Relative ergodicity results for product groups

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Contents

1 Introduction

These notes present various relative ergodicity results, to the effect that actions of Bernoulli shifts of a comparatively larger number of copies of the free group must be relatively ergodic to free actions of smaller numbers of copies of the free group. Ultimately this is used to give a "quote-unquote" elementary proof of a theorem previously established by Adams and Kechris, that there are $\leq_B$-incomparable countable Borel equivalence relations.

2 Definitions

Definition We let E_0 be the equivalence relation of eventual agreement on infinite binary sequences; thus for $f, g \in \{0,1\}^{\mathbb{N}}$ we set

$$fE_0 g$$

if there is some N such that for all $n > N$

$$f(n) = g(n).$$

Definition Given groups standard Borel groups G, H, with G acting in a Borel manner on standard Borel X, we say that

$$\alpha : X \times G \to H$$

is a *cocycle* if it is Borel and for all $x \in X$, $g_1, g_2 \in G$

$$\alpha(x, g_1 g_2) = \alpha(g_2 \cdot x, g_1)\alpha(g_1, x).$$

Two cocycles

$$\alpha, \beta : X \times G \to H$$

are *cohomologous* if there is a Borel

$$\psi : X \to H$$

such that for all $x \in X, g \in G$

$$\beta(x, g) = \psi(g \cdot x)\alpha(x, g)\psi(x)^{-1}.$$

Frequently we will have a Borel probability measure μ on X and we can talk about a μ-measurable cocycle, which will be a cocycle defined on some conull subset of (X, μ), and two cocycles being μ-measurably cohomologous if they are outright cohomologous on a conull set. When there is no ambiguity about the measure, we will simply write *measurable* in place of the more lengthy μ-measurable.

In the present paper we will only really be considering the case of G and H both countable. Admittedly many of the results extend to the wider context of locally compact.

Definition Let E and F be equivalence relations on standard Borel X and Y. A Borel function

$$\rho : X \to Y$$

is a *homomorphism* of E to F if for all $x_1, x_2 \in X$

$$x_1 E x_2 \Rightarrow \rho(x_1) F \rho(x_2);$$

in other words, $\rho \times \rho[E] \subset F$. An equivalence relation E on a standard Borel probability space (X, μ) is said to be *F-ergodic* if whenever ρ is a homomorphism from E to F then there is a single y_0 such that for μ-a.e. $x \in X$ we have $\rho(x) F y_0$; in other words, $[\rho(x)]_F$ is constant on a conull set.

We write $E \leq_B F$, *E is Borel reducible to F,* if there is a homomorphism ρ from E to F which has the stronger property that $x_1 E x_2$ if *and only if* $\rho(x_1) F \rho(x_2)$; in other words, ρ can be thought of as pushing down to an injection between the quotient spaces,

$$X/E \hookrightarrow Y/F.$$

It is known that an equivalence relation with all classes countable is hyperfinite, that is to say writeable as an increasing union of Borel equivalence relations with finite classes, if and only if it is Borel reducible to E_0.

At some point it will be convenient to consider classes of reductions somewhat broader than Borel. For instance, we say that E is *universally measurably reducible to F* if there is a universally measurable

$$\rho : X \to Y$$

such that for all $x_1, x_2 \in X$

$$x_1 E x_2 \Leftrightarrow \rho(x_1) F \rho(x_2).$$

Among the *countable* (that is to say, all classes are countable) Borel equivalence relations there are maximal ones with respect to the ordering of Borel reducibility. We arbitrarily choose one of these, and denote it by E_∞ and assume it lives on the standard Borel space X_∞. While it is immediate from the definitions that any countable Borel equivalence relations E must be Borel reducible to E_∞, it is not true that $E \leq_B E_\infty$ implies E is countable; in this case we say that E is *essentially* countable.

Proofs of these and related facts can be found in [6] and [7].

A homomorphism ρ from E on X to F on Y is said to *factor through* a third equivalence relation F' on Y' if there are homomorphisms

$$\rho_1 : X \to Y',$$

$$\rho_2 : Y' \to Y,$$

from E to F' and from F' to F, with

$$\rho = \rho_2 \circ \rho_1.$$

An important example of a cocycle arises from a homomorphism of one orbit equivalence relation into another. Suppose G acts in a Borel manner on X and H acts *freely* and in a Borel manner on Y and

$$\rho : X \to Y$$

is a homomorphism of E_G to E_H. Then we obtain an *induced cocyle*

$$\alpha : X \times G \to H$$

by letting

$$\alpha(x, g) = h$$

if $h \in H$ is the unique group element with

$$h \cdot \rho(x) = \rho(g \cdot x).$$

The characterstic identity of the definition of a cocyle is easily checked. We do need to argue that $\alpha(\cdot, \cdot)$ is indeed Borel, but this follows by the classical uniformization theorem for Borel sets in the plane with countable sections (see for instance the relevant passages of [10]).

3 Relative ergodicity result

Lemma 3.1 *Let G and H be countable groups. Suppose G acts by measure preserving transformations on standard Borel (X, μ) and H acts freely by Borel automorphisms on standard Borel Y. Suppose that the induced cocycle*

$$\alpha : X \times G \to H$$

is cohomologous to a cocycle into an abelian subgroup of H and suppose further that E_G is E_0-ergodic.

Then there is a single $y_0 \in Y$ such that for μ-a.e. $x \in X$ we have

$$\rho(x) E_H y_0.$$

Proof Restating the assumptions of the lemma gives us some abelian $H_0 < H$ and Borel

$$\psi : X \to H$$

such that for all $x \in X$, $g \in G$

$$\psi(g \cdot x)\alpha(x, g)\psi(x)^{-1} \in H_0.$$

Therefore if we define

$$\rho_0 : X \to Y$$

$$x \mapsto \psi(x) \cdot \rho(x),$$

then for any x, g and

$$h_0 = \psi(g \cdot x)\alpha(x, g)\psi(x)^{-1}$$

we must have

$$h_0 \cdot \rho_0(x) = \psi(g \cdot x)\alpha(x, g)\psi(x)^{-1}\rho_0(x)$$
$$= \psi(g \cdot x)\alpha(x, g)\psi(x)^{-1}\psi(x)\rho(x) = \psi(g \cdot x)\alpha(x, g)\rho(x)$$
$$= \psi(g \cdot x)\rho(g \cdot x)$$

by definition of $\alpha(\cdot, \cdot)$, which in turn equals

$$\rho_0(g \cdot x).$$

Thus the induced cocycle for the *new* homomorphism ρ_0 of E_G to E_H always takes its values inside H_0.

