we have either:

(i) for all

$$\mathcal{M} \models \bar{A}((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in \bar{g}_0}, (a_{\vec{s}})_{\vec{s}\in f_1}, ...(b_v)_{v\in \bar{g}_{\bar{n}}})$$

we have

$$\mathcal{M} \models \forall y \neg\psi(\vec{c}, y);$$

or

(ii) there is some $i \leq \bar{n}$ *and* $v \in \bar{g}_i$ *such that for all*

$$\mathcal{M} \models \bar{A}((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in \bar{g}_0}, (a_{\vec{s}})_{\vec{s}\in f_1}, ...(b_v)_{v\in \bar{g}_{\bar{n}}})$$

we have

$$\mathcal{M} \models \psi(\vec{c}, b_v).$$

Proof It suffices to prove this for a single $\vec{c} \in \{a_{\vec{s}}, b_v : \vec{s} \in \bigcup_{j\leq n} f_j, v \in \bigcup_{j\leq n} g_j\}$. So let us fix $\vec{c}$ and

$$\mathcal{M}_0 \models A((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in \bar{g}_0}, (a_{\vec{s}})_{\vec{s}\in f_1}, ...(b_v)_{v\in g_n}).$$

Case(1). $\mathcal{M}_0 \models \forall y \neg\psi(\vec{c}, y)$.

Then we can simply tighten up A to $\bar{A}$ so that

$$\mathcal{M} \models \bar{A}((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, (a_{\vec{s}})_{\vec{s}\in f_1}, ...(b_v)_{v\in g_n})$$

if and only if

$$\mathcal{M} \models A((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, (a_{\vec{s}})_{\vec{s}\in f_1}, ...(b_v)_{v\in g_n})$$

and

$$\mathcal{M} \models \forall y \neg\psi(\vec{c}, y).$$

Case(2). There is some $d \in \mathcal{M}_0$ with $\mathcal{M}_0 \models \psi(\vec{c}, d)$.

Then after possibly applying 12.13 we may assume that

$$\rho^{\infty}_{F,\mathcal{M}}(0, d) < \gamma_{n+1}.$$

So we can choose a fragment $\bar{H} \supset H$ with

$$\rho^{\bar{H}}_{F,\mathcal{M}}(0,d) = \rho^{\infty}_{F,\mathcal{M}}(0,d) < \gamma_{n+1}.$$

Now for $i \leq n$ with

$$\gamma_i \leq \rho^{\bar{H}}_{F,\mathcal{M}}(0,d) < \gamma_{i+1}$$

we can simply extend g_i to $\bar{g}_i$, where $\{b_v : v \in \bar{g}_i\} = \{b_v : v \in g_i\} \cup \{d\}$ and let $\bar{A}$ be given by

$$\mathcal{M} \models \bar{A}((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, (a_{\vec{s}})_{\vec{s}\in f_1}, ...(b_v)_{v\in g_n})$$

if and only if

$$\mathcal{M} \models A((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, (a_{\vec{s}})_{\vec{s}\in f_1}, ...(b_v)_{v\in g_n})$$

and

$$\mathcal{M} \models \psi(\vec{c}, b_{v*}),$$

where $\bar{g}_i = g_i \cup \{v*\}$.

□

Notation From now until the end of the section fix an $L_{\omega_2^{\mathrm{ck}}}$-generic filter $G_1 \subset \mathbb{Q}_1$.

Definition For $\vec{x} \in (2^\omega)^\omega$ we define a countable structure $\mathcal{N}_{\vec{x}}$; we define the underlying set now as in section 11, simply letting

$$|\mathcal{N}_{\vec{x}}| = |\mathcal{M}_{\vec{x}}|,$$

where $\mathcal{M}_{\vec{x}}$ is as given just before lemma 11.7.

Lemma 12.15 *For $R \in \mathcal{L}$, $\vec{x} \in (2^\omega)^\omega$, and*

$$a_{\vec{y}^0}, a_{\vec{y}^1}, ..., a_{\vec{y}^{k(0}}, b_{w_0}, b_{w_1}, ..., b_{w_{k(1)}}$$

in $|\mathcal{M}_{\vec{x}}|$ we have one of the following two possibilities:

(i) for cofinally many

$$p = \langle A, H, \vec{\gamma}, (a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (a_{\vec{s}})_{\vec{s}\in f_n}, (b_v)_{v\in g_n}\rangle \in G_1,$$

with each $f_j = 2^{l_0} \times 2^{l_1} \times ... \times 2^{l_j}$, and $\vec{l} = \langle l_0, l_1, ..., l_n\rangle$, we have that for all

$$\mathcal{M} \models A((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (b_v)_{v\in g_n})$$

it is the case that

$$\mathcal{M} \models R(a_{\pi_{\vec{l}}(\vec{y}^0)}, ..., a_{\pi_{\vec{l}}(\vec{y}^{k(0)})}, b_{\pi_{\vec{l}}(w_0)}, ..., b_{\pi_{\vec{l}}(w_{k(1)})});$$

or

(ii) for cofinally many

$$p = \langle A, H, \vec{\gamma}, (a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (a_{\vec{s}})_{\vec{s}\in f_n}, (b_v)_{v\in g_n})\rangle \in G_1,$$

with each $f_j = 2^{l_0} \times 2^{l_1} \times ... \times 2^{l_j}$, *and* $\vec{l} = \langle l_0, l_1, ..., l_n\rangle$, *for all*

$$\mathcal{M} \models A((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (b_v)_{v\in g_n})$$

we have

$$\mathcal{M} \models \neg R(a_{\pi_{\vec{l}}(\vec{y}^0)}, ..., a_{\pi_{\vec{l}}(\vec{y}^{k(0)})}, b_{\pi_{\vec{l}}(w_0)}, ..., b_{\pi_{\vec{l}}(w_{k(1)})}).$$

Proof Just as in 11.7.

□

Definition We provide $\mathcal{N}_{\vec{x}}$ with an $\mathcal{L}$-structure as follows. For R a relation symbol in the language,

$$\mathcal{N}_{\vec{x}} \models R(a_{\vec{y}^0}, a_{\vec{y}^1}, ..., b_{w_0}, b_{w_1}, ...)$$

if and only if the following two things happen:

(a) $a_{\vec{y}^0}, a_{\vec{y}^1}, ..., b_{w_0}, b_{w_1}, ...$ are in $|\mathcal{N}_{\vec{x}}|$;

(b) for all sufficiently large $\vec{l}$ and for cofinally many

$$p = \langle A, H, \vec{\gamma}, (a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (a_{\vec{s}})_{\vec{s}\in f_n}, (b_v)_{v\in g_n})\rangle \in G_1,$$

with each $f_j = 2^{l_0} \times 2^{l_1} \times ... \times 2^{l_j}$, and $\vec{l} = \langle l_0, l_1, ..., l_n\rangle$, we have that for all

$$\mathcal{M} \models A((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (b_v)_{v\in g_n})$$

it is the case that

$$\mathcal{M} \models R(a_{\pi_{\vec{l}}(\vec{y}^0)}, ..., a_{\pi_{\vec{l}}(\vec{y}^{k(0)})}, b_{\pi_{\vec{l}}(w_0)}, ..., b_{\pi_{\vec{l}}(w_{k(1)})}).$$

Lemma 12.16 *For* $\psi \in \mathcal{L}_{\omega_1,\omega} \cap L_{\omega_1^{\mathrm{ck}}}$ *and* $a_{\vec{y}^0}, a_{\vec{y}^1}, ..., b_{w_0}, b_{w_1}, ...$ *in* $|\mathcal{N}_{\vec{x}}|$, *we have*

$$\mathcal{N}_{\vec{x}} \models \psi(a_{\vec{y}^0}, a_{\vec{y}^1}, ..., b_{w_0}, b_{w_1}, ...)$$

if and only for all sufficiently large $\vec{l}$ *and cofinally many*

$$p = \langle A, H, \vec{\gamma}, (a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (a_{\vec{s}})_{\vec{s}\in f_n}, (b_v)_{v\in g_n}\rangle \in G_0,$$

with each $f_j = 2^{l_0} \times 2^{l_1} \times ... \times 2^{l_j}$, *and* $\vec{l} = \langle l_0, l_1, ..., l_n\rangle$, *we have that:*

for all $\mathcal{M} \in \mathrm{Mod}_\Omega(\sigma)$, *if*

$$\mathcal{M} \models A((a_{\vec{s}})_{\vec{s}\in f_0}, (b_v)_{v\in g_0}, ..., (b_v)_{v\in g_n}),$$

then

$$\mathcal{M} \models \neg\psi(a_{\pi_{\vec{l}}(\vec{y}^0)}, ..., a_{\pi_{\vec{l}}(\vec{y}^{k(0)})}, b_{\pi_{\vec{l}}(w_0)}, ..., b_{\pi_{\vec{l}}(w_{k(1)})}).$$

Proof This follows from 12.14 as 11.9 followed from 11.6.

□

Lemma 12.17 *For all $\vec{x}, \vec{y} \in (2^\omega)^\omega$,*

$$\vec{x}(E_0)^\omega \vec{y}$$

implies

$$\mathcal{N}_{\vec{x}} = \mathcal{N}_{\vec{y}}.$$

Proof As with 11.10 this follows immediately from the definitions.

□

Lemma 12.18 *For all $\vec{x}, \vec{y} \in (2^\omega)^\omega$,*

$$\mathcal{N}_{\vec{x}} \cong \mathcal{N}_{\vec{y}}$$

implies

$$\vec{x}(E_0)^\omega \vec{y}.$$

Proof Almost word for word as in the proof of 11.11.

□

And again as at 11.12 and 11.13 we have:

Lemma 12.19 *There is a Borel function*

$$\theta : (2^\omega)^\omega \to \mathrm{Mod}(\sigma)$$

such that for all $\vec{x} \in (2^\omega)^\omega$

$$\mathcal{N}_{\vec{x}} \cong \theta(\vec{x}).$$

Finally:

Lemma 12.20 *Under the assumptions of case(IIb) we have*

$$(E_0)^\omega \leq_B \cong|_{\mathrm{Mod}(\sigma)}.$$

13 Proof for case(IIa)

From now on let us assume that for all fragments F there is some fragment $H \supset H_F$ such that
$$\forall \mathcal{M} \in \mathrm{Mod}(\sigma) \exists \vec{c} \in \mathcal{M}(\sup_{\vec{b} \in \mathcal{M}} \rho^H_{F,\mathcal{M}}(\vec{c}, \vec{b}) < \omega).$$

Note that if $\sup_{\vec{b} \in \mathcal{M}} \rho^H_{F,\mathcal{M}}(\vec{c}, \vec{b}) = k$ some finite k, then for any $\vec{c}^\frown \vec{d} \in T^F_{\mathcal{M}}$ with $lh(\vec{d}) = k$ we have
$$\sup_{\vec{b} \in \mathcal{M}} \rho^H_{F,\mathcal{M}}(\vec{c}^\frown \vec{d}, \vec{b}) = 0,$$
which states that every F-type is H-definable over $\mathcal{M}$ from $\vec{a}^\frown \vec{d}$. Note moreover that there is a canonical formula in $\mathcal{L}_{\omega_1,\omega} \cap L_{\omega_1^{\mathrm{ck}}}$ which holds about $\vec{c}^\frown \vec{d}$ exactly when every F-type realized in $\mathcal{M}$ is H-definable from $\vec{c}^\frown \vec{d}$.

Thus for every $F \in L_{\omega_1^{\mathrm{ck}}}$ we can find fragments
$$F^{++} \supset F^+ \supset H_F$$
such that for all $\mathcal{M} \in \mathrm{Mod}(\sigma)$:

(a) there exists $\vec{a} \in \mathcal{M}$ such that for all $\vec{b} \in \mathcal{M}$ there exists $\psi_{\vec{b}} \in F^+$ with
$$\mathcal{M} \models \psi_{\vec{b}}(\vec{a}, \vec{b})$$
and for all $\varphi \in F$
$$\mathcal{M} \models \forall \vec{c}^0, \vec{c}^1((\psi_{\vec{b}}(\vec{a}, \vec{c}^0) \wedge \psi_{\vec{b}}(\vec{a}, \vec{c}^1)) \Rightarrow (\varphi(\vec{c}^0) \Leftrightarrow \varphi(\vec{c}^1)));$$

(b) we have $\Delta_F \in F^{++}$, the canonical $\mathcal{L}_{\omega_1,\omega}$ formula such that for all $\mathcal{N}, \vec{e} \in \mathcal{N}$,
$$\mathcal{N} \models \Delta_F(\vec{e})$$
if and only if for all $\vec{b} \in \mathcal{N}$ there is some $\psi \in F^+$ with
$$\mathcal{N} \models \psi(\vec{e}, \vec{b})$$
and for all $\varphi \in F$
$$\mathcal{M} \models \forall \vec{c}^0, \vec{c}^1((\psi(\vec{e}, \vec{c}^0) \wedge \psi(\vec{e}, \vec{c}^1)) \Rightarrow (\varphi(\vec{c}^0) \Leftrightarrow \varphi(\vec{c}^1))).$$

The properties indicated for the tripe (F^+, F^{++}, Δ_F) are uniformly Π^1_1 in any natural number encoding a well ordering of order type α where F is the αth object in $L_{\omega_1^{\mathrm{ck}}}$. Thus we may assume the functions
$$F \mapsto F^{++}$$
$$F \mapsto \Delta_F$$

are uniformly Σ_1 over $L_{\omega_1^{\mathrm{ck}}}$. Thus we may iterate, using that transfinite recursion applied to Σ_1 over an admissible set leads to functions which are again Σ_1 over that admissible; thus we obtain a Σ_1 over $L_{\omega_1^{\mathrm{ck}}}$ function

$$\omega_1^{\mathrm{ck}} \to L_{\omega_1^{\mathrm{ck}}}$$

$$\alpha \mapsto H_\alpha,$$

where

$$H_0 = \mathcal{L}_{\omega,\omega},$$

$$H_{\alpha+1} = (H_\alpha)^{++},$$

and at λ a limit,

$$H_\lambda = \bigcup_{\alpha\in\lambda} H_\alpha.$$

Recall that at section 5 we define the notion of Σ^0_α and Π^0_α formula.

Lemma 13.1 *Suppose* $\varphi \in \mathcal{L}_{\omega_1,\omega} \cap (\Sigma^0_\alpha \cup \Pi^0_\alpha)$*,* $\alpha < \omega_1^{\mathrm{ck}}$*, and* $\mathcal{M},\mathcal{N} \in \mathrm{Mod}(\sigma)$*, with* $\vec{a} \in \mathcal{M}$*,* $\vec{c} \in \mathcal{N}$ *such that*

$$\mathcal{M} \models \Delta_{H_\alpha}(\vec{a}),$$

$$\mathcal{N} \models \Delta_{H_\alpha}(\vec{c}),$$

$$H_{\alpha+1}\text{−type}_{\mathcal{M}}(\vec{a}) = H_{\alpha+1}\text{−type}_{\mathcal{N}}(\vec{c}).$$

Then

$$\mathcal{M} \models \varphi(\vec{a})$$

if and only if

$$\mathcal{N} \models \varphi(\vec{c}).$$

Proof We prove this by transfinite induction on α. The base case of $\alpha = 0$ is trivial as is the passage through limit ordinals.

For successor step, suppose the lemma is true at α and $\{\varphi_i(\vec{x},\vec{y}) : i \in \omega\}$ are all either Σ^0_α or Π^0_α and suppose

$$\beta = \alpha + 1$$

with $\mathcal{M},\mathcal{N} \in \mathrm{Mod}(\sigma)$, and $\vec{a} \in \mathcal{M}$, $\vec{c} \in \mathcal{N}$, and

$$\mathcal{M} \models \Delta_{H_\beta}(\vec{a}),$$

$$\mathcal{N} \models \Delta_{H_\beta}(\vec{c}),$$

$$H_{\beta+1}\text{−type}_{\mathcal{M}}(\vec{a}) = H_{\beta+1}\text{−type}_{\mathcal{N}}(\vec{c}).$$

We want to show

$$\mathcal{M} \models \bigvee_{i\in\omega} \exists\vec{y}\varphi_i(\vec{a},\vec{y})$$

if and only if
$$\mathcal{N} \models \bigvee_{i \in \omega} \exists \vec{y} \varphi_i(\vec{c}, \vec{y}).$$

We may as well assume that $\mathcal{M} \models \exists \vec{y} \varphi_i(\vec{a}, \vec{y})$, some i, the other case being symmetrical. Fix $\vec{a}^0, \vec{a}^1 \in \mathcal{M}$ with
$$\mathcal{M} \models \varphi_i(\vec{a}, \vec{a}^0),$$
$$\mathcal{M} \models \Delta_{H_{\alpha+1}}(\vec{a}, \vec{a}^0, \vec{a}^1).$$

By the definition of $\Delta_{H_{\alpha+1}}$ we can find some $\psi \in H_{\beta+1}$ such that
$$\mathcal{M} \models \psi(\vec{a}, \vec{a}^0, \vec{a}^1)'$$
$$\mathcal{M} \models \forall \vec{z}^0, \vec{z}^1 (\psi(\vec{a}, \vec{z}^0, \vec{z}^1) \Rightarrow \bigwedge_{\theta \in H_{\alpha+1}} (\theta(\vec{a}, \vec{z}^0, \vec{z}^1) \Leftrightarrow \theta(\vec{a}, \vec{a}^0, \vec{a}^1))).$$

Since $H_{\beta+1}-\text{type}_{\mathcal{M}}(\vec{a}) = H_{\beta+1}-\text{type}_{\mathcal{N}}(\vec{c})$ we can find $\vec{c}^0, \vec{c}^1 \in \mathcal{N}$ with
$$\mathcal{N} \models \psi(\vec{c}, \vec{c}^0, \vec{c}^1).$$

Again by $H_{\beta+1}-\text{type}_{\mathcal{M}}(\vec{a}) = H_{\beta+1}-\text{type}_{\mathcal{N}}(\vec{c})$ we have that for all $\theta \in H_{\alpha+1}$ with
$$\mathcal{M} \models \theta(\vec{a}, \vec{a}^0, \vec{a}^1)$$
one has
$$\mathcal{N} \models \forall \vec{z}^0, \vec{z}^1 (\psi(\vec{c}, \vec{z}^0, \vec{z}^1) \Rightarrow \theta(\vec{c}, \vec{z}^0, \vec{z}^1)).$$
Thus we have $H_{\alpha+1}-\text{type}_{\mathcal{M}}(\vec{a}, \vec{a}^0, \vec{a}^1) = H_{\alpha+1}-\text{type}_{\mathcal{N}}(\vec{c}, \vec{c}^0, \vec{c}^1)$, and we can appeal to the inductive assumption.

□

In the next corollary we use for the first and only time that $\cong|_{\mathrm{Mod}(\sigma)}$ is Borel.

Corollary 13.2 *There is a single $\delta < \omega_1^{\mathrm{ck}}$ such that for all $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(\sigma)$*
$$\mathcal{M} \cong \mathcal{N}$$
if and only if there exist $\vec{a} \in \mathcal{M}$, $\vec{c} \in \mathcal{N}$, and
$$\mathcal{M} \models \Delta_{H_\delta}(\vec{a}),$$
$$\mathcal{N} \models \Delta_{H_\delta}(\vec{c}),$$
$$H_{\delta+1}-\text{type}_{\mathcal{M}}(\vec{a}) = H_{\delta+1}-\text{type}_{\mathcal{N}}(\vec{c}).$$

Proof Choose $\delta < \omega_1^{\mathrm{ck}}$ such that each isomorphism type is $\underset{\sim}{\Pi}^0_\delta$. Then by [23] we have that for all $\mathcal{M} \in \mathrm{Mod}(\sigma)$ there is some

$$\varphi_{\mathcal{M}} \in \mathcal{L}_{\omega_1,\omega} \cap \Pi^0_\delta$$

with

$$\{\mathcal{N} \in \mathrm{Mod}(\sigma) : \mathcal{N} \models \varphi_{\mathcal{M}}\} = \{\mathcal{N} \in \mathrm{Mod}(\sigma) : \mathcal{N} \cong \mathcal{M}\}.$$

Now the corollary follows from the lemma.

□

Corollary 13.3 *Under the case assumptions of case(IIa),* $\cong|_{\mathrm{Mod}}(\sigma)$ *is Borel reducible to a countable Borel equivalence relation.*

Proof Fix $\delta < \omega_1^{\mathrm{ck}}$ as in 13.2. $H_{\delta+1}$ is a countable fragment and for each $\mathcal{M} \in \mathrm{Mod}(\sigma)$ there is by 13.2 $\vec{a} \in \mathcal{M}$ with the $H_{\delta+1}$-theory of $\langle \mathcal{M}; \vec{a}$ being $\aleph_0$-categorical. Thus we have by 4.3 of [13] that $\cong|_{\mathrm{Mod}}(\sigma)$ is reducible to a countable Borel equivalence relation

□

There is an alternative way to finish up 13.3. We can define a Borel function

$$f : \mathrm{Mod}(\sigma) \to 2^{H_{\delta+1}}$$

so that for all $\mathcal{M} \in \mathrm{Mod}(\sigma)$ there exists $\vec{a} \in \mathcal{M}$ with

$$\mathcal{M} \models \Delta_{H_\delta}(\vec{a}),$$

$$H_{\delta+1}\mathrm{-type}_{\mathcal{M}}(\vec{a}) = f(\mathcal{M}).$$

Then we may alternatively quote from 5.1 of [11]

Lemma 13.4 *(Kechris) Let F be a Borel equivalence relation on a Polish space X and*

$$g : X \to Y$$

a Borel function to any other Polish space Y. If
(i) for all $x \in X$ we have $|\{g(z) : zFx\}| \leq \aleph_0$, and
(ii) for all $x_1, x_2 \in X$

$$g(x_1) = g(x_2) \Rightarrow x_1 F x_2,$$

then F is Borel reducible to a countable Borel equivalence relation.

14 Variations and modifications

We have proved:

Theorem 14.1 *Let $\sigma \in \mathcal{L}_{\omega_1,\omega}$ be such that the isomorphism relation $\cong|_{\mathrm{Mod}(\sigma)}$ on its countable models is Borel. Then exactly one of the following occurs:*
(I) $\cong|_{\mathrm{Mod}(\sigma)} \leq_B F$ for some countable Borel equivalence relation F;
(II) $(E_0)^\omega \leq_B \cong|_{\mathrm{Mod}(\sigma)}$.

There are various refinements which can be obtained more or less immediately by inspection of the proof. The first of these is that we may replace (II) by the nominally stronger
(II*) $(E_0)^\omega \leq_c \cong|_{\mathrm{Mod}(\sigma)}$,
since the process of constructing the model $\mathcal{M}_{\vec{x}}$ in case(I) and $\mathcal{N}_{\vec{x}}$ in case(IIb) is clearly continuous in $\vec{x}$. In fact one can even have
(II*) $(E_0)^\omega \sqsubseteq_c \cong|_{\mathrm{Mod}(\sigma)}$.
Indeed an easy Baire category argument shows that:

Fact 14.2 *If F is an equivalence relation with*

$$(E_0)^\omega \leq_B F,$$

then

$$(E_0)^\omega \sqsubseteq_c F.$$

A rather different direction in which the main theorem can be sharpened is to work on $\underset{\sim}{\Sigma}^1_1$ subsets of the space of countable $\mathcal{L}$-structures, as against only those which are defined by some $\sigma \in \mathcal{L}_{\omega_1,\omega}$. If $B \in \Sigma^1_1$ and

$$B \cap X_F \neq 0$$

then the proof of case(I) at 11 goes as before to obtain $(E_0)^\omega \leq_B \cong|_B$. For the other cases we have that the application of boundedness at 12.11 works when restricted to a Σ^1_1 set and case(IIa) is unaffected as long as we are working inside some set on which the isomorphism relation is Borel.

Using for instance

Lemma 14.3 *(H. Friedman, [5]) If E is a Borel equivalence relation with $E \leq_B \cong|_{\mathrm{Mod}(\mathcal{L})}$ some countable language $\mathcal{L}$ then there is $\sigma \in \mathcal{L}_{\omega_1,\omega}$ with*
(i) $\cong|_{\mathrm{Mod}(\sigma)}$ Borel;
(ii) $E \leq_B \cong|_{\mathrm{Mod}(\sigma)}$

one obtains

Theorem 14.4 *Let E be a Borel equivalence relation with $E \leq_B \cong|_{\mathrm{Mod}(\mathcal{L})}$ some countable language $\mathcal{L}$. Then exactly one of the following occurs:*
(I) $E \leq_B F$ for some countable Borel equivalence relation F;
(II) $(E_0)^\omega \leq_B E$.

It is possible to extend these results in some form to the case when $\cong|_{\mathrm{Mod}(\sigma)}$ is non-Borel. This is a little more than a routine paraphrasing of the arguments above. Essentially one asks if there is any fragment F appearing in some $L_{\omega_1^{\mathrm{ck}(z)}}[z]$ for which we arrive at case(I) or case(IIb); if so, then we obtain $(E_0)^\omega \leq_B \cong|_{\mathrm{Mod}(\sigma)}$ by the arguments given there. The difficulty comes about if there is no such fragment.

It becomes necessary to use something along the lines of lemma 4.2 of [12]. One can in some sense perform the construction of section 13 "in the codes". In this situation we obtain a weaker conclusion:

Theorem 14.5 *Let $\sigma \in \mathcal{L}_{\omega_1,\omega}$ for some countable language $\mathcal{L}$. Then exactly one of the following holds:*

(I) there is a $\underset{\sim}{\Delta}^1_2$ in the codes function

$$f : \mathrm{Mod}(\sigma) \to 2^{<\aleph_1}$$

such that for

(i) for all $\mathcal{M} \in \mathrm{Mod}(\sigma)$ we have $|\{g(\mathcal{N}) : \mathcal{N} \cong \mathcal{M}\}| \leq \aleph_0$, and

(ii) for all $\mathcal{N}_1, \mathcal{N}_2 \in \mathrm{Mod}(\sigma)$

$$g(\mathcal{N}_1) = g(\mathcal{N}_2) \Rightarrow \mathcal{N}_1 \cong \mathcal{N}_2;$$

(II) $(E_0)^\omega \leq_B \cong|_{\mathrm{Mod}(\sigma)}$.

This has an obvious parallel with 13.4.

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