An oscillation theorem for groups of isometries*

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Abstract

This paper answers a question raised by Vladimir Pestov.

Let G be a non-trivial group of isometries of a complete, separable metric space. Let $\hat{G}$ be the semi-group arising from its completion with respect to the left uniformity. Then there is a uniformly continuous function from $\hat{G}$ to the unit interval which assumes both extreme points on any non-empty right ideal.

Greg Hjorth[1] July 25, 2006

This paper has one main theorem. If G is a group of isometries of a space X, $\hat{G}$ the semi-group that arises as its completion in a certain natural sense, then we can find uniformly continuous functions on X^2 that display high oscillation on the image of any $\rho \in \hat{G}$. There are some details to be added to make this statement precise, but they are unsurprising, and can be postponed. There are some further issues about when X^2 can be replaced by X, and what one can say for specific classes of X and G, and these need also to be eventually addressed.

Before any of these housekeeping chores, I want to give an extended aside on two seemingly unrelated results in combinatorics, which can actually be viewed as lying at the heart of this theorem. Although these results are well known, the proofs are sufficiently short to be included, and the proof of the second of these will be needed at a later point.

Theorem 0.1 *(Ramsey's theorem) Let*

$$f : \mathbb{N}^2 \to \{0, 1\}.$$

Then there is an infinite

$$A \subset \mathbb{N}$$

and $i \in \{0, 1\}$ such that for any $a < b$ we have

$$f(a, b) = i.$$

Proof For each $a \in \mathbb{N}$ and infinite $B \subset \mathbb{N}$ we may certainly find an $i \in \{0, 1\}$ and an infinite $B' \subset B$ such that for all $b \in B'$

$$f(a, b) = i_a.$$

Applying this we successively choose $a_0 \in \mathbb{N}, B_{a_0} \subset \mathbb{N}, a_1 \in B_{a_0}, B_{a_1} \subset B_{a_0}, a_2 \in B_{a_1}, ...$ and so on, to obtain an increasing infinite sequence of a_i's and a sequence of infinite sets

$$B_{a_0} \supset B_{a_1} \supset B_{a_2} \supset ...$$

such that at each n we have some i_{a_n} such that every $b \in B_{a_n}$ has $f(a_n, b) = i_{a_n}$.

At this point we can find some infinite subsequence $(a_{n(k)})_{k \in \mathbb{N}}$ such that the corresponding $(i_{a_{n(k)}})_{k \in \mathbb{N}}$ is constant. □

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The next theorem is sometimes associated with the name of Devlin, though his real contribution is not this folklore result but the far more non-trivial task of computing certain precise bounds for partitions of $\mathbb{Q}^n$ at $n > 2$. There are various proofs for this theorem. The specific proof given below will be important.

Theorem 0.2 *(Folklore) There is a function*

$$f : \mathbb{Q} \to \{0, 1\}$$

such that for any order preserving injection

$$\rho : \mathbb{Q} \to \mathbb{Q}$$

there exist

$$q_1 < q_2, r_1 < r_2,$$

all in the image of ρ, with

$$f(q_1, q_2) \neq f(r_1, r_2).$$

Proof Let $(x_n)_{n\in\mathbb{N}}$ be some enumeration of $\mathbb{Q}$. For any $x_n < x_m$ in the rationals set

$$f(x_n, x_m) = 0$$

if $n < m$ and equal to 1 otherwise.

Now let $\rho : \mathbb{Q} \to \mathbb{Q}$ be an order preserving injection. Since ρ is order preserving we can find two real numbers $\alpha_1 < \alpha_2$ which are limits of increasing sequences from the image of ρ. Then choose sequences $(k(n))_{n\in\mathbb{N}}, (\ell(n))_{n\in\mathbb{N}}$ with

$$x_{k(n)} \to \alpha_1,$$

$$x_{\ell(n)} \to \alpha_2,$$

each $x_{k(n)}, x_{\ell(n)}$ in the image of ρ, each $x_{k(n)} < \alpha_1, x_{\ell(n)} < \alpha_2$.

We then let n be arbitrary and choose n' so that $\ell(n') > k(n)$ and then n'' with $k(n'') > \ell(n')$ to obtain

$$f(x_{k(n)}, x_{\ell(n')}) = 0,$$
$$f(x_{k(n'')}, x_{\ell(n')}) = 1.$$

□

The conclusions of Ramsey's and Devlin's theorem contrast one another. We obtain one kind of behavior for the ordering of $(\mathbb{N}, <)$ and the total opposite for $(\mathbb{Q}, <)$.

A distinctive property of $(\mathbb{Q}, <)$ is that for any two finite increasing sequences of the same size

$$q_1 < q_2 < ... < q_\ell,$$

$$r_1 < r_2 < ... < r_\ell,$$

we can find an automorphism of $(\mathbb{Q}, <)$ which sends the first sequence to the second. In this sense $(\mathbb{Q}, <)$ is *ultrahomogeneous*. Clearly one could phrase a related notion for graphs – a graph (V, E) is *ultrahomogeneous* if any isomorphism between finite subgraphs extends to an automorphism of the whole graph – and after that arbitrary countable structures in some relational language. Here there should be the analog of considering the values of π on increasing pairs, and for general ultrahomogenous structures the notion is that of *having the same type*; for the moment this can be roughly defined as having the same orbit under the automorphism group of the structure. Then instead of letting ρ range over order preserving maps we need to consider ρ which preserve the entire flank of relations associated to the structure.

Theorem 0.3 *Let $\mathcal{M}$ be an ultrahomogeneous relational structure whose automorphism group has size at least two. Then there is a subset $S \subset \mathcal{M}^2$ on which the automorphism group acts transitively and a function*

$$f : S \to \{0, 1\}$$

such that for any monomorphism

$$\rho : \mathcal{M} \to \mathcal{M}$$

we have a_1, a_2, b_1, b_2 in the image of ρ with

$$(a_1, a_2), (b_1, b_2) \in S,$$

$$f(a_1, a_2) \neq f(b_1, b_2).$$

0.3 is not the ultimate goal, and while a weaker form can be derived from the final theorem, its proof is easier. The main result of this paper is to prove a kind of generalization for separable complete metric spaces.

Let X be a complete separable metric space; at this stage we will assume just for convenience that the metric is bounded by one. Let G be a group of isometries, and for $\{x_i : i \in \mathbb{N}\}$ a dense subset of X we let

$$\hat{D}(\pi_1, \pi_2) = \sum_{n \in \mathbb{N}} 2^{-n} d(\pi_1(x_n), \pi_2(x_n))$$

provide a metric on G and we let $\hat{G}$ be the completion of G with respect to this metric; $\hat{G}$ can be identified with a semigroup of isometries from X to X.

Theorem 0.4 *For $G \leq \mathrm{Isom}(X)$ as above, G of size at least two, there are $x_0, x_1 \in X$ and uniformly continuous*

$$f : \overline{\{(\pi \cdot x_0, \pi \cdot x_1) : \pi \in G\}} \to [0, 1]$$

such that for any $\rho \in \hat{G}$ there exist

$$(y_0, y_1), (z_0, z_1) \in \{(\rho(\pi(x_0)), \rho(\pi(x_1))) : \pi \in G\}$$

with

$$f(y_0, y_1) = 0,$$
$$f(z_0, z_1) = 1.$$

A corollary answers a question raised in [6].

Corollary 0.5 *Let G be a Polish group of size bigger than one and let $\hat{G}$ be its completion with respect to the left uniformity. Then there is a bounded, uniformly left continuous*

$$f : \hat{G} \to [0, 1]$$

such for any non-empty right ideal

$$\mathcal{I} \subset \hat{G}$$

we can find

$$\rho_0 \neq \rho_1 \in \mathcal{I}$$

with $f(\rho_0) = 0, f(\rho_1) = 1$.

The connection between 0.5 and 0.3 is that the latter can be used to prove the former in the case that G is a closed subgroup of S_∞, the group of all permutations of $\mathbb{N}$ in the topology of pointwise convergence.

We prove 0.3 in §1.1. It consists in analyzing the relation of which tuples inside the structure can, up to some finite error, be defined from which other tuples. There are several cases, most of which turn out to be trivial, and in the one non-trivial case we construct the *irrepressibly* oscillating function f as in the proof of 0.2. §1.2 explores how variations might succeed or fail for other classes of countable structures. §1.2, unlike the rest of the paper, requires some familiarity with first order logic.

§2.1 gives some background information about groups of isometries and their ideals. We present the theorem in §2.2. The central idea of the proof is the notion of ϵ-capturing. This is something like an analog for metric spaces of the concept of definability over countable structures, and indeed there are parallels between the proof of this theorem and the proof of 0.3. There are also certain technical problems arising from the use of approximation arguments which have no precursor in the case of countable structures. It may be easier to understand the proof in §2.2 having first seen the proof of 1.1 in §1.1.

The 0.5, along with associated notational background, is presented in §2.3. §2.4 describes the connection between the main theorem and 1.1. §2.5 discusses some examples and specific situations in which a more optimal result may be possible. §2.6 explores connections with distortion theorems such as the Odell-Schlumprecht theorem for Hilbert space.

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1 Devlin type theorems

1.1 Ultrahomogeneous structures

Definition Let $\mathcal{M} = (M; R_0, R_1, R_2, ...)$ be a relational structure, with underlying set M, and each R_i a relation of arity $n(i)$, which is to say a subset of $M^{n(i)}$. (We allow either finitely many relations or countably infinitely many.) We will make the technically convenient assumption that one of the relations is equality.

$\pi : M \to M$ is said to be an *automorphism* of $\mathcal{M}$ if it is a bijection and for any $\vec{a} = (a_0, a_1, ..., a_{\ell-1}) \in M^{n(i)}$ we have

$$\vec{a} \in R_i \Leftrightarrow (\pi(a_0), \pi(a_1), ..., \pi(a_{n(i)-1}) \in R_i.$$

$\pi : \mathcal{M} \to \mathcal{M}$ is a *monomorphism* if it simply respects the relations in this way without necessarily being onto. (Since equality is hardwired into our language, such a π will necessarily be one to one.) We use $\mathrm{Aut}(\mathcal{M})$ to denote the automorphism group and for $\vec{a}$ a finite sequence from M we use $\mathrm{Aut}(\mathcal{M}, \vec{a})$ to denote the subgroup that fixes $\vec{a}$ pointwise.

For $\vec{a} = (a_0, a_1, ..., a_{\ell-1}) \in M^{\ell}$ we say that the *quantifier free type of $\vec{a}$ in M*, written $\text{q.f.type}_{\mathcal{M}}(\vec{a})$, is the set of pairs $(\vec{s}, i)$, where $\vec{s} \in \{0, 1, ...\ell\}^{n(i)}$, and $(a_{s(0)}, a_{s(1)}, ..., a_{s(n(i)-1)}) \in R_i$. In other words, the quantifier free type of a tuple $\vec{a}$ is the complete description of which finite sequences chosen from $\vec{a}$, with repetitions allowed, satisfy which relations from the language of $\mathcal{M}$.

We then say that a structure $\mathcal{M}$ is *ultrahomogeneous* if whenever $\vec{a}, \vec{b}$ are sequences of the same length in $\mathcal{M}$ with the same quantifier free type, then there is an automorphism

$$\pi : \mathcal{M} \to \mathcal{M}$$

with $\pi(a_i) = b_i$ at each applicable i.

In the following we will generally observe the usual lazy, convenient, but arguably inaccurate, custom, and identify $\mathcal{M}$ with its underlying set M.

Example The *random graph* can be defined in the following way. Consider the complete loopless undirected graph on the vertex set $\mathbb{N}$, with exactly one edge between any two distinct vertices. Assign to each a $\frac{1}{2}$ probability of being removed, and then go through the graph removing various edges with independent probability.

As can be found in [5], the isomorphism type of the resulting structure is constant on a conull set, and we use the term *random graph* to describe any representative of this isomorphism class. This structure is ultrahomogeneous, as can be shown by a "back-and-forth" argument, presented for instance in [5].

Example $(\mathbb{Q}, <)$, the rationals in their usual ordering are ultrahomogeneous. Again, see [5] for details.

Example Two examples which will be especially relevant are the *rational Urysohn metric space* and the *bounded rational Urysohn metric space*. The first of these can be characterized up to isometry as follows. It is the countable metric space such that the distance between any two points is rational and it satisfies the following extension axiom: given any finite sequence $x_0, x_1, ..., x_{\ell-1}$ from the space and rational distances $q_0, q_1, ..., q_{\ell-1}$ with

$$d(x_i, x_j) \leq q_i + q_j,$$

$$q_i \leq d(x_i, x_j) + q_j,$$

at all i, j, there exists y in the space such that each $d(x_i, y) = q_i$. Its cousin, the bounded Urysohn space, is similar, but allowing only distances ≤ 1 and only minding this extension axiom when each $q_i \leq 1$. It is again a routine back-and-forth argument to show isomorphisms between finite pieces extend to automorphisms of the whole space.

Example A degenerate example is provided by the structure where every point has a different name. So let $\mathcal{M} = (\mathbb{N}; R_0, R_1, R_2, ...)$ be the relational structure which has each R_i, $i > 0$, of arity 1 with

$$R_i = \{i + 1\}.$$

In this way distinct points have distinct quantifier free types, and thus the only way for q.f.type$_{\mathcal{M}}(\vec{a})$ to equal q.f.type$_{\mathcal{M}}(\vec{b})$ is for the sequences to be equal and then the identity map is the required automorphism.

Example A non-example is provided by $(\mathbb{N}; <)$. Clearly the quantifier free type of the sequence $(2, 3)$ is the same as $(5, 10)$, since all we can say is that the first point is smaller than another and these two sequences have that much in common. However there is no automorphism of $(\mathbb{N}; <)$ other than the trivial one which fixes every point.

Definition For $\mathcal{M}$ a countable relational structure and $\vec{a}$ a finite sequence from $\mathcal{M}$, we say that b is *in the algebraic closure of* $\vec{a}$, written

$$b \in \mathrm{acl}_{\mathcal{M}}(\vec{a}),$$

if the image of b under $\mathrm{Aut}(\mathcal{M}, \vec{a})$, the group of automorphisms of $\mathcal{M}$ fixing $\vec{a}$ pointwise, is finite.

Note that in the case that $\mathcal{M}$ is ultrahomogeneous, q.f.type$_{\mathcal{M}}(\vec{a}^\frown b)$ determines orbit in the automorphism group, and thus $b \in \mathrm{acl}_{\mathcal{M}}(\vec{a})$ if and only there are just finitely many c with

$$\text{q.f.type}_{\mathcal{M}}(\vec{a}^\frown b) = \text{q.f.type}_{\mathcal{M}}(\vec{a}^\frown c).$$

Theorem 1.1 *Let $\mathcal{M}$ be an ultrahomogeneous structure with $|\mathrm{Aut}(\mathcal{M})| > 1$. Then there is a pair $\vec{a} = (a_0, a_1)$ from $\mathcal{M}$ and*

$$f : \mathcal{M}^2 \to \{0, 1\}$$

such that for any monomorphism

$$\rho : \mathcal{M} \to \mathcal{M}$$

we can find $\vec{b}, \vec{c}$ in the image of ρ, both having the same quantifier free type as $\vec{a}$, but with

$$f(\vec{b}) = 0,$$

$$f(\vec{c}) = 1.$$

Proof We split the proof into cases, the first being the simplest.

Case(I): Every element of $\mathcal{M}$ is in $\mathrm{acl}(\emptyset)$, the algebraic closure of the empty sequence.

Fix some $\pi \in \mathrm{Aut}(\mathcal{M})$ not equal to the identity. Fix $a \in \mathcal{M}$ with $\pi(a) \neq a$ and let $b = \pi(a)$.

Our case assumptions imply that every point has only finitely many other points with the same quantifier free type. Since every monomorphism is injective and preserves quantifier free type, it follows that all monomorphisms are onto, and so we have a and b in the image of any such ρ. Thus if we simply define

$$f(a, x) = 0$$

$$f(b, x) = 1$$

for any $x \in \mathcal{M}$, and f arbitrary in all other cases, then f along with the quantifier free type of (a, a) is as required.

Case(II): Some $a \in \mathcal{M}$ is not in the algebraic closure of the empty sequence.

Let A be the collection of all points in $\mathcal{M}$ with the same quantifier free type as a. There is a further split into subcases.

Subcase(IIa): $\mathrm{acl}_{\mathcal{M}}(a) \cap A$ is infinite.

This implies that for any $b \in A$ we again have $\mathrm{acl}_{\mathcal{M}}(b) \cap A$ infinite, since the automorphism group of $\mathcal{M}$ acts transitively on A. Let $(c_i)_{i \in \mathbb{N}}$ enumerate A and choose successively points $d_i \neq e_i \in A$ such that

$$d_i, e_i \in \mathrm{acl}_{\mathcal{M}}(c_i) \cap A$$

and

$$d_i, e_i \notin \{d_j, e_j : j < i\}.$$

We can do this since each $\mathrm{acl}_{\mathcal{M}}(c_i) \cap A$ is infinite.

Then, parallel to before, we define at each i

$$f(d_i, x) = 0,$$

$$f(e_i, x) = 1$$

all x, arbitrary else wise. Any monomorphism must pick up some c_i in its image, and after that some corresponding d_i, e_i.

There is a sense in which these two cases are trivial, since we could have found the function $f : \mathcal{M} \to \{0, 1\}$, which implies the existence of a function on pairs as described by the theorem. The one case left is non-trivial in that it actually requires pairs, and that case tracks the argument from 0.2.

Subcase(IIb): $\mathrm{acl}_{\mathcal{M}}(a) \cap A$ is finite.

Then since the automorphism group acts transitively on A there will be some single bound n_0 such that for every $b \in A$

$$|\mathrm{acl}_{\mathcal{M}}(a) \cap A| \leq n_0.$$

Since A is infinite we can find a sequence of length $n_0 + 1$

$$b_0, b_1, ..., b_{n_0},$$

and another point

$$c \in A \setminus (\mathrm{acl}_{\mathcal{M}}(b_0) \cup \mathrm{acl}_{\mathcal{M}}(b_1) \cup ... \cup \mathrm{acl}_{\mathcal{M}}(b_{n_0})).$$

One of these b_i's will fail to be in the algebraic closure of c, and in this way we have a pair (c, b_i), which it will be convenient to rename,

$$(a_0, a_1),$$

with

$$a_0 \notin \mathrm{acl}_{\mathcal{M}}(a_1),$$
$$a_1 \notin \mathrm{acl}_{\mathcal{M}}(a_0).$$

We let $(x_i)_{i\in\mathbb{N}}$ enumerate A. For (x_i, x_j) having the same type as

$$(a_0, a_1)$$

we let

$$f(x_i, x_j) = 0$$

if $i < j$ and

$$f(x_i, x_j) = 1$$

if $i > j$. f is again arbitrary off of the type of (a_0, a_1).

The claim is that f and the type of (a_0, a_1) is as required, and so suppose $\rho : \mathcal{M} \to \mathcal{M}$ is a monomorphism. We let

$$x_i = \rho(a_0),$$
$$x_j = \rho(a_1).$$

There are two possibilities, the first being $i < j$ and $f(\rho(a_0), \rho(a_1)) = 0$. Then note that since there are infinitely many d's with (d, a_1) having the same quantifier free type as (a_0, a_1) we can find some such d with

$$\rho(d) \notin \{x_0, x_1, ..., x_j\}$$

and then $\rho(d) = x_k$ some $k > j$ and

$$f(\rho(d), \rho(a_1)) = 1$$

as required.

The other possibility is $i > j$, but then, similarly, since $a_1 \notin \mathrm{acl}_{\mathcal{M}}(a_0)$ we can replace a_1 with some suitable d to make $f(\rho(a_0), \rho(d)) = 0$. □

1.2 Examples and counterexamples for first order logic and beyond

This section differs from the other parts of the paper in having considerable prerequisites from logic which are not always made explicit. This section can be bypassed. None of the results resurface later in the paper.

We consider various ways in which 1.1 might be modified. From the point of view of a logician it is natural to consider the situation of elementary embeddings between first order structure. While there are two theorems which follow directly from 1.1, there is a counterexample to the most naive formulations.

Example We build a structure $\mathcal{M}$ admitting many pairs with the same first order type and such that for any set of tuples with the same type

$$S \subset \mathcal{M}^2$$

and

$$f : S \to \{0, 1\}$$

we can find an elementary embedding of $\mathcal{M}$ into itself with the image homogeneous for f.

We first let $\{Q_n : n \in \mathbb{N}\}$ be mutually dense co-dense subsets of the half open interval $(0,1]$ such that any two distinct rationals are separated by some Q_n; we do this so that for any distinct non-overlapping finite sets $I_0, I_1 \subset \mathbb{N}$ the intersection

$$\bigcup_{n \in I_0} Q_n \setminus (\bigcup_{n \in I_1} Q_n)$$

is dense in the interval $(0,1)$. (The required Q_n's can be produced by a routine diagonalization argument, which we omit.) We then extend these subsets to $P_n \subset \mathbb{R}$ by setting $\ell + r \in P_n$ if and only if $r \in Q_n$, where $\ell \in \mathbb{Z}$, $r \in (0,1]$. We let $\mathcal{M}$ be the structure whose underlying set is $\mathbb{Q}^+$, the positive rationals, equipped with the usual ordering $<$ of the rationals, but also the countably many unary predicates, $\{P_n : n \in \mathbb{N}\}$, with the interpretation given above. Note then that for each $q \in \mathcal{M}$ and semi-open interval $(\ell, \ell+1]$, $\ell \in \mathbb{N}$, there is exactly one $q' \in (\ell, \ell+1]$ with the same type as q.

Now given $S \subset \mathcal{M}^2$ consisting of pairs of the same type, we can find some $q, q' \in \mathbb{Q}$ such that $S(a,b)$ implies $a \equiv q \bmod \mathbb{Z}$ and $b \equiv q' \bmod \mathbb{Z}$. After possibly replacing S by its reflection we can assume $S(x_0, x_1) \Rightarrow x_0 < x_1$. Then let

$$f : S \to \{0,1\}.$$

Thus the domain of f consists of all pairs

$$(\ell + q, k + q')$$

which have $\ell + q < k + q'$.

From f we define a new function

$$f_{\mathbb{N}} : [\mathbb{N}]^2 \to \{0,1\}$$

with

$$f_{\mathbb{N}}(\ell, k) = f(\ell + q, k + q').$$

Applying Ramsey's theorem we find a

$$A \subset \mathbb{N}$$

and a single $i \in \{0,1\}$ such that whenever $k < \ell$ are both in A

$$f_{\mathbb{N}}(k, \ell) = i.$$

We let

$$k_0 < k_1 < k_2 < \dots$$

enumerate A in increasing order. Then let $\hat{\mathcal{M}}$ be the set of rationals which are in a semi-open interval of the form

$$[k_{2n}, k_{2n} + q)$$

($(k_0, k_0 + q)$ in the case $n = 0$) or

$$[k_{2n+1} + q, k_{2n+1} + 1).$$

It follows from the assumptions on A that f is constant on $S \cap \hat{\mathcal{M}}^2$. Standard model theoretic arguments show that

$$\hat{\mathcal{M}} \prec \mathcal{M},$$

and the inclusion map from $\hat{\mathcal{M}}$ into $\mathcal{M}$ is elementary. (Using a back and forth argument one can show that if $\mathcal{L}_n$ is the finite language consisting of $(<, P_0, P_1, \dots, P_n)$ then for any $\vec{a} \in \hat{\mathcal{M}}$

$$(\hat{\mathcal{M}}, \vec{a})|_{\mathcal{L}_n} \cong (\mathcal{M}, \vec{a})|_{\mathcal{L}_n},$$

which clearly suffices.)

To have a sensible analog of 1.1 we must either pass to infinitary logic or content ourselves with atomic models.

Recall that $\mathcal{L}_{\omega_1,\omega}$ is the infinitary logic obtained by allowing countable conjunctions and disjunctions.

Proposition 1.2 *Let $\mathcal{M}$ be a countable first order structure with non-trivial automorphism group. Then there is an automorphism invariant set $S \subset \mathcal{M}^2$ and*

$$f : S \to \{0,1\}$$

such that for any $\mathcal{L}_{\omega_1,\omega}$ elementary substructure

$$\mathcal{N} \prec_{\mathcal{L}_{\omega_1,\omega}} \mathcal{M}$$

f picks up both possible values on

$$\mathcal{N}^2 \cap S.$$

Proof Following Scott's theorem, as found in say [7], we can find a countable fragment $F \subset \mathcal{L}_{\omega_1,\omega}$ such that for any $\vec{a} \in \mathcal{M}$ there is some $\varphi(\vec{x}) \in F$ such that for any other $\vec{b}$

$$(\mathcal{M}, \vec{a}) \cong (\mathcal{M}, \vec{b})$$

if and only if $\mathcal{M} \models \varphi(\vec{b})$.

Then let $\{R_\varphi : \varphi \in F\}$ be a collection of fresh predicates, and let $\hat{\mathcal{M}}$ be the expansion of $\mathcal{M}$ obtained by adding these predicates to $\mathcal{M}$ with the obvious interpretation. It follows from the assumptions on F that this new expanded model $\hat{\mathcal{M}}$ is ultrahomogeneous, and in fact this is true in the strong sense that any tuple $\vec{a}$ has its orbit under $\mathrm{Aut}(\mathcal{M})$ determined by a single quantifier free formula $R_\varphi(\vec{a})$. It also follows from the usual kinds of back and forth arguments that for any $\mathcal{L}_{\omega_1,\omega}$ elementary substructure

$$\mathcal{N} \prec_{\mathcal{L}_{\omega_1,\omega}} \mathcal{M}$$

we can find an $\mathcal{L}_{\omega_1,\omega}$-elementary

$$\rho : \mathcal{M} \cong \mathcal{N}.$$

This all said, the proposition follows from 1.1. □

Proposition 1.3 *Let $\mathcal{M}$ be a countable atomic model with non-trivial automorphism group. Then there is a type $\tau(x_0, x_1)$ and*

$$f : \{(a,b) \in \mathcal{M}^2 : \mathcal{M} \models \tau(a,b)\} \to \{0,1\}$$

such that for an elementary

$$\mathcal{N} \prec \mathcal{M}$$

we have f picking up both values on $\mathcal{N}^2$.

Proof This follows from the proof of the last proposition. $\mathcal{M}$ being atomic means that the fragment F from that argument can be taken to be the collection of first order formulas. □

Another detail which arises from the proof of 1.1 is that frequently we can take the irrepressible function f to be one which is defined not on pairs but on single points. In both case(I) and case(IIa) from 1.1 we actually obtain

$$f : \mathcal{M} \to \{0,1\}$$

such that for any elementary $\rho : \mathcal{M} \to \mathcal{M}$ one has both values assumed by f on $\rho[\mathcal{M}]$. There are some atomic structures for which the oscillating f is unavoidably defined on pairs – for instance, $(\mathbb{N}; =)$, a countably infinite set equipped only with the relation of equality, or even $(\mathbb{Q}; <)$. On the other hand the partially ordered space obtained from $(\mathbb{Q}; <)$ by bifurcating every point into an antichain of size two admits an irrepressibly oscillating f defined just on single points.

It is not clear whether one can characterize the atomic models for which the f of 1.3 can be defined on $\mathcal{M}$ as against $\mathcal{M}^2$.

2 The oscillation theorem

2.1 Groups and semigroups of isometries: Definitions, lemmas, and so on

Definition Given a metric space (X, d) we let $\mathrm{Isom}(X)$ be the group of isometries of X onto itself and $\mathrm{In}(X)$ the semi-group of isometries from X into itself. We define the *canonical topologies* on $\mathrm{In}(X)$ as follows. Given a $z_0, z_1, ..., z_m, y_0, ..., y_m \in X, \epsilon > 0$ we let

$$\mathcal{O}(z_0, ..., z_m, y_0, ..., y_m, \epsilon)$$

be the set of $\rho \in \mathrm{In}(X)$ with

$$d(\rho(z_i), y_i) < \epsilon$$

at each $i \leq m$, and we give $\mathrm{In}(X)$ the topology with the basis all open sets of this form. $\mathrm{Isom}(X)$ is then given the induced subspace topology.

Lemma 2.1 *If X is separable and completely metrizable, then both $\mathrm{In}(X)$ and $\mathrm{Isom}(X)$ are Polish in the indicated topology.*

Proof Let $\{x_i : i \in \mathbb{N}\}$ be some arbitrary dense subset of X. We define D on $\mathrm{Isom}(X)$ with

$$D(\pi_1, \pi_2) = \sum_{n \in \mathbb{N}} 2^{-n} \max(1, d(\pi_1(x_n), \pi_2(x_n)) + d(\pi_1^{-1}(x_n), \pi_2^{-1}(x_n)))$$

and $\hat{D}$ on $\mathrm{In}(X)$ by

$$\hat{D}(\rho_1, \rho_2) = \sum_{n \in \mathbb{N}} 2^{-n} \max(1, d(\rho_1(x_n), \rho_2(x_n))).$$

It is routine to verify that these metrics have the required properties of completeness and separability. □

Definition For $G \leq \mathrm{Isom}(X)$ a subgroup, we let $\hat{G}$ be its closure in $\mathrm{In}(X)$.

Even when G is a group, there is no necessity for its closure in the semi-group $\mathrm{In}(X)$ to form a group. For instance, if $X = \mathbb{N}$ and d is the complete metric, then for $G = S_\infty$ the group of all permutations of $\mathbb{N}$ we have $\hat{G}$ equal to all injections from $\mathbb{N}$ to $\mathbb{N}$. In general we also have $\hat{G}$ much smaller than $\mathrm{In}(X)$ even when G equals $\mathrm{Isom}(X)$. For instance if X equals $\mathbb{N}$ with $d(m, n) = |m - n|$, then the isometry group is trivial, whilst $\mathrm{In}(X)$ consists of infinitely many possible shifts.

For the rest of this section fix some complete separable metric space (X, d) and let G be some group of isometries.

Definition For $\rho_0 \in \hat{G}$ we let $\mathcal{J}(\rho_0)$ be the $\hat{D}$ closure of the set

$$\{\rho_0 \cdot \pi : \pi \in G\}.$$

Lemma 2.2 *ρ is in $\mathcal{J}(\rho_0)$ if and only if for every $\epsilon > 0$ and $x_1, ..., x_m \in X$ there exists $\pi \in G$ such that*

$$d(\rho_0(\pi(x_i)), \rho(x_i)) < \epsilon$$

at every $i \leq m$.

Proof At once from the definition of the topology. □

Lemma 2.3 *Let $\rho_0 \in \hat{G}$ and use X_0 to denote $\rho_0[X]$. Let $x_1, ..., x_m \in X, y_1, ..., y_m \in X_0$. Then the following three things are equivalent:*

(i) There exists $\rho \in \{\rho_0 \cdot \pi : \pi \in G\}$ with

$$d(\rho(x_i), y_i) < \epsilon$$

all $i \leq m$.
(ii) There there exists $\rho \in \mathcal{J}(\rho_0)$ with

$$d(\rho(x_i), y_i) < \epsilon$$

all $i \leq m$.
(iii) There exists $\pi \in G$ with

$$d(\pi(x_i), y_i) < \epsilon$$

all $i \leq m$.

Proof (i) and (ii) are clearly equivalent in virtue of the definition of $\mathcal{J}(\rho_0)$, thus it suffices to show (ii) if and only if (iii). The only if direction is immediate from ρ_0 being in the $\hat{D}$-closure of G, and so let us assume instead that $\pi \in G$ and $d(\pi(x_i), y_i) < \epsilon$ all $i \leq m$. Let z_i be such that $\rho_0(z_i) = y_i$ all $i \leq m$. Choose $\sigma_n \in G$ with $\sigma_n \to \rho_0$ in $\hat{G}$; this gives in particular that at each i

$$\sigma_n(z_i) \to y_i$$

at each i, and then since G acts by isometries

$$\sigma_n^{-1}(y_i) \to z_i,$$

and then at large enough n

$$\begin{aligned} d(\sigma_n^{-1}(\pi(x_i)), z_i) &\leq d(\sigma_n^{-1}(\pi(x_i)), \sigma_n^{-1}(y_i)) + d(\sigma_n^{-1}(y_i), z_i) \\ &= d(\pi(x_i), y_i) + d(\sigma_n^{-1}(y_i), z_i) < \epsilon. \end{aligned}$$

Then since ρ_0 is an isometry, at large enough n

$$d(\rho_0(\sigma_n^{-1}(\pi(x_i))), \rho_0(z_i)) = d(\rho_0(\sigma_n^{-1}(\pi(x_i))), y_i) < \epsilon,$$

as required for $\rho = \rho_0 \circ \sigma_n^{-1}$ to verify the lemma. □

Corollary 2.4 *$\mathcal{J}(\rho_0)$ is closed under multiplication.*

Proof Let $\gamma_0, \gamma_1 \in \mathcal{J}(\rho_0)$. Since ρ_0 is a limit of elements in G we can find sequences $(\sigma_n^0)_n, (\sigma_n^1)_n$ from G with

$$\sigma_n^0 \to \gamma_0,$$
$$\sigma_n^1 \to \gamma_1.$$

Appealing to 2.2 we need only show that for all $x_1, ..., x_m \in X$, $\epsilon > 0$, if we set $y_i = \gamma_0(\gamma_1(x_i)))$, then we can produce $\pi \in G$ with

$$d(\rho_0(\pi(x_i)), y_i) < \epsilon$$

at each $i \leq m$. By appealing to 2.3, it suffices to show that there is some $\hat{\pi} \in G$ with

$$d(\hat{\pi}(x_i), y_i) < \epsilon$$

all $i \leq m$.

But if $z_i = \gamma_1(x_i)$, then we obtain

$$\sigma_n^1(x_i) \to \gamma_1(x_i)$$

and then

$$\sigma_n^2(z_i) \to \gamma_2(z_i) = y_i$$

by the definition of $\hat{D}$ convergence. From this and the assumption that each σ_n^1 is an isometry we obtain

$$\sigma_n^1 \circ \sigma_n^2(x_i) \to y_i$$

as $n \to \infty$, and so we take $\hat{\pi}$ to be $\sigma_n^1 \circ \sigma_n^2$ for n sufficiently large. □

Definition $\mathcal{J} \subset \hat{G}$ is a *right ideal* if it is closed under multiplication on the right by G and it is closed under composition.

Thus 2.4 shows that any $\mathcal{J}(\rho_0)$ is a right ideal.

2.2 The theorem

Definition Let G be a group of isometries of (X, d) and $\epsilon, \delta > 0$. For $x, y \in X$ we say that x ϵ, δ-*captures* y relative to G if there is some $N \in \mathbb{N}$ such that we cannot find N disjoint balls of the form

$$B_\epsilon(\pi(y))$$

for $\pi \in G$ with

$$d(\pi(x), x) < \delta.$$

Here $B_\epsilon(\pi(y))$ denotes the ball of all points z with $d(\pi(y), z) < \epsilon$. Thus in loose terms we could say that x ϵ, δ-captures y if it not possible to keep x very still while simultaneously moving y around to many different places.

Theorem 2.5 *Let (X, d) be a complete metric space. Let $G \leq \mathrm{Isom}(X, d)$ be a group of isometries and let $\hat{G} \subset \mathrm{In}(X, d)$ be its closure. Assume G has size bigger than one.*

Then there exists $x_0, x_1 \in X$ and uniformly continuous

$$f : \overline{\{(\pi \cdot x_0, \pi \cdot x_1) : \pi \in G\}} \to [0, 1]$$

such that for any $\rho_0 \in \hat{G}$ there exist

$$(y_0, y_1), (z_0, z_1) \in \{(\rho_0(\pi(x_0)), \rho_0(\pi(x_1))) : \pi \in G\}$$

with

$$f(y_0, y_1) = 0,$$
$$f(z_0, z_1) = 1.$$

Proof It suffices to show that there is a uniformly continuous $f : X \times X \to [0, 3]$ and $x_0, x_1 \in X$ such that for any $\rho_0 \in \hat{G}$ there are $\pi_1, \pi_2 \in G$ with

$$f(\rho_0(\pi_1(x_0)), \rho_0(\pi_1(x_1))) < 1,$$

$$f(\rho_0(\pi_2(x_0)), \rho_0(\pi_2(x_1))) > 2.$$

Given an f with this property, we can obtained the f as above by truncating to the interval $[1, 2]$ and subtracting 1. Thus we head towards constructing $f : X \times X \to [0, 3]$. Once the problem has been put in this form we can always assume that G is a closed subgroup of $\mathrm{Isom}(X)$, since this present form of the statement of the theorem will pass downwards from the closure of G to G itself.

Case(I): Every $x \in X$ has precompact orbit under G.

That means we can write

$$X = \bigcup_{x \in X} C_x,$$

each $C_x \subset X$ with $C_x \supset [x]_G$, compact and

$$\pi[C_x] \subset C_x$$

all $\pi \in G$, and hence

$$\rho[C_x] \subset C_x$$

all $\rho \in \hat{G}$. But since any isometry of a compact metric space into itself will be onto, it follows that any $\rho \in \hat{G}$ will actually be an isometry, and we are done once we choose some $(x_0, x_1) \in X \times X$ with non-trivial orbit under G and an *onto* uniformly continuous $f : \overline{(\pi(x_0), \pi(x_1)) : \pi \in G\}} \to [0, 3]$.

Case(II): Some $x \in X$ does not have precompact orbit in X.

Fix such an x_0 without precompact orbit. Fix ϵ such that there are infinitely many disjoint balls of the form

$$B_\epsilon(\pi(x_0))$$

for $\pi \in G$. The rest of the argument involves rather messy approximations, but in general terms we want to replace the notion of algebraic closure by the notion of being $\frac{\epsilon}{20}, \frac{\epsilon}{100}$-captured. To avoid constantly needing to mention these unpleasant constants, from now on we use the following completely ad hoc definition.

Definition We say that x *ϵ-captures* y if it $\frac{\epsilon}{20}, \frac{\epsilon}{100}$-captures y in the original sense given prior to the statement of the theorem.

Subcase(IIa): There exists $x_1 \in G \cdot x_0$ such that x_1 does not ϵ-capture x_0 and x_0 does not ϵ-capture x_1.

Note that since G acts by isometries, for any $x \in G \cdot x_0$ there exists $x' \in G \cdot x = G \cdot x_0$ with neither of x, x' ϵ-capturing the other.

Fix such an x_1 as in subcase hypothesis. Let $\{z_n : n \in \mathbb{N}\}$ be dense in $G.x_0$. At each $n \in \mathbb{N}$ we define continuous

$$g_n : X \to [0, 3]$$

by

$$g_n(y) = \max\{0, 3 - \frac{150}{\epsilon}(d(y, B_{\frac{\epsilon}{50}}(z_n)))\}.$$

(Here $d(y, B_{\frac{\epsilon}{50}}(z_n))$ denotes the infimum over $d(y, z)$ for z chosen with $d(z, z_n) < \frac{\epsilon}{50}$.) Thus g_n is uniformly continuous uniformly in n, and constantly assumes the value 3 on $B_{\frac{\epsilon}{50}}(z_n)$ and the value zero outside $B_{\frac{\epsilon}{25}}(z_n)$.

We then define $f : X \times X \to [0, 3]$ by

$$f(y, y') = \max(0, \sup\{g_n(y) - \max_{m \leq n} g_m(y') : n \in \mathbb{N}\}.$$

Given the uniformity of the continuity of the $g_n's$ we obtain that f is uniformly continuous. Unwrapping the definitions we have that f constantly assume 3 on

$$B_{\frac{\epsilon}{50}}(z_n) \times (X \setminus \bigcup_{m \leq n} B_{\frac{\epsilon}{25}}(z_m))$$

and zero on

$$(X \setminus \bigcup_{m < n} B_{\frac{\epsilon}{25}}(z_m)) \times B_{\frac{\epsilon}{50}}(z_n).$$

I claim that f is as required.

For this purpose fix $\rho_0 \in \hat{G}$ and

$$\sigma_n \to \rho_0$$

in G. We need to show that f attains $[0, 1)$ and $(2, 3]$ inside

$$\{(\rho_0 \circ \pi(x_0), \rho_0 \circ \pi(x_1)) : \pi \in G\}.$$

Choose $m(x_0), m(x_1)$ so that

$$\rho_0(x_0) \in B_{\frac{\epsilon}{100}}(z_{m(x_0)}),$$

$$\rho_0(x_1) \in B_{\frac{\epsilon}{100}}(z_{m(x_1)}).$$

Let $k = \max\{m(x_0), m(x_1)\} + 1$. Let $\pi_1, ..., \pi_k, \pi'_1, ..., \pi'_k \in G$ be such that at each $i \leq k$ we have

$$d(\pi_i(x_0), x_0) < \frac{\epsilon}{100},$$

$$d(\pi'_i(x_1), x_1) < \frac{\epsilon}{100},$$

but for $i \neq j$

$$B_{\frac{\epsilon}{20}}(\pi_i(x_1)) \cap B_{\frac{\epsilon}{20}}(\pi_j(x_1)) = 0,$$
$$B_{\frac{\epsilon}{20}}(\pi'_i(x_0)) \cap B_{\frac{\epsilon}{20}}(\pi'_j(x_0)) = 0.$$

Since ρ_0 is an isometry arising as a limit of invertible isometries in G these imply $B_{\frac{\epsilon}{20}}(\rho(\pi_i(x_1)))\cap B_{\frac{\epsilon}{20}}(\rho(\pi_j(x_1)) = 0$ and $B_{\frac{\epsilon}{20}}(\rho(\pi'_i(x_0))) \cap B_{\frac{\epsilon}{20}}(\rho(\pi'_j(x_0)) = 0$.

Choose $\delta > 0$ with

$$\rho_0(x_0) \in B_{(\frac{\epsilon}{100}-2\delta)}(z_{m(x_0)}),$$
$$\rho_0(x_1) \in B_{(\frac{\epsilon}{100}-2\delta)}(z_{m(x_1)}),$$

and $\delta < \frac{\epsilon}{200}$. Choose n large enough that at each $i \leq k$

$$d(\sigma_n(\pi_i(x_0)), \rho_0(\pi_i(x_0))),\ \ d(\sigma_n(\pi_i(x_1)), \rho_0(\pi_i(x_1))) < \frac{\epsilon}{200},$$

$$d(\sigma_n(x_0), \rho_0(x_0)),\ \ d(\sigma_n(x_1), \rho_0(x_1)) < \delta.$$

Let $\hat{\pi}_i = \sigma_n \pi_i \sigma_n^{-1}$, $\hat{\pi}'_i = \sigma_n \pi'_i \sigma_n^{-1}$.

Then at each $i \leq k$ we have

$$d(\hat{\pi}_i(\rho_0(x_0)), \rho_0(x_0)) \leq d(\hat{\pi}_i(\rho_0(x_0)), \hat{\pi}_i(\sigma_n(x_0))) + d(\hat{\pi}_i(\sigma_n(x_0)), \sigma_n(x_0)) + d(\sigma_n(x_0), \rho_0(x_0)),$$

which in virtue of $\hat{\pi}_i$ being an isometry equals

$$d(\rho_0(x_0), \sigma_n(x_0)) + d(\hat{\pi}_i(\sigma_n(x_0)), \sigma_n(x_0)) + d(\sigma_n(x_0), \rho_0(x_0)),$$

which in turn by the definition of $\hat{\pi}_i$ equals

$$d(\rho_0(x_0), \sigma_n(x_0)) + d(\sigma_n(\pi_i(x_0)), \sigma_n(x_0)) + d(\sigma_n(x_0), \rho_0(x_0)).$$

Then since σ_n is an isometry this equals

$$d(\rho_0(x_0), \sigma_n(x_0)) + d(\pi_i(x_0), x_0) + d(\sigma_n(x_0), \rho_0(x_0))$$
$$< \delta + \frac{\epsilon}{100} + \delta = 2\delta + \frac{\epsilon}{100}.$$

A similar calculation gives

$$d(\hat{\pi}_i(\rho_0(x_1)), \rho_0(\pi_i(x_1))) \leq d(\hat{\pi}_i(\rho_0(x_1)), \hat{\pi}_i(\sigma_n(x_1))) + d(\hat{\pi}_i(\sigma_n(x_1)), \rho_0(\pi_i(x_1)))$$
$$= d(\rho_0(x_1), \sigma_n(x_1)) + d(\hat{\pi}_i(\sigma_n(x_1)), \rho_0(\pi_i(x_1))) < \frac{\epsilon}{200} + \frac{\epsilon}{200} = \frac{\epsilon}{100}.$$

And of course we have the same calculations on the other side, replacing $\hat{\pi}_i$ by $\hat{\pi}'_i$, x_0 by x_1, and x_1 by x_0. Thus at some high enough m we have

$$d(\hat{\pi}_i(\sigma_m(x_0)), \rho_0(x_0)) < 2\delta + \frac{\epsilon}{100},$$
$$d(\hat{\pi}_i(\sigma_m(x_1)), \rho_0(\pi_i(x_1))) < \frac{\epsilon}{100},$$
$$d(\hat{\pi}'_i(\sigma_m(x_1)), \rho_0(x_1)) < 2\delta + \frac{\epsilon}{100},$$
$$d(\hat{\pi}'_i(\sigma_m(x_0)), \rho_0(\pi_i(x_0))) < \frac{\epsilon}{100}.$$

At this point we can appeal to lemma 2.3 and find $\rho_1, ..., \rho_k, \rho'_1, ..., \rho'_k \in \{\rho_0 \circ \pi : \pi \in G\}$ such that at each i

$$d(\rho_i(x_0), \rho_0(x_0)) < 2\delta + \frac{\epsilon}{100},$$
$$d(\rho_i(x_1), \rho_0(\pi_i(x_1))) < \frac{\epsilon}{100},$$

$$d(\rho_i'(x_1), \rho_0(x_1)) < 2\delta + \frac{\epsilon}{100},$$
$$d(\rho_i'(x_0)), \rho_0(\pi_i(x_0))) < \frac{\epsilon}{100}.$$

Then
$$B_{\frac{\epsilon}{25}}(\rho_i(x_1)) \subset B_{(\frac{\epsilon}{25}+\frac{\epsilon}{100})}(\rho_0(\pi_i(x_1))) = B_{\frac{\epsilon}{20}}(\rho_0(\pi_i(x_1))),$$
and hence
$$B_{\frac{\epsilon}{25}}(\rho_i(x_1)) \cap B_{\frac{\epsilon}{25}}(\rho_j(x_1)) = 0$$
for $i \neq j$. On the other hand
$$d(\rho_i(x_0), z_{m(x_0)}) \leq d(\rho_i(x_0), \rho_0(x_0)) + d(\rho_0(x_0), z_{m(x_0)})$$
$$< (2\delta + \frac{\epsilon}{100}) + (\frac{\epsilon}{100} - 2\delta) = \frac{\epsilon}{50}$$
$$\therefore \rho_i(x_0) \in B_{\frac{\epsilon}{50}}(z_{m(x_0)}).$$
Similarly at each $i \neq j$
$$B_{\frac{\epsilon}{25}}(\rho_i'(x_0)) \cap B_{\frac{\epsilon}{25}}(\rho_j'(x_0)) = 0$$
whilst
$$\rho_i'(x_1) \in B_{\frac{\epsilon}{50}}(z_{m(x_1)}).$$

Since each z_m can be in at most one of the balls $B_{\frac{\epsilon}{25}}(\rho_i(x_1))$, $i \leq k$, and since $k > m(x_0)$, we may find specific i_0 with
$$\rho_{i_0}(x_1) \notin B_{\frac{\epsilon}{25}}(z_m)$$
all $m \leq m(x_0)$. Similarly we can fix a j_0 with
$$\rho_{j_0}'(x_0) \notin B_{\frac{\epsilon}{25}}(z_m)$$
all $m \leq m(x_1)$. We then let
$$(y_0, y_1) = (\rho_{i_0}(x_0), \rho_{i_0}(x_1)),$$
$$(z_0, z_1) = (\rho_{j_0}'(x_1), \rho_{j_0}'(x_1)).$$

Since $y_1 \notin \bigcup_{m \leq m(x_0)} B_{\frac{\epsilon}{25}}(z_m)$ whilst $y_0 \in B_{\frac{\epsilon}{50}}(z_{m(x_0)})$ we get
$$f(y_0, y_1) = 3,$$
whilst since $z_0 \notin \bigcup_{m \leq m(x_1)} B_{\frac{\epsilon}{25}}(z_m)$ whilst $z_1 \in B_{\frac{\epsilon}{50}}(z_{m(x_1)})$ we alternatively obtain
$$f(z_0, z_1) = 0.$$

Subcase(IIb): There does *not* exist $x_1 \in G \cdot x_0$ with neither x_0, x_1 ϵ-capturing the other.

Subsubcase(IIbi): The set of $y \in G \cdot x_0$ such that x_0 ϵ-captures y can be covered by finitely many balls of radius $\frac{\epsilon}{4}$.

Fix N such that the y's in $G \cdot x_0$ that are ϵ-captured by x_0 can be covered by N balls of radius $\frac{\epsilon}{4}$. Since G is acting by isometries we have that for every $x \in G \cdot x_0$ the set of y's ϵ-captured by x can be covered by N many balls of radius $\frac{\epsilon}{4}$.

Claim: For any $\rho \in \hat{G}$ and $x' \in G \cdot x_0$ there exists $y \in \rho[X]$ with $d(y, x') < \frac{\epsilon}{10}$.

Proof of Claim: Fix $\sigma_n \to \rho$ in G. Fix $y_1, ..., y_N, y_{N+1}$ in $G \cdot x_0$ with
$$B_\epsilon(y_i) \cap B_\epsilon(y_j) = 0$$
for $i \neq j$. We then pass to a large enough M that
$$d(\sigma_n(y_i), \sigma_m(y_i)) < \frac{\epsilon}{100}$$

all $m, n \geq M$, $i \leq N+1$ and let
$$z = \sigma_M^{-1}(x').$$

By the subsubcase assumptions at (IIbi) there will be some $i_0 \leq N+1$ with x' not ϵ-capturing $\sigma_M(y_{i_0})$. But then the subcase assumptions at (IIb) we have $\sigma_M(y_{i_0})$ ϵ-capturing $x' = \sigma_M(z)$.

Then for all $n \geq M$

$$d((\sigma_n \circ \sigma_M^{-1})(\sigma_M(y_{i_0})), \sigma_M(y_{i_0})) = d(\sigma_n(y_{i_0}), \sigma_M(y_{i_0})) < \frac{\epsilon}{100},$$

and thus by $\sigma_M(y_{i_0})$ ϵ-capturing $x' = \sigma_M(z)$ we have

$$d(\sigma_n(z), x') = d(\sigma_n(\sigma_M^{-1}(x')), x') < \frac{\epsilon}{20}.$$

Since this happens at all n sufficiently large,

$$d(\rho_{i_0}(z), x') \leq \frac{\epsilon}{20},$$

and we may complete the claim by taking $y = \rho(z)$. (Claim$\square$)

But now we simply choose $q_0, q_1 \in G \cdot x_0$ with

$$B_\epsilon(q_0) \cap B_\epsilon(q_1) = 0$$

and define
$$f_0 : X \to [0, 3]$$
$$w \mapsto \min(0, \max(3, \frac{6}{\epsilon} d(w, B_{\frac{\epsilon}{2}}(q_1)).$$
We let x_1 be arbitrary and define

$$f : \overline{(\pi(x_0), \pi(x_1)) : \pi \in G\}} \to [0, 3]$$

with $f(w_0, w_1) = f_0(w_0)$.

For any $\rho_0 \in \hat{G}$ we can appeal to the claim and lemma 2.3 to find $\pi_0, \pi_1 \in G$ with

$$\rho_0(\pi_0(x_0)) \in B_{\frac{\epsilon}{2}}(q_0),$$

$$\rho_0(\pi_1(x_0)) \in B_{\frac{\epsilon}{2}}(q_1),$$
and take $(y_0, y_1) = (\rho_0(\pi_0(x_0)), \rho_0(\pi_0(x_1))), (z_0, z_1) = (\rho_0(\pi_1(x_0)), \rho_0(\pi_1(x_1)))$.

Subsubcase(IIbii): The set of $y \in G \cdot x_0$ that are ϵ-captured by x_0 *cannot* be covered by finitely many balls of radius $\frac{\epsilon}{4}$.

Let $\{x_n : n \in \mathbb{N}\}$ enumerate a dense subset of $G \cdot x_0$.

Claim: There exists sequences $(k'_n)_{n \in \mathbb{N}}$, $(k''_n)_{n \in \mathbb{N}}$, $(\ell'_n)_{n \in \mathbb{N}}$, $(\ell''_n)_{n \in \mathbb{N}}$ and arrays

$$(y'_{j,i,m})_{m \in \mathbb{N}, i < k'_m, j < \ell'_m},$$

$$(y''_{j,i,m})_{m \in \mathbb{N}, i < k''_m, j < \ell''_m}$$
of elements in $G \cdot x_0$ such that at each $n \in \mathbb{N}$:

(a) if $\pi \in G$ has $d(\pi(x_m), x_m) < \frac{\epsilon}{100}$ and $j' \leq \ell'_m, j'' \leq \ell''_m$ then at some $p' \leq k'_m, p'' \leq k''_m$

$$d(\pi(y'_{j',0,m}), y'_{j',p',m}) < \frac{\epsilon}{10},$$

$$d(\pi(y''_{j'',0,m}), y''_{j'',p'',m}) < \frac{\epsilon}{10};$$

(b) for $j \neq j' \leq \ell'_m$

$$d(y'_{j,0,m}, y'_{j',0,m}) \geq \frac{\epsilon}{4};$$

(c) for $j \neq j' \leq \ell''_m$

$$d(y''_{j,0,m}, y''_{j',0,m}) \geq \frac{\epsilon}{4};$$

(d)

$$\ell'_n > \sum_{m<n} \ell'_m \cdot k'_m + \sum_{m<n} \ell''_m \cdot k''_m;$$

(e)

$$\ell''_n > \sum_{m\leq n} \ell'_m \cdot k'_m + \sum_{m<n} \ell''_m \cdot k''_m.$$

Proof of Claim: This is a matter of grinding down the assumptions of subsubcase(IIbii). We build the sequences inductively. Given that we have decided on

$$(y'_{j,i,m})_{m<n, i<k'_m, j<\ell'_m}, (y''_{j,i,m})_{m<n, i<k''_m, j<\ell''_m}$$

we choose some very large ℓ'_n as indicated at (d) and then a sequence of $(y'_{j,0,n})_{j<\ell'_n}$ which are all ϵ-captured by x_0 and are distance at least $\frac{\epsilon}{4}$ apart. At each $j < \ell'_n$ we can choose finitely many $y'_{j,i,m}$'s in (a) by the definition of ϵ-capturing. We take k'_n to be the maximum of the number of $y'_{j,i,m}$'s we need at the various $i < \ell'_n$. Then construction of the $(y_{j,i,n})_{i<k''_n, j<\ell''_n}$ is exactly similar. (Claim$\square$)

Our interest in this claim will be in terms of an application of the pigeonhole principle. If π is an isometry of X then at some $q' < \ell'_n$ we have

$$\pi(y'_{q',0,n}) \notin \bigcup_{m<n, i<k'_m, j<\ell'_m} B_{\frac{\epsilon}{9}}(y'_{j,i,m}) \cup \bigcup_{m<n, i<k''_m, j<\ell''_m} B_{\frac{\epsilon}{9}}(y''_{j,i,m})$$

and at some $q'' < \ell'_n$ we have

$$\pi(y'_{q'',0,n}) \notin \bigcup_{m\leq n, i<k'_m, j<\ell'_m} B_{\frac{\epsilon}{9}}(y'_{j,i,m}) \cup \bigcup_{m<n, i<k''_m, j<\ell''_m} B_{\frac{\epsilon}{9}}(y''_{j,i,m}).$$

Now at each m we let

$$A_m = \bigcup_{i<k'_m, j<\ell'_m} B_{\frac{\epsilon}{9}}(y'_{j,i,m}),$$

$$B_m = \bigcup_{i<k''_m, j<\ell''_m} B_{\frac{\epsilon}{9}}(y''_{j,i,m}),$$

and then

$$C_n = \bigcup_{i<k'_n, j<\ell'_n} B_{\frac{\epsilon}{10}}(y'_{j,i,n}) \setminus (\bigcup_{m<n} A_m \cup \bigcup_{m<n} B_m),$$

$$D_n = \bigcup_{i<k''_n, j<\ell''_n} B_{\frac{\epsilon}{10}}(y''_{j,i,n}) \setminus (\bigcup_{m\leq n} A_m \cup \bigcup_{m<n} B_m).$$

It follows from clauses (b) and (d) above that each C_n is non-empty, and from (c) and (e) that each D_n is non-empty. The way in which these sets have been defined guarantees that

$$d(C_n, D_m) \geq \frac{\epsilon}{9} - \frac{\epsilon}{10} = \frac{\epsilon}{90}$$

at *every* m, n, and thus $d(C, D) \geq \frac{\epsilon}{90}$, and thus we can find a uniformly continuous

$$f_0 : X \to [0, 3]$$

with $f_0|_C \equiv 3,\ f_0|_D \equiv 0$.

Claim: For any $\rho_0 \in \hat{G}$ there exists $z' \in \overline{C} \cap \rho_0[G \cdot x_0]$.

Proof of Claim: Choose $\sigma_n \to \rho_0$ in G, and go to N_0 such that at all later $n \geq N_0$ we have

$$d(\sigma_n(x_0), \sigma_{N_0}(x_0)) < \frac{\epsilon}{300}.$$

Choose N_1 with $d(\sigma_{N_0}(x_0), x_{N_1}) < \frac{\epsilon}{300}$ and then observe that at all $n \geq N_0$

$$\begin{aligned} d(\sigma_n\sigma_{N_0}^{-1}(x_{N_1}), x_{N_1}) &\leq d(\sigma_n\sigma_{N_0}^{-1}(x_{N_1}), \sigma_n(x_0)) + d(\sigma_n(x_0), \sigma_{N_0}(x_0)) + d(\sigma_{N_0}(x_0), x_{N_1}) \\ &= d(\sigma_n\sigma_{N_0}^{-1}(x_{N_1}), \sigma_n\sigma_{N_0}^{-1}(\sigma_{N_0}(x_0)) + d(\sigma_n(x_0), \sigma_{N_0}(x_0)) + d(\sigma_{N_0}(x_0), x_{N_1}) \\ &= d(x_{N_1}, \sigma_{N_0}(x_0)) + d(\sigma_n(x_0), \sigma_{N_0}(x_0)) + d(\sigma_{N_0}(x_0), x_{N_1}) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{\epsilon}{100}. \end{aligned}$$

Thus by assumption (a) on our construction we can find at each $n \geq N_0$ and $j < \ell'_{N_1}$ we can find some corresponding $p_j < k'_{N_1}$ with

$$d(\sigma_n\sigma_{N_0}^{-1}(y'_{j,0,N_1}), y'_{j,p_j,N_1}) < \frac{\epsilon}{10}.$$

(b) gives for $j < j' \leq \ell'_{N_1}$ we have

$$d(\sigma_n\sigma_{N_0}^{-1}(y'_{j,0,N_1}), \sigma_n\sigma_{N_0}^{-1}(y'_{j',0,N_1})) \geq \frac{\epsilon}{4},$$

and so by applying the pigeonhole principle to clause (d) from our construction we can find some specific $j(n) < \ell'_{N_1}$ with

$$\sigma_n\sigma_{N_0}^{-1}(y'_{j(n),0,N_1}) \notin \bigcup_{n<N_1} A_m \cup \bigcup_{m<N_1} B_m,$$

and hence $\sigma_n\sigma_{N_0}^{-1}(y'_{j(n),0,N_1}) \in C$. Thinning out the sequence $(\sigma_n)_{n\in\mathbb{N}}$ we may assume there is a fixed j with $j = j(n)$ all n. Thus for

$$z' = \sigma_{N_0}^{-1}(y'_{j,0,N_1})$$

we have each $\sigma_n(z') \in C$ and hence in the limit $\rho_0(z') \in \overline{C}$. (Claim□)

An exactly similar claim gives the existence of some z'' with $\rho_0(z'') \in \overline{D}$, thereby completing the proof for this last remaining subsubcase, and hence the proof of the theorem. □

Corollary 2.6 *Let $(X, d), G, \hat{G}$ be as above, again with $|G| > 1$.*

Then there exists uniformly continuous

$$f : \overline{\{(\pi \cdot x_0, \pi \cdot x_1) : \pi \in G\}} \to [0, 1]$$

such that for any non-empty right ideal $\mathcal{I} \subset \hat{G}$ there are

$$\rho, \gamma \in \mathcal{I}$$

with

$$f(\rho(x_0), \rho(x_1)) = 0,$$
$$f(\gamma(x_0), \gamma(x_1)) = 1.$$

Proof We can assume that the ideal has the form $\mathcal{I} = \mathcal{J}(\rho_0)$ for some $\rho_0 \in \hat{G}$, and then 2.6 follows directly from the definitions and the theorem. □

Corollary 2.7 *Let $(X,d), G, \hat{G}$ be as above, again with $|G|>1$. Define $\hat{D}$ to be the usual metric on $\hat{G}$, with*

$$\hat{D}(\rho_0,\rho_1)=\sum_{n\in\mathbb{N}}2^{-n}\max(1,d(\rho_0(z_n),\rho_1(z_n)))$$

for some dense set $\{z_n : n\in\mathbb{N}\}\subset X$.

Then there exists uniformly continuous

$$\hat{f}:\hat{G}\to[0,1]$$

such that for any non-empty right ideal $\mathcal{I}\subset\hat{G}$ there are

$$\rho,\gamma\in\mathcal{I}$$

with

$$\hat{f}(\rho)=0,$$
$$\hat{f}(\gamma)=1.$$

Proof Take f and x_0, x_1 as in 2.6 and let

$$\hat{f}(\rho)=f(\rho(x_0),\rho(x_1)).$$

It is immediate from the assumptions on f that $\hat{f}$ has high oscillation on any non-trivial ideal and the definition of $\hat{D}$ implies that $\hat{f}$ is uniformly continuous. □

Remark In each of 2.5, 2.6, 2.7 we can ask that f not just be uniformly continuous but in fact Lipschitz. For instance in 2.5 we can replace f by

$$f^*:X^2\to[0,1]$$

with

$$f^*(z_0,z_1)=\min(1,\inf\{d(z_0,y_0)+d(z_1,y_1): f(y_0,y_1)=0)\}).$$

f^* is clearly Lipschitz and there will be some positive $\epsilon\leq 1$, depending on the uniformity of continuity of f, and fixed $x_0,x_1\in X$ such that f^* will assume the value 0 and at least the value ϵ on

$$\{(\rho_0(\pi(x_0)),\rho_0(\pi(x_1))) : \pi\in G\}$$

for any $\rho_0\in\hat{G}$. Then we can simply rescale and truncate f^*, replacing it by

$$(z_0,z_1)\mapsto\min(1,\frac{f^*(z_0,z_1)}{\epsilon}).$$

2.3 Oscillation for Polish groups

Theorem 2.5 allows us to conclude with a kind of distortion theorem for Polish groups. If G is a Polish group and d_L is a left invariant metric, and $\hat{G}$ is the Cauchy completion of G with respect to d_L, then there is a uniformly continuous

$$f:\hat{G}\to[0,1]$$

such that for any non-empty ideal $\mathcal{I}\subset\hat{G}$ we have f attaining both 0 and 1 on $\mathcal{I}$.

There are a couple of remarks and definitions before we can give the proof.

Theorem 2.8 *(Birkhoff-Kakutani) Every Polish group has a compatible left invariant metric.*

Proof See [1]. □

This is *not* to say that every Polish group admits a *complete* left invariant metric. S_∞ for instance, the group of all permutations of the natural numbers with the topology of pointwise convergence, does not.

Lemma 2.9 *Any two compatible left invariant metrics have the same Cauchy sequences.*

Proof Note that a sequence $(g_n)_{n\in\mathbb{N}}$ is Cauchy with respect to one compatible left invariant metric (and hence all compatible left invariant metrics) if and only if for every open neighborhood U of the identity there is an N such that for all $m, n > N$

$$g_m^{-1} g_n \in U.$$

$\square$

Thus the *left completion* of a Polish group does not depend on the choice of a left invariant metric. This gives a way of exactly matching up the notions from §2.2 with the present situation.

Notation For G a Polish group, let $\hat{G}$ be the Cauchy completion of G with respect to some (equivalently, any) choice of a compatible left invariant metric d_L on G.

Lemma 2.10 *If $(g_n)_{n\in\mathbb{N}}, (h_n)_{n\in\mathbb{N}}$ are Cauchy with respect to a left invariant metric on a Polish group G then $(g_n h_n)_{n\in\mathbb{N}}$ is Cauchy with respect to the left invariant metric.*

Proof Nodding back to the proof of the last lemma, note that for any open neighborhood U of the identity we can find an $N_1 < N_2$ such that all bigger $n, m \geq N_2$ have

$$g_m^{-1} g_n \in h_{N_1} U h_{N_1}^{-1},$$

and for all $m, n \geq N_1$

$$h_m^{-1} h_n \in U.$$

Then at any $m, n \geq N_2$

$$(g_m h_m)^{-1}(g_n h_n) = (h_m^{-1} h_{N_1})(g_m h_{N_1})^{-1} g_n h_{N_1}(h_{N_1}^{-1} h_n) \in U^3.$$

Letting U shrink to radius zero proves the lemma. $\square$

Thus G acts on its left completion by multiplication on the right.

Definition A subset $\mathcal{I} \subset \hat{G}$ is said to be a *right ideal* if it is closed under composition and under multiplication on the right by elements in G.

Note that we can identify each $g \in G$ with an isometry

$$\pi_g : h \mapsto gh$$

of (G, d_L) for any left invariant metric. Since G is dense in $\hat{G}$, any such isometry extends to an isometry of $\hat{G}$.

Lemma 2.11 *Under the natural identification of G with a subgroup of* $\mathrm{Isom}(\hat{G}, d_L)$ *and $\hat{G}$ with a subset of* $\mathrm{In}(\hat{G}, d_L)$*, and for $\hat{D}$ defined as in §2.1, we have that $\hat{G}$ equals the $\hat{D}$-closure of G and the composition of elements in $\hat{G}$ in the sense of lemma 2.1 coincides with their composition as elements of* $\mathrm{In}(\hat{G}, d_L)$ *– and in fact, for each $\epsilon > 0$ there exist δ_1, δ_2 such that*

$$\hat{D}(\rho_0, \rho_1) < \delta_1 \Rightarrow d_L(\rho_0, \rho_1) < \epsilon,$$

$$d_L(\rho_0, \rho_1) < \delta_2 \Rightarrow \hat{D}(\rho_0, \rho_1) < \epsilon.$$

Proof $\hat{D}$ provides a left invariant metric on G, and hence gives rise to the same completion by lemma 2.9. For checking the compositionality, note that the composition of the isometries associated to Cauchy $(g_n)_n, (h_n)_n$ sends each $h \in G$ to the limit of $(g_n h_n h)_n$.

It remains to show the equivalence of the two metrics defined on $\hat{G}$, as either the Cauchy completion of G or the closure of $\{\pi_g : g \in G\}$ in $\mathrm{In}(\hat{G}, d_L)$. We fix a dense subset $\{z_i : i \in \mathbb{N}\}$ of $\hat{G}$ and define the metric $\hat{D}$ as usual. We can assume that this dense subset lies inside G, since G is dense in its Cauchy completion, and we can also assume that $z_1 = e$, the identity in G. It is clear that the topology obtained by viewing $\hat{G}$ as a subset of $\mathrm{In}(\hat{G}, d_L)$ is at least as strong the topology given by viewing as the Cauchy completion, since for any $\epsilon > 0$ we have that if

$$\hat{D}(\rho_0, \rho_1) < \frac{\epsilon}{2}$$

then

$$d_L(\rho_0(z_1), \rho_1(z_1)) = d_L(\rho_0 z_1, \rho_1 z_1) = d_L(\rho_0, \rho_1) < \epsilon.$$

For the converse, consider $\epsilon > 0$. We can find $\delta > 0$ and an initial segment $z_1, ..., z_N$ of our enumerated dense subset, such that for any $\rho_0, \rho_1 \in \hat{G}$ with

$$d_L(\rho_0(z_i), \rho_1(z_i)) < \delta$$

all $i \leq N$ we have

$$\hat{D}(\rho_0, \rho_1) < \epsilon.$$

Then we can find some δ_1 such that

$$d_L(g, e) < \delta_1 \Rightarrow \forall i \leq N(d_L(z_i^{-1} g z_i, e) < \delta,$$

$$\therefore \forall g_0, g_1 \in G \forall i \leq N(d_L(g_0, g_1) < \delta_1 \Rightarrow d_L(z_i^{-1}(g_1^{-1} g_0) z_i, e) = d_L(g_0 z_i, g_1 z_i) < \delta,$$

$$\therefore \forall \rho_0, \rho_1 \in \hat{G} \forall i \leq N(d_L(\rho_0, \rho_1) < \delta_1 \Rightarrow d_L(\rho_0 z_i, \rho z_i) < \delta),$$

as required. □

Thus in particular the definition of *closed right ideal* in $\hat{G}$ given above agrees with the definition presented in §2.1.

Theorem 2.12 *Let G be a Polish group of size bigger than one, and $\hat{G}$ its left completion with respect to some left invariant metric d_L. Then there is a uniformly continuous*

$$f : (\hat{G}, d_L) \to [0, 1]$$

such that on any non-empty ideal $\mathcal{I} \subset \hat{G}$ we have f obtaining both zero and one.

Proof As in the proof of theorem 2.5 it suffices to show that we can obtain $f : \hat{G} \to [0, 3]$ such that on any non-empty ideal it hits values less than one and greater than 2. Following 2.5 we can choose uniformly continuous $(\gamma_0, \gamma_1) \in \hat{G}$ and uniformly continuous

$$f_0 : \overline{\{(g \cdot \gamma_0, g \cdot \gamma_1) : g \in G\}} \to [0, 3]$$

such that for any $\rho_0 \in \hat{G}$ and induced ideal $\mathcal{J}(\rho_0) \subset \mathrm{In}(\hat{G}, d_L)$ we attain values in both $[0, 1)$ and $(2, 3]$ on the set

$$\{(\rho \cdot \gamma_0, \rho \cdot \gamma_1) : \rho \in \mathcal{J}(\rho_0)\}.$$

This induces a function

$$f : \hat{G} \to [0, 3]$$

with the specification that

$$f(\rho) = f_0(\rho \cdot \gamma_0, \rho \cdot \gamma_1).$$

It follows immediately from 2.11 that f is uniformly left continuous as a function on $(\hat{G}, d_L)$ and that f obtains the required oscillation on any non-empty ideal. □

This theorem is in the form posed as a problem in [6].

2.4 Connecting 1.1 to the oscillation theorem

Definition S_∞ is the group of all permutations of $\mathbb{N}$ with the topology of pointwise convergence.

We specifically include permutations with infinite support in S_∞. A basic open set in this topological group consists in all $\pi \in S_\infty$ such that

$$\pi(i) = k_i$$

all $i \leq N$, some choice of $k_1, ..., k_N$.

Theorem 2.13 *(Folklore) Any closed subgroup of S_∞ equals Aut($\mathcal{M}$) for some ultrahomogeneous structure whose underlying set is $\mathbb{N}$.*

Proof See [1]. □

Notation Given a countable ultrahomogeneous structure $\mathcal{M}$, let $\mathrm{In}(\mathcal{M})$ be the set of all monomorphisms

$$\rho : \mathcal{M} \hookrightarrow \mathcal{M}.$$

We place a complete metric on $\mathrm{In}(\mathcal{M})$ as follows. We first assume without loss that the underlying set of $\mathcal{M}$ is the set of natural numbers. Then for $\rho_0 \neq \rho_1$ we let $\Delta(\rho_0, \rho_1)$ be the first n with $\rho_0(n) \neq \rho_1(n)$, and then define

$$d(\rho_0, \rho_1) = 2^{-\Delta(\rho_0, \rho_1)}.$$

Recall that in §2.3 we defined $\hat{G}$ for G a Polish group to be the Cauchy completion with respect to any compatible left invariant metric. We now have that d provides a left invariant metric on $\mathrm{In}(\mathcal{M})$ and hence $\mathrm{Aut}(\mathcal{M})$, and so for $G = \mathrm{Aut}(\mathcal{M})$ we can identify $\hat{G}$ with the closure of $\mathrm{Aut}(\mathcal{M})$ with respect to d.

Using ultrahomogeneity this $\hat{G}$ actually turns out to be the full semi-group of monomorphisms from $\mathcal{M}$ to itself.

Lemma 2.14 *For $\mathcal{M}$ ultrahomogeneous, $G = \mathrm{Aut}(\mathcal{M})$, we have that $\hat{G}$ equals $\mathrm{In}(\mathcal{M})$.*

Proof Immediately from the definition of ultrahomogeneous we have that for any $\rho \in \mathrm{In}(\mathcal{M})$ and $\vec{a}, \vec{b} \in \mathcal{M}^{<\infty}$, if $\rho(\vec{a}) = \vec{b}$ then there is an automorphism π with $\pi(\vec{a}) = \vec{b}$. Since this holds for all $\vec{a}, \vec{b}$ we have $\rho \in \hat{G}$. □

Lemma 2.15 *Let $\rho_0 \in \mathrm{In}(\mathcal{M})$. Then the ideal*

$$\mathcal{J}(\rho_0) = \overline{\{\rho_0 \circ \pi : \pi \in \mathrm{Aut}(\mathcal{M})\}}$$

equals all $\gamma \in \mathrm{In}(\mathcal{M})$ with $\mathrm{range}(\gamma) \subset \mathrm{range}(\rho_0)$.

Proof It is clear from the definitions that if $\gamma \in \mathcal{J}(\rho_0))$ then its image is included in that of ρ_0. Conversely, if the $\mathrm{range}(\gamma) \subset \mathrm{range}(\rho_0)$ then at each $\vec{a} \in \mathcal{M}^{<\infty}$ we can first find $\vec{c}$ with $\rho_0(\vec{c}) = \gamma(\vec{a})$ and then, since $\vec{a}, \vec{c}$ have the same type, we can find $\pi \in \mathrm{Aut}(\mathcal{M})$ with $\pi(\vec{a}) = \vec{c}$ to obtain

$$\rho_0 \circ \pi(\vec{a}) = \gamma(\vec{a}).$$

Since this holds for all such $\vec{a}$ we have that γ is in the closure of $\{\rho_0 \circ \pi : \pi \in \mathrm{Aut}(\mathcal{M})\}$. □

With all this background out of the way, we can obtain theorem 2.12 for G a closed subgroup of S_∞ as follows.

Applying 2.13 we can assume $G = \mathrm{Aut}(\mathcal{M})$ for some ultrahomogeneous $\mathcal{M}$. We use 1.1 to find some $(a_0, a_1) \in \mathcal{M}^2$ and

$$f : \mathcal{M}^2 \to \{0, 1\}$$

such that for S the set of all pairs with the same type as (a_0, a_1), we have f obtaining both possible values on $S \cap \rho_0[\mathcal{M}] \times \rho_0[\mathcal{M}]$ for any monomorphism $\rho_0 : \mathcal{M} \to \mathcal{M}$. Then we can define

$$\hat{f} : \hat{G} \to \{0, 1\}$$

by $\hat{f}(\gamma) = f(\gamma(a_0), \gamma(a_1))$. It is trivial to verify that this is left uniformly continuous Given any $\rho_0 \in \hat{G}$ we choose $\vec{b} = (b_0, b_1), \vec{c} = (c_0, c_1)$ in range$(\rho) \cap S$ with $f(\vec{b}) = 0, f(\vec{c}) = 1$. We let b_0', b_1', c_0', c_1' be the preimages under ρ of b_0, b_1, c_0, c_1 respectively. Appealing to ultrahomogeneity we can find $\pi, \sigma \in \mathrm{Aut}(\mathcal{M})$ with

$$\pi(a_0) = b_0', \;\; \pi(a_1) = b_1',$$

$$\sigma(a_0) = c_0', \;\; \sigma(a_1) = c_1',$$

and hence $\rho \circ \pi(\vec{a}) = \vec{b}$, $\rho \circ \sigma(\vec{a}) = \vec{b}$, and thus

$$\hat{f}(\rho \circ \pi) = f(\vec{b}) = 0,$$

$$\hat{f}(\rho \circ \sigma) = f(\vec{c}) = 1,$$

as required.

2.5 $n = 1$ and $n = 2$ spaces

Definition Let (X, d) be a complete separable metric space and let G be the isometry group of all onto isometries from X to X. As previously defined, let $\hat{G}$ be its closure in the space of all isometries from X into X. We say that X is an $n = 1$ *space* if there is a single point $x_0 \in X$ and uniformly continuous

$$f : \overline{\{\pi(x_0) : \pi \in G\}} \to [0, 1]$$

such that for any $\rho_0 \in \hat{G}$ we have f achieving both extreme values on

$$\overline{\{\rho_0(\pi(x_0)) : \pi \in G\}}.$$

Otherwise we say that X is an $n = 2$ *space*.

Thus this terminology is motivated by 2.5 which says that X is either an $n = 1$ space or there is an

$$f : \overline{\{(\pi(x_0), \pi(x_1)) : \pi \in G\}} \to [0, 1]$$

which picks up both extreme values on any

$$\overline{\{(\rho_0(\pi(x_0)), \rho_0(\pi(x_1))) : \pi \in G\}}.$$

While there are some $n = 2$ spaces, most of the examples I know are trivial.

Example Let (X, d) be a countably infinite set equipped with the discrete metric. This is an $n = 2$ space.

Example Recall from say [3] that the *Urysohn space* $\mathbb{U}$ is the complete, separable, universal metric space which is ultrahomogeneous, in the sense that any isometry between finite subsets extends to an isometry of the whole space onto itself.

The Urysohn space is an $n = 1$ space: Fix some arbitrary z_0 in the space. We choose rapidly increasing $(k_n)_{n \in \mathbb{N}}$ with

$$k_n > \sum_{m<n} 2k_m,$$

and we let $B_m = \{z : k_m < d(z, z_0) < 2k_m\}$. We can certainly find a uniformly continuous function f which is 0 on each B_{2m} and 1 on each B_{2m+1}, and then any isometric embedding

$$\rho : \mathbb{U} \to \mathbb{U}$$

must pick up all but finitely many of the B_m's in its image.

This argument arose in the course of conversations with Vladimir Pestov, and it is clear that it modifies to show any unbounded metric space to be an $n = 1$ space.

Example A more mysterious example is the *bounded Urysohn space*,$\mathbb{U}_{\leq 1}$ – the complete separable ultrahomogeneous metric space which is universal for metric spaces bounded by 1 ($d(x, x') \leq 1$ all x, x').

Question Is the bounded Urysohn space an $n = 1$ space?

The next example might give some hope.

Example Let $\mathbb{U}_{\{0,1,2\}}$ be the shaved down version $\mathbb{U}$ allowing only the distances $0, 1, 2$. This can be characterized as follows: It is the *countable* metric space where $d(x, x')$ is always either 0, 1, or 2, and which satisfies the following *extension* axiom: For any disjoint finite sequences $x_1, ..., x_N$ and $y_1, ..., y_M$ we can find some z with

$$d(x_i, z) = 1,$$

$$d(y_j, z) = 2,$$

all $i \leq N, j \leq M$. (It is a routine back-and-forth argument to show that $\mathbb{U}_{\{0,1,2\}}$ is the unique up to isometry countable ultrahomogeneous metric space universal among countable spaces realizing only distances 0, 1, 2. Compare for instance chapter 2 of [1].)

$\mathbb{U}_{\{0,1,2\}}$ is an $n = 2$ space. The main point is that given any $x_1, ..., x_N, y_1, ..., y_M$ as above, the set of z which satisfy $d(x_i, z) = 1$, $d(y_j, z) = 2$, all $i \leq N, j \leq M$ is already isometric to $\mathbb{U}_{\{0,1,2\}}$. Thus given $A \subset \mathbb{U}$ we either have that A satisfies the extension axiom, or there is some $x_1, ..., x_N, y_1, ..., y_M$ in A such that the set of z's with $d(x_i, z) = 1$, $d(y_j, z) = 2$, all $i \leq N, j \leq M$ forms a copy of $\mathbb{U}_{\{0,1,2\}}$ disjoint from A.

Thus in trying to understand the situation with the bounded Urysohn space one might want to ask about the *rational bounded Urysohn space.*

Example Let $\mathbb{U}_{\mathbb{Q},\leq 1}$ be the *countable* ultrahomogeneous metric space which is universal among all countable metric spaces realizing only rational distances bounded by one. Although this space is not complete, it is natural from the model theoretic point of view. If we introduce a relation R_q for each $q \in \mathbb{Q} \cap [0, 1]$ with

$$R_q(x, x')$$

exactly when $d(x, x') = q$ then we obtain a countable ultrahomogeneous structure.

In the first draft of this paper the following question was posed:

Question Does any partition of $\mathbb{U}_{\mathbb{Q},\leq 1}$ into finitely many pieces have one piece in which there is an isometric copy of the whole space?

This has since been answered negatively by [2]. Their argument seems to leave unresolved the earlier problem of whether the bounded Urysohn space is an $n = 1$ space.

2.6 The ragged end: Unclear connections with Banach space theory

Definition Let $\mathbb{B}$ a Banach space and $(\mathbb{B}_1, d)$ the unit ball, with the obvious metric $d(x, x') = \|x - x'\|$. Let G be the group of all isometries of $(\mathbb{B}_1, d)$ which are the restrictions of linear isometries of $\mathbb{B}$ onto itself. In the usual way we form a metric $\hat{D}$ on the space $\mathrm{In}(X, d)$ of isometries from X into itself with

$$\hat{D}(\rho_0, \rho_1) = \sum_{n \in \mathbb{N}} 2^{-n} d(\rho_0(x_n), \rho_1(x_n))$$

for some dense subset $\{x_n : n \in \mathbb{N}\}$ and let $\hat{G}$ be the $\hat{D}$-closure of G.

I will say that $\mathbb{B}$ is an $n = 1$ *Banach space* if there exists point $x_0 \in \mathbb{B}_1$ and a uniformly continuous

$$f : \overline{G \cdot x_0} \to [0, 1]$$

such that for any $\rho \in \hat{G}$ there are $y_0, z_0 \in \rho[G \cdot x_0]$ with

$$f(y_0) = 0,$$

$$f(z_0) = 1.$$

In the case of Hilbert space ℓ^2 we can find for each infinite dimensional subspace $H \subset \ell^2$ a sequence of isometries $\sigma_n : \ell^2 \to \ell^2$ along with $x(\xi)_{\xi \in H}$ such that

$$\sigma_n(x(\xi)) \to \xi$$

and

$$\{x(\xi) : \xi \in H\} = \ell^2.$$

Thus we obtain $\rho \in \hat{G}$ as the limit of the $\sigma_n|_{\mathbb{B}_1}$ with $\rho[\mathbb{B}_1] = H$. Thus the assertion that Hilbert space is an $n = 1$ Banach space amounts to the assertion there is a uniformly continuous function from the unit ball to $[0, 1]$ which realizes both extreme points on any infinite dimensional subspace, an assertion which has indeed been proved in Odell and Schlumprecht's celebrated solution to the *distortion problem.*

Theorem 2.16 *(Odell-Schlumprecht; see [8]) If $\mathbb{B} = \ell^p$ for $1 < 0 < \infty$ then there is a uniformly continuous function from the unit ball*

$$f : \mathbb{B}_1 \to [0, 1]$$

which assumes both 0 and 1 on every infinite dimensional subspace.

This should compared with:

Theorem 2.17 *(Gowers; see [4]) If $\mathbb{B} = c_0$, then for any uniformly continuous function from the unit ball*

$$f : \mathbb{B}_1 \to [0, 1]$$

and any $\epsilon > 0$ there is an isometric copy of c_0 appearing as a subspace on which $|f(x) - f(x')|$ is bounded by ϵ.

However in contrast to ℓ^2 there are isometric embeddings of c_0 into itself which do not appear in $\hat{G}$, the closure of the *onto* isometries. It is not hard to see that any onto isometry of c_0 must move take the canonical basis back to itself with possible multiplications by -1, and thus for c_0 we actually have that the onto isometries are *already* closed in the space of isometries, and thus, for almost trivial reasons, c_0 is an $n = 1$ Banach space. It is therefore natural to ask:

Question Does there exist a separable Banach space which is not an $n = 1$ Banach space?

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