In other words, ρ_0 is in fact a homomorphism from E_G to E_{H_0}, from the orbit equivalence relation induced by the action of G on X to the equivalence relation induced by the action of H_0 on Y.

We can accordingly define a measure

$$\nu = (\rho_0)_*(\mu)$$

on Y by

$$\nu(B) = \mu(\rho_0^{-1}[B]).$$

As remarked in [7], it follows from [2] that any countable abelian group of Borel automorphisms gives rise to an equivalence relation which is hyperfinite, and thus Borel reducible to E_0. In particular, $E_{H_0} \leq_B E_0$.

The lemma now follows by E_0-ergodicity of E_G. □

Lemma 3.2 *Suppose G and H are countable groups and $K \triangleleft G$ is normal. Suppose that G acts by measure preserving transformations on probability space (X, μ) and the action of K is by itself already ergodic.*

Suppose that

$$\alpha : X \times G \to H$$

is a cocycle with

$$\alpha(x, k) = 1$$

for all $k \in K$ and μ-a.e. $x \in X$.

Then there is a homomorphism

$$\pi : G \to H$$

such that for all $g \in G$ and μ-a.e. $x \in X$

$$\alpha(x, g) = \pi(g).$$

Proof For a.e. x and any $g \in G$, $k \in K$ we have

$$\alpha(k \cdot x, g) = \alpha(k \cdot x, g)\alpha(x, k)$$

$$= \alpha(x, gk) = \alpha(x, \bar{k}g),$$

for $\bar{k} = gkg^{-1} \in K$, which in turn on a conull set equals

$$\alpha(g \cdot x, \bar{k})\alpha(x, g) = \alpha(x, g).$$

Thus the function

$$x \mapsto \alpha(x, g)$$

is K-invariant, and so by the ergodicity of K constant almost everywhere.

If we define $\pi(g)$ to be the almost everywhere constant value of $\alpha(x,g)$ then we obtain for any $g_1, g_2 \in G$ that for a conull set of $x \in X$

$$\pi(g_1 g_2) = \alpha(x, g_1 g_2)$$
$$= \alpha(g_2 \cdot x, g_1)\alpha(x, g_2) = \pi(g_1)\pi(g_2).$$

$\square$

As a word towards notation, if G is a discrete group and S a discrete set, we let $\{0,1\}^{G\times S}$ be the collection of functions

$$f : G \times S \to \{0,1\}$$

equipped with the product topology. We allow G to act on this space of functions by

$$(\sigma \cdot f)(\tau, s) = f(\sigma^{-1}\tau, s).$$

Theorem 3.3 *Let S be a countable set. Let $G = (\mathbb{F}_2)^n \times \mathbb{Z}$ act freely on*

$$X =_{\text{df}} \{0,1\}^{G\times S} = \prod_{G\times S} \{0,1\}$$

by shift; view X as coming equipped with the usual G-invariant product measure.

Let $m < n$ and suppose that $H = (\mathbb{F}_2)^m \times \mathbb{Z}$ acts freely by Borel automorphisms on standard Borel Y.

Then E_G is E_H-ergodic.

Proof We prove this by induction on n, with the base case $n = 1$ following by 3.1 and appendix C. Assuming now that $n > 1$ let us suppose the theorem is true for all $n' < n$.

At each $i \leq n, j \leq m$ let

$$G_i = \{1\} \times \{1\} \times ... \mathbb{F}_2{}^{(\text{i}^{\text{th}}\text{place})} \times \{1\} ... \times \{0\} \triangleleft G,$$

the i^{th} copy of the free group in G,

$$H_j = \{1\} \times \{1\} \times ... \mathbb{F}_2{}^{(\text{j}^{\text{th}}\text{place})} \times \{1\} ... \times \{0\} \triangleleft H,$$

the j^{th} copy of the free group in H, and let

$$K = \{1\} \times \{1\} ... \times \{1\} \times \mathbb{Z} \triangleleft G.$$

We take

$$\beta : X \times G \to H$$

to be the induced cocyle, and at each $i \leq n$ we let

$$\beta_i : X \times G_i \to H$$

be the restriction $\beta|_{X\times G_i}$. Defining at each $j \leq m$

$$\pi_j : H \twoheadrightarrow H_j$$

to be the canonical surjection, we let

$$\beta_{i,j} : X \times G_i \to H_j$$

to be the cocycle given by

$$\beta_{i,j} = \pi_j \circ \alpha_i.$$

We similarly let

$$\hat{\beta} : X \times K \to H$$

be the cocycle induced for K and let $\hat{\beta}_j : X \times K \to H_j$ be defined by $\hat{\beta}_j = \pi_j \circ \hat{\beta}$.

For now on let us fix some $i \leq n$.

Case 1 At each $j \leq m$ we have that $\beta_{i,j}$ is measurably cohomologous to a cocycle into an abelian subgroup of H_j.

Then at each j fix measurable

$$\psi_j : X \to H_j$$

and abelian $H_j^0 < H_j$ such that

$$\psi_j(g \cdot x)\alpha_{i,j}(x, g)\psi_j(x)^{-1} \in H_j^0$$

for all $g \in G_i$ and almost every $x \in X$. Thus if we define

$$\vec{\psi} : X \to H$$

$$x \mapsto \psi_1(x)\rho_2(x) \cdots \psi_m(x)$$

then we have that

$$\vec{\psi}(g \cdot x)\beta_i(x, g)\vec{\psi}(x)^{-1} \in H_1^0 \times H_2^0 \times \cdots H_m^0 \times Z.$$

In particular, $\beta_i(\cdot, \cdot)$ is (almost everywhere) cohomologous to a cocycle into an abelian subgroup of H.

With this established, the lemma follows by 3.1 and appendix C.

Case 2 There is some $j \leq m$ with $\beta_{i,j}$ not measurably cohomologous to a cocycle into an abelian subgroup of H_j.

From now until the finish we fix this j. The next argument parallels a similar point in [11].

Claim: We can find a measurable assignment

$$x \mapsto \mu_x$$

$$X \to M_{1,3}^+(H_j)$$

with

$$\hat{\beta}_j(x, k) \cdot \mu_x = \mu_{k \cdot x}$$

for all $k \in K$ and a.e. $x \in X$.

Proof of claim: Following appendix B we can certainly find a measurable assignment

$$x \mapsto \mu_x$$

$$X \to M_1^+(\partial H_j)$$

with

$$\hat{\beta}_j(x, k) \cdot \mu_x = \mu_{k \cdot x}$$

for all $k \in K$ and a.e. $x \in X$. Let us assume for a contradiction that we cannot find such an assignment with the measures not just concentrating on at most 2 ends almost everywhere. Appealing to the ergodicity of the action of K on X we have that either almost everywhere μ_x concentrates on two ends or almost everywhere the measure concentrates on exactly one measure. Which ever case holds, we may assume that there is no other assignment concentrating on a greater number of ends.

For organizational simplicity, we will simply assume that μ_x almost everywhere concentrates on two ends, $\{e_x, e'_x\}$; without any loss we can further simplify the situation by assuming $\mu_x(\{e_x\}) = \mu_x(\{e'_x\}) = \frac{1}{2}$. We then have that

$$\{\hat{\beta}_j(x,k)\cdot e_x, \hat{\beta}_j(x,k)\cdot e'_x\} = \{e_{k\cdot x}, e'_{k\cdot x}\}$$

all k, a.e. x, and that for any other such assignment

$$X \to [\partial H_j]^2$$

$$x \mapsto \{f_x, f'_x\}$$

with $\{\hat{\beta}_j(x,k)\cdot f_x, \hat{\beta}_j(x,k)\cdot f'_x\} = \{f_{k\cdot x}, f'_{k\cdot x}\}$ all k, a.e. x we have

$$\{e_x, e'_x\} = \{f_x, f'_x\}$$

almost everywhere. Now note the following:

For any $g \in G_i$, if we let

$$\{f_x, f'_x\} = \{\beta_{i,j}(x,g)^{-1}e_{g\cdot x}, \beta_{i,j}(x,g)^{-1}e'_{g\cdot x}\}$$

then for all $k \in K$ and a.e. $x \in X$

$$\{\hat{\beta}_j(x,k)\cdot f_x, \hat{\beta}_j(x,k)\cdot f'_x\} = \{f_{k\cdot x}, f'_{k\cdot x}\}.$$

To see this we use that $gk = kg$ and appeal to the defining identity of cocycles to obtain

$$\beta(x,gk) = \beta(x,kg)$$

$$\therefore \beta(k\cdot x, g)\beta(x,k) = \beta(g\cdot x, k)\beta(x,g)$$
$$\therefore \beta(k\cdot x, g)\beta(x,k)\beta(x,g)^{-1} = \beta(g\cdot x, k)$$
$$\therefore \beta(x,k)\beta(x,g)^{-1} = \beta(k\cdot x, g)^{-1}\beta(g\cdot x, k),$$

which in particular gives

$$\hat{\beta}_j(x,k)\beta_{i,j}(x,g)^{-1} = \beta_{i,j}(k\cdot x, g)^{-1}\hat{\beta}_j(g\cdot x, k),$$

and hence

$$\{\hat{\beta}_j(x,k)\cdot f_x, \hat{\beta}_j(x,k)\cdot f'_x\} =_{\mathrm{df}} \{\hat{\beta}_j(x,k)\beta_{i,j}(x,g)^{-1}e_{g\cdot x}, \hat{\beta}_j(x,k)\beta_{i,j}(x,g)^{-1}e'_{g\cdot x}\}$$
$$= \{\beta(k\cdot x, g)^{-1}\beta(g\cdot x, k)\cdot e_{g\cdot x}, \beta(k\cdot x, g)^{-1}\beta(g\cdot x, k)\cdot e'_{g\cdot x}\}$$
$$= \{\beta(k\cdot x, g)^{-1}\cdot e_{kg\cdot x}, \beta(k\cdot x, g)^{-1}\cdot e'_{kg\cdot x}\}$$
$$= \{\beta(k\cdot x, g)^{-1}\cdot e_{g\cdot(k\cdot x)}, \beta(k\cdot x, g)^{-1}\cdot e'_{g\cdot(k\cdot x)}\} =_{\mathrm{df}} \{f_{k\cdot x}, f'_{k\cdot x}\}.$$

That point then granted, we have by our assumptions on $\{e_x, e'_x\}$ that

$$\{e_x, e'_x\} = \{f_x, f'_x\}$$

almost everywhere. Since our choice of $g \in G_i$ was arbitrary, we have discovered that for every $g \in G_i$ that

$$\mu_x = \beta_{i,j}(x,g)^{-1}\mu_{g\cdot x};$$

in other words,

$$\mu_{g\cdot x} = \beta_{i,j}(x,g)\mu_x.$$

In particular the map

$$x \mapsto \mu_x$$
$$X \to M_1^+(\partial\mathbb{F}_2)$$

is a homomorphism from $E_{G_i}^X$ to the equivalence relation $E_{H_j}^{M_1^+(\partial H_j)}$. From A.6 the latter equivalence relation is hyperfinite, and then from C we have therefore that there is a single $\hat{\mu} \in M_1^+(\partial H_j)$ such that for almost all $x \in X$

$$\mu_x F_{H_j}^{M_1^+(\partial H_j)} \hat{\mu}.$$

Thus we may in a Borel manner assign a group element h_x to each x such that almost everywhere

$$h_x \cdot \mu_x = \hat{\mu}.$$

Thus if we replace $\beta_{i,j}(\cdot,\cdot)$ by the cohomologous

$$\bar{\beta} : X \times G_i \to H_j$$
$$(x, g) \mapsto h_{g\cdot x}\beta_{i,j}(x, g)h_x^{-1}$$

then for all g and a.e. x

$$\bar{\beta}(x, g) \cdot \hat{\mu} = h_{g\cdot x}\beta_{i,j}(x, g)h_x^{-1} \cdot \hat{\mu}$$
$$= h_{g\cdot x}\beta_{i,j}(x, g) \cdot \mu_x = h_{g\cdot x} \cdot \mu_{g\cdot x} = \hat{\mu}.$$

Thus almost everywhere $\bar{\beta}$ maps into the stabilizer of $\hat{\mu}$, which from A.6 must be abelian. With a contradiction to the assumption on i and j. (☐Claim)

We can now quote A.1 and find a Borel

$$s : M_{1,3}^+(\partial H_j) \to H_j$$

such that $s(h \cdot \mu) = hs(\mu)$, $h \in H_j$, $\mu \in M_{1,3}^+(\partial H_j)$, and then appealing to the claim we define measurable

$$\hat{s} : X \to H_j$$
$$x \mapsto s(\mu_x).$$

We have for all $k \in K$ and a.e. $x \in X$

$$\hat{s}(k \cdot x) = s(\mu_{k\cdot x}) = s(\hat{\beta}_j(x, 2) \cdot \mu_x)$$
$$= \hat{\beta}_j(x, k) \cdot s(\mu_x) = \hat{\beta}_j(x, k) \cdot \hat{s}(x)$$
$$\therefore \hat{s}(k \cdot x)^{-1}\hat{\beta}_j(x, k)\hat{s}(x) = 1.$$

Thus let us replace the homomorphism

$$\rho : X \to Y$$

by

$$\hat{\rho} : X \to Y$$
$$x \mapsto \hat{s}^{-1} \cdot \rho(x);$$

then we have a new homomorphism from E_G^X to E_H^Y which has the property that $\hat{\rho}(x) = \rho(x)$ almost everywhere. Consequently it suffices to show to prove the theorem for this new homomorphism $\hat{\rho}$. By analogy with before we let

$$\gamma : X \times G \to H$$

to be the induced cocyle, and let

$$\gamma_j : X \times G \to H_j$$

be defined by $\gamma_j = \pi_j \circ \gamma$.

Claim: $\gamma_j|_{X\times K}$ is trivial almost everywhere.

Note then for almost all x, all $k \in K$, if we choose $\vec{h} = \gamma(x,k)$ then

$$\vec{h} \cdot \hat{\rho}(x) = \hat{\rho}(k \cdot x),$$

$$\therefore \vec{h} \cdot \hat{s}(x)^{-1} \cdot \rho(x) = \hat{s}(k \cdot x)^{-1} \rho(k \cdot x)$$

$$\therefore \hat{s}(k \cdot x)\vec{h} \cdot \hat{s}(x)^{-1} \cdot \rho(x) = \rho(k \cdot x)$$

which implies that

$$\hat{s}(k \cdot x)\vec{h} \cdot \hat{s}(x)^{-1} = \hat{\beta}(x,k)$$

$$\therefore \pi_j(\vec{h}) = \pi_j(\hat{s}(k \cdot x)^{-1}\hat{\beta}(x,k)\hat{s}(x))$$

$$= \hat{s}(k \cdot x)^{-1}\hat{\beta}_j(x,k)\hat{s}(x) = 1.$$

($\Box$Claim)

Thus by 3.2 we obtain an actual homomorphism

$$\hat{\pi} : G \to H_j$$

such that almost every where

$$\gamma_j(x,g) = \hat{\pi}(g).$$

We want to show that $\hat{\pi}[G_{i'}] = \{1\}$ for $i' \neq i$. For this purpose it suffices to prove the following claim:

Claim: If $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{F}_2$, all non-trivial, and σ_1 and σ_2 commute as do σ_1 and σ_3, then σ_2 and σ_3 commute as well.

Proof of claim: Since subgroups of free groups are free, we have $\langle \sigma_1, \sigma_2 \rangle \cong \mathbb{Z}$, and thus we can find some $w\tau w^{-1}$ such that

$$\langle w\tau w^{-1} \rangle \supset \langle \sigma_1, \sigma_2 \rangle.$$

We may assume by taking τ as short as possible that $\tau \neq \sigma^k$ for any $k > 1$, $\sigma \in \mathbb{F}_2$; and then we may assume that $\tau \neq v\sigma v^{-1}$ any $\sigma \in \mathbb{F}_2$ and $v \neq 1$. Thus there will be ℓ_1, ℓ_2 with

$$\sigma_1 = w\tau^{\ell_1}w^{-1},$$

$$\sigma_2 = w\tau^{\ell_2}w^{-1}.$$

Applying the same analysis to σ_1, σ_3 we may find u, δ such that

$$\langle u\delta u^{-1} \rangle \supset \langle \sigma_1, \sigma_3 \rangle,$$

but $\delta \neq v\sigma v^{-1}$, any $v \neq 1$, $\delta \neq \sigma^k$ any $k > 1$. Then we have

$$\sigma_1 = u\delta^{k_1}u^{-1},$$

$$\sigma_3 = u\delta^{k_3}u^{-1},$$

some k_1, k_3. Thus in particular

$$\tau^{\ell_1} w^{-1} = u\delta^{k_1} u^{-1},$$

and then the minimality assumptions on τ and δ give

$$w = u,$$

$$\tau = \delta,$$

and we are happy. (□Claim)

Thus we have established that the cocycle γ_j, derived from $\hat{\rho}$ and projection to the j^{th} copy of the free group in H, is almost everywhere trivial for elements of G which have the identity on the i^{th} coordinate. In other words, if we let

$$\hat{G} = \mathbb{F}_2 \times \mathbb{F}_2 \times ...\{1\}^{(\text{i}^{\text{th}}\text{place})} \times \mathbb{F}_2 ... \times \mathbb{F}_2 \triangleleft G,$$

and

$$\hat{H} = \mathbb{F}_2 \times \mathbb{F}_2 \times ...\{1\}^{(\text{j}^{\text{th}}\text{place})} \times \mathbb{F}_2 ... \times \mathbb{F}_2 \triangleleft H,$$

then for all $\vec{g} \in \hat{G}$ and almost all $x \in X$ there will exist some $\vec{h} \in \hat{H}$ with

$$\vec{h} \cdot \hat{\rho}(x) = \hat{\rho}(\vec{g} \cdot x).$$

In otherwords, $\hat{\rho}$ is a measurable homomorphism of $E^X_{\hat{G}}$ to $E^Y_{\hat{Y}}$, and we can appeal to the inductive assumption.

□

Standing back somewhat from the specifics of the proof one can abstract the following:

Theorem 3.4 *Let $G = G_1 \times G_2 \times ...G_{n+1}$ and $H = H_1 \times ...H_m$ be countable groups where $m \leq n$ and each H_i is either the free group or abelian and G_{n+1} is amenable.*

Suppose G acts by mpts on standard Borel probability space (X, μ), where the action of G_{n+1} is ergodic and the action of each G_i, $i \leq n$ is E_0-ergodic. Suppose H acts freely by Borel transformations on a standard Borel space Y.

Then E^X_G is E^Y_H-ergodic.

Corollary 3.5 *For $m \leq n+1$ and S countable, the shift action of $(\mathbb{F}_2)^{n+1}$ on $\{0,1\}^{(\mathbb{F}_2)^{n+1} \times S)}$ is relatively ergodic with respect to any $E^{(\mathbb{F}_2)^m}_Y$, where $(\mathbb{F}_2)^m$ acts freely by Borel transformations on standard Borel automorphisms.*

Corollary 3.6 *For m, n, Y as above, and*

$$\pi : (\mathbb{F}_2)^{n+1} \twoheadrightarrow (\mathbb{F}_2)^{m+1}$$

the projection onto the first $m+1$ coordinates, if we let $(\mathbb{F}_2)^{n+1}$ act on

$$X = \{0,1\}^{(\mathbb{F}_2)^{n+1}} \times \{0,1\}^{(\mathbb{F}_2)^{m+1}}$$

by

$$g \cdot (x_1, x_2) = (g \cdot x_1, \pi(g) \cdot x_2),$$

then $E^X_{(\mathbb{F}_2)^{n+1}}$ is $E^Y_{(\mathbb{F}_2)^m}$-ergodic.

Proof Write $(\mathbb{F}_2)^{n+1}$ as $(\mathbb{F}_2)^{m+1} \times S_0$, where $S_0 = (\mathbb{F}_2)^{m-n}$. We take one more step in this direction and let $S = S_0 \cup \{*\}$, where $*$ is any point not in S_0, and observe that, under its natural inclusion into $(\mathbb{F}_2)^{n+1}$, the action of $(F_2)^{m+1}$ on X is isomorphic in a measure preserving fashion to the shift action of $(\mathbb{F}_2)^{m+1}$ on

$$\{0,1\}^{(\mathbb{F}_2)^{m+1}\times S}.$$

□

Thus we obtain a funny kind of "invariant" which can be used to distinguish between different kinds of equivalence relations induced by the free measure preserving actions of some $(\mathbb{F}_2)^\ell$ on a standard Borel probability space. We can consider the largest $m < \ell$ such that we obtain a relative ergodicity result with respect to free actions of $(\mathbb{F}_2)^m$.

The example of the last corollary shows that this largest m can be any natural number less than ℓ.

4 Amenable classes of structures

In what follows we will adopt the convention that whenever we consider a class of structures $\mathcal{C}$ we have in mind that they are all on some fixed countable set. Most of the time that set could be taken to be $\mathbb{N}$, but it is notationally convenient to allow domains such as $\mathbb{N}^n$.

Definition We say that a countable Borel equivalence relation E on X is *amenable* if there is an E-invariant assignment

$$x \mapsto \varphi_x$$

assigning a mean to each equivalence class $[x]_E$ which has the property that whenever μ is a Borel probability measure on X and

$$f : E \to [-1,1]$$

is Borel then

$$x \mapsto \varphi_x(f_x)$$

is μ-measurable, where $f_x : [x]_E \to [-1,1]$ is defined by

$$y \mapsto f(x,y).$$

Definition We say that a standard Borel class of structures $\mathcal{C}$ is *essentially countable* if

$$\cong|_{\mathcal{C}} \leq_B E_\infty;$$

that is to say, if its isomorphism relation is essentially countable.

Definition We say that a class of countable structures $\mathcal{C}$, each having underlying set $\mathbb{N}^n$, some $n \in \mathbb{N}$, is *amenable* if there is an assignment

$$\mathcal{M} \mapsto \varphi_{\mathcal{M}}$$

of means on $\mathbb{N}^n$ to structures in $\mathcal{C}$ such that whenever $\pi : \mathcal{M} \cong \mathcal{N}$ is an isomorphism, then

$$\varphi_{\mathcal{N}} \circ \pi = \varphi_{\mathcal{M}},$$

and moreover whenever μ is a standard Borel probability measure on X and $f : \mathcal{C} \times \mathbb{N}^n \to [-1,1]$ is Borel then

$$\mathcal{M} \mapsto \varphi_{\mathcal{M}}(f_{\mathcal{M}})$$

is μ-measurable.

Example (Compare [7].) Assume the continuum hypothesis, and let $\mathcal{C}$ be the structures on $\mathbb{N}$ which are isomorphic to

$$(\mathbb{Z}, <, U),$$

the integers under the usual ordering, equipped with some unary predicate $U \subset \mathbb{Z}$ which will depend on the particular $\mathcal{M} \in \mathcal{C}$. For each $\mathcal{M} \in \mathcal{C}$ we have may in a Borel manner choose an isomorphism

$$\psi : \mathcal{M} \cong \mathbb{Z},$$

and then at each n let

$$\varphi_{\mathcal{M},n} : [-1,1]^{\mathcal{M}} \to [-1,1]$$
$$f \mapsto \frac{1}{2n+1} \sum_{i=-n}^{i=n} f(i).$$

Using Modobodzki's limit medial we may find a universally measurable ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and for each $\mathcal{M}$ let

$$\varphi_m = \int_{\mathcal{U}} \varphi_{\mathcal{M},n} :: [-1,1]^{\mathcal{M}} \to [-1,1]$$
$$f \mapsto \int_{\mathcal{U}} \frac{1}{2n+1} \sum_{i=-n}^{i=n} f(i).$$

We have to face up to the fact that $\mathcal{C}$ may be amenable without being essentially countable: For instance consider the models of [13], consisting of expansions of $(\mathbb{N} \cdot \mathbb{Z}, <^{\mathrm{lex}})$; as discussed in [3], this isomorphism relation is far from essentially countable, but if we assume CH then we can, in parallel to the example above, use Modobodzki to obtain an invariant and suitably measurable assignment of means. Worse: The countable class of structures may be amenable and essentially countable without $\cong|_{\mathcal{C}}$ being Borel reducible or universally measurably reducible to an amenable countable Borel equivalence relation. If we take any class of structures $\mathcal{B}$ whatsoever and replace it by $\mathcal{C} = \{\mathcal{M}\dot{\cup}(\mathbb{Z}, <) : \mathcal{M} \in \mathcal{B}\}$ then under CH we may assign suitably invariant means by concentrating on the $(\mathbb{Z}, <)$, while maintaining $\cong|_{\mathcal{B}} \leq_B \cong|_{\mathcal{C}}$.

This difficulty is the subject of some rather tortured lemmas below. Under suitable assumptions amenability, to the effect that the amenability is suitably ubiquitous, we do indeed obtain that essentially countability will entail reduction to an amenable countable Borel equivalence relation.

The second problem is that without CH there is no known way to obtain any examples of measure amenable structures.

This will ultimately be bypassed by a standard metamathematical trick. The statements we will be proving are all sufficiently absolute, and it turns out that we may simply impose the assumption that the continuum hypothesis holds.

Perhaps there is a more elegant solution, and a better way to formulate a relevant concept of amenability.

The next lemma is implicit in [5].

Lemma 4.1 *Let $\mathcal{C}$ be a class of structures with*

$$\theta : \mathcal{C} \to X_\infty$$

witnessing $\cong|_{\mathcal{C}} \leq_B E_\infty$. Then we may find a countable Borel equivalence relation F on standard Borel Y with $\gamma : \mathcal{C} \to Y$ witnessing $\cong|_{\mathcal{C}} \leq_B F$ and we may in a Borel manner assign to each $\mathcal{M} \in \mathcal{C}$ a pair $(A_{\mathcal{M}}, \rho_m)$ such that

(a) $A_{\mathcal{M}} \subset \bigcup_n \mathcal{M}^n$;

(b) $\rho_{\mathcal{M}} : A_{\mathcal{M}} \to Y$;

(c) $\mathcal{M} \mapsto (A_{\mathcal{M}}, \rho_{\mathcal{M}})$ *is* $\cong$*-invariant, in the sense that if* $\pi : \mathcal{M} \cong \mathcal{N}$ *then*

$$\pi[A_{\mathcal{M}}] = A_{\mathcal{N}},$$

$$\rho_{\mathcal{M}} = \rho_{\mathcal{N}} \circ \pi;$$

(d) $\rho_{\mathcal{M}}(\vec{a})F(\mathcal{M})$ *all* $\vec{a} \in A_{\mathcal{M}}$*;*
(e) Y *equals the range of* γ*;*
(f) there is Borel $\psi : Y \to \mathcal{C}$ *such that* $\psi(\gamma(\mathcal{M})) \cong \mathcal{M}$ *all* $\mathcal{M} \in \mathcal{C}$*.*

Proof Following [5] we may find a countable fragment $F \subset \mathcal{L}_{\omega_1,\omega}$ such that for each $\mathcal{M} \in \mathcal{C}$ there will be some $\vec{a} \in \mathcal{M}^{<\mathbb{N}}$ such that $\langle \mathcal{M}; \vec{a} \rangle$ is F-atomic. The following are all routine consequences of the definition of atomicity:

(i) the set $\{\langle \mathcal{M}, \vec{a} \rangle : \langle \mathcal{M}, \vec{a} \rangle$ is F-atomic $\}$ is Borel;

(ii) in a given fragment $F_n = F(c_1, ..., c_n)$ (where $c_1, ..., c_n$ are fresh constant symbols), the collection of $T \subset F_n$ which are complete and admit an atomic model is Borel;

(iii) for each T as in (ii) we may in a Borel way choose some $\langle \mathcal{M}, \vec{a} \rangle$ with $\langle \mathcal{M}, \vec{a} \rangle \models T$.

Given these facts, we let

$$A_{\mathcal{M}} = \{\vec{a} : \langle \mathcal{M}, \vec{a} \rangle \, \mathrm{is} F\mathrm{-atomic}\},$$

$$\rho_{\mathcal{M}}(\vec{a}) = \mathrm{Th}_F(\langle \mathcal{M}, \vec{a} \rangle),$$

$$Y = \{\mathrm{Th}_F(\langle \mathcal{M}, \vec{a} \rangle) : \vec{a} \in A_{\mathcal{M}}\},$$

and for each $T \in Y$ we choose $\mathcal{M} \in \mathcal{C}$ as (iii). □

Definition For $\mathcal{C}$ a class of structures, we let $\mathcal{C}^{[n]} = \{\mathcal{M}^n : \mathcal{M} \in \mathcal{C}\}$, the class of all structures arising as n-fold products of a structure from $\mathcal{C}$.

Lemma 4.2 *If* $\mathcal{C}$ *is an amenable class of structures on* $\mathbb{N}$ *then* $\mathcal{C}^{[n]}$ *is an amenable class of structures on* $\mathbb{N}^n$*.*

Proof Let $\mathcal{M} \mapsto \varphi_{\mathcal{M}}$ be the invariant and universally measurable assignment of means. We then define $\varphi^{\ell}_{\mathcal{M}}$ recursively on ℓ, beginning with

$$\varphi^1_{\mathcal{M}} = \varphi_{\mathcal{M}}$$

and continuing with

$$\varphi^{\ell+1}_{\mathcal{M}}(f) = \varphi_{\mathcal{M}}(a \mapsto \varphi^{\ell}_{\mathcal{M}}(f_a)),$$

where $f_a : \mathcal{M}^{\ell} \to [-1, 1]$ is defined by $f_a(b_1, b_2, ..., b_\ell) = f(a, b_1, b_2, ..., b_\ell)$. The invariance property for the various $\varphi^{\ell}_{\mathcal{M}}$'s is easily checked. The main concern is measurability.

Given any measure μ on $\mathcal{C}^{[n]}$, for any $a_1, ..., a_{n-1} \in \mathbb{N}$ we have

$$\mathcal{M} \mapsto \varphi_{\mathcal{M}}(a_n \mapsto f(a_1, a_2, ..., a_n))$$

μ-measurable, and hence we can find some μ-conull set $A_1 \subset \mathcal{C}$ such these functions are all Borel on this set. Then the functions

$$A_1 \to [-1, 1]$$

$$\mathcal{M} \mapsto \varphi_{\mathcal{M}}(a_{n-1} \mapsto \varphi_{\mathcal{M}}(a_n \mapsto \varphi_{\mathcal{M}}(f(a_1, a_2, ...a_{n-1}, a_n)))$$

are μ-measurable, and we again can go to a conull $A_2 \subset A_1$ on which they are all Borel, and continue in this fashion until we reach A_n. □

Lemma 4.3 *Let $\mathcal{C}$ be an essentially countable class of amenable structures on $\mathbb{N}$ with the futher property that given any Borel and $\cong$-invariant assignment*

$$\mathcal{C} \to \mathcal{P}(\mathbb{N}^n)$$

$$\mathcal{M} \mapsto A_{\mathcal{M}}$$

we can choose an assignment of means

$$\mathcal{M} \mapsto \varphi_{\mathcal{M}}$$

which not only witnesses amenability but has the further property of always concentrating on $A_{\mathcal{M}}$ – that is to say, $\varphi_{\mathcal{M}}(\chi_{A_{\mathcal{M}}})$ is always 1.

Then $\cong|_{\mathcal{C}}$ is Borel reducible to an amenable countable Borel equivalence relation.

Proof Following 4.1 we may choose $\mathcal{M} \mapsto (A_{\mathcal{M}}, \rho_{\mathcal{M}}), \gamma, \psi, Y, F$ all as indicated there. At each n let $Y_n = \{y \in Y : \text{ for } \langle \mathcal{M}, \vec{a}\rangle = \gamma(y), \vec{a} \in \mathcal{M}^n\}$; it suffices to show that each $F|_{Y_n}$ is amenable.

But for this purpose we can choose an assignment of means

$$\mathcal{M} \mapsto \varphi_{\mathcal{M}}$$

such that each $\varphi_{\mathcal{M}}$ concentrates on $\mathcal{M}^n \cap A_{\mathcal{M}}$. (In the case that $\mathcal{M}^n \cap A_{\mathcal{M}}$ is empty, then $\mathcal{M}$ plays no part in Y_n and we could for instance ask that $\varphi_{\mathcal{M}}$ simply concentrate on $\mathcal{M}^n$.)

Then for $y \in Y_n$ we let

$$\varphi_y : [-1,1]^{[y]_F} \to [-1,1]$$

$$f \mapsto \varphi_{\psi}(\vec{a} \mapsto f(\rho_{\mathcal{M}}(\vec{a}))).$$

$\square$

Corollary 4.4 *Let $\mathcal{C}$ be a class of amenable structures and suppose that for all $\mathcal{M} \in \mathcal{C}$ and $a \in \mathcal{M}$*

$$\mathcal{M} \models \forall b \bigvee_{t \text{ a term}} t(a) = b;$$

that is to say, the algebraic closure of any $a \in \mathcal{M}$ is the whole structure.

Then $\cong|_{\mathcal{C}}$ is Borel reducible to an amenable countable Borel equivalence relation.

Proof It is easily seen, and observed in the course of [5], that since structures in $\mathcal{C}$ are finitely generated we must have $\cong|_{\mathcal{C}}$ essentially countable. Let $(t_i)_{i \in \mathcal{N}}$ be some fixed enumeration of terms.

Given any invariant Borel assignment $\mathcal{M} \mapsto A_{\mathcal{M}} \subset \mathcal{M}^n$ and invariant measurable

$$\mathcal{M} \mapsto \varphi_{\mathcal{M}}$$

we can define for each $\mathcal{M} \in \mathcal{C}$

$$\sigma_{\mathcal{M}} : \mathcal{M} \mapsto A_{\mathcal{M}}$$

by

$$\sigma_{\mathcal{M}}(a) = (t_{i(1)}(a), t_{i(2)}(a), ..., t_{i(n)}(a))$$

where the sequence $i(1), i(2), ..., i(n)$ is chosen lexicographically least to ensure

$$(t_{i(1)}(a), t_{i(2)}(a), ..., t_{i(n)}(a)) \in A_{\mathcal{M}}.$$

$\square$

5 Non-reduction

Throughout this section we assume the continuum hypothesis. We do this without entertaining any philosophical bias either for or against CH, but simply because the statements we are attempting to prove are all $\underset{\sim}{\Pi}^1_1$ and easily absolute between the universe and some extension where we have forced a wellorder of $2^{\aleph_0}$ of ordertype $\aleph_1$.

We first need a technical result. This result can be most easily proved by collecting together the observations of the last section.

Lemma 5.1 *Let G and $H = H_1 \times H_2 \times ...H_N$ be countable groups acting by Borel transformations on standard Borel X and Y, with the H acting freely, and let*

$$\rho : X \to Y$$

be a homomorphism of E^X_G to E^Y_H with

$$\beta : X \times G \to H$$

the induced cocycle, and

$$\beta_j = \pi_j \circ \beta : X \times G \to H_j$$

the induced cocycle for the various H_j.

Suppose that at each $j \leq N$ we have either

(a) β_j maps into an abelian subgroup of H_j, or

(b) there is a hyperfinite equivalence relation F_j on some standard Borel Z_j and a homomorphism $\hat{\rho} : X \to Z_j$ from E^X_G to F_j with the property that for all $x \in X, g \in G$

$$\hat{\rho}_j(g \cdot x) = \hat{\rho}_j(x) \Leftrightarrow \beta_j(x, g) = 1.$$

Then there is an amenable countable Borel equivalence relation $\hat{F}$ on $\hat{Z}$ such that ρ factors through $\hat{F}$.

Proof We may as well assume that $N = 2$, H_1 is abelian, and that there is homomorphism

$$\hat{\rho} : X \to Z$$

from E^X_G to some hyperfinite F with the additional property for all $x \in X, g \in G$

$$\hat{\rho}(g \cdot x) = \hat{\rho}(x) \Leftrightarrow \beta_2(x, g) = 1.$$

We make the further harmless assumption that each $\hat{\rho}[[x]_G]$ is infinite; the case for those which are finite can be dealt with separately using an argument which is similar to and somewhat easier than the one below.

Since hyperfinite equivalence relations are induced by $\mathbb{Z}$-actions (see [6], [7]), we may in a Borel fashion assign

$$z \mapsto <_z,$$

where each $<_z$ is a linear order of ordertype $\mathbb{Z}$ on $[z]_F$ with the the invariance property that

$$z_1 F z_2 \Rightarrow <_{z_1} = <_{z_2}.$$

We then write

$$x R_\ell x'$$

if $\hat{\rho}(x) F \hat{\rho}(x')$ and $\hat{\rho}(x)$, $\hat{\rho}(x')$ are exactly ℓ many places apart in the linear order $<_{\hat{\rho}(x)} = <_{\hat{\rho}(x')}$ restricted to $\hat{\rho}[[x]_G]$. In other words, if there are $x_0, x_1, ..., x_\ell \in [x]_G$ with

$$x_0 = x,$$

$$x_\ell = x',$$

$$\hat{\rho}(x_0) = \hat{\rho}(x),$$

$$\hat{\rho}(x_\ell) = \hat{\rho}(x'),$$

each

$$\hat{\rho}(x_i) <_{\hat{\rho}(x)} \hat{\rho}(x_{i+1})$$

and for all $\hat{x} \in [x]_G$ with

$$\hat{\rho}(x_0) <_{\hat{\rho}(x)} \hat{\rho}(\hat{x}) <_{\hat{\rho}(x)} \hat{\rho}(x_\ell)$$

there will be some i with

$$\hat{\rho}(\hat{x}) = \hat{\rho}(x_i).$$

This definition makes sense for $\ell \geq 0$, and we can extend it in the natural way to$\ell < 0$ by setting

$$x' R_\ell x$$

if $x R_{-\ell} x'$.

We will build a class of amenable structures $\mathcal{C}$ and then a class of expansions $\mathcal{D}$. We will show that ρ factors through $\cong |_{\mathcal{D}}$ and then appeal to 4.4 to argue that $\cong |_{\mathcal{D}}$ is bi-borel reducible to an amenable countable Borel equivalence relation.

The language for $\mathcal{C}$ will have unary functions $(F_\ell)_{\ell \in \mathbb{Z}}$, $(\hat{F}_h)_{h \in H_1}$, and an equivalence relation E^*. The structures in $\mathcal{C}$ are those structures on $\mathbb{N}$ that satisfy:

(i) $\forall a([a]_{E^*} = \{\hat{F}_h(a) : h \in H_1\})$; for all $h_1, h_2 \in H_1$

$$\hat{F}_{h_1} \circ \hat{F}_{h_2} = \hat{F}_{h_1+h_2},$$

$$\forall b(\hat{F}_0(b) = b);$$

(ii) $\forall a, b$

$$\bigvee_{\ell \in \mathbb{Z}} \bigvee_{h \in H_1} (\hat{F}_h \circ F_\ell(a) = b);$$

(iii) for all $\ell \in \mathbb{Z}, h_1, h_2, h_3, h_4 \in H_1$

$$\forall a, b \bigvee_{\bar{h} \in H_1} (\hat{F}_{\bar{h}} \circ \hat{F}_{h_1} \circ F_\ell \circ \hat{F}_{h_2}(a) = \hat{F}_{h_3} \circ F_\ell \circ \hat{F}_{h_4}(a));$$

(iv)

$$\forall a \bigvee_{h \in H_1} (\hat{F}_h(a) = F_0(a));$$

(v) for all $\ell_1, \ell_2 \in \mathbb{Z}$

$$\forall a \bigvee_{h \in H_1} (\hat{F}_h \circ F_{\ell_1+\ell_2}(a) = F_{\ell_1} \circ F_{\ell_2}(a)).$$

In essence (i) states that the $\{\hat{F}_h : h \in H_1\}$ act transitively on each E^*-equivalence class, and that the action respects the group structure suggested by H_1. (ii) states in particular that the algebraic closure of any point is the whole structure. (iii)-(v) state that in essence that $\mathbb{Z}$, via the assignment $\ell \mapsto F_\ell$, acts on the structure's collection of all E^* equivalence classes.

We will in fact that this class of structures is not only amenable but "2-amenable" in the sense of [7].

We begin with a Folner sequence $(D_n)_{n \in \mathbb{N}}$ for H_1, that is to say a collection with the property that

$$D_n \subset D_{n+1},$$

$$H_1 = \bigcup_n D_n,$$

and for all $h \in H_1$

$$\frac{|D_n h \Delta D_n|}{|D_n|} \to 0.$$

For $\mathcal{N} \in \mathbb{N}$, $\mathcal{M} \in \mathcal{C}$, $a \in \mathcal{M}$, we first let

$$f_n^{a,\mathcal{M}} = \frac{1}{|D_n|}\chi_{\{F_h(a):h\in D_n\}};$$

that is to say, $f_n^{a,\mathcal{M}}(b)$ equals $1/D_n$ if there exists $h \in D_n$ with $\mathcal{M} \models \hat{F}_h(a) = b$, and equals zero otherwise. We iterate and define for each n, m

$$f_{n,m}^{a,\mathcal{M}} : \mathcal{M} \to [0,1]$$

$$f_{n,m}^{a,\mathcal{M}} = \frac{1}{2n+1}(\sum_{\ell\in[-m,m]} f_n^{F_\ell(a),m}).$$

At each $\mathcal{M} \in \mathcal{C}$ and $a, b \in \mathcal{M}$ with aE^*b, it follows from (i) and the properties of the Folner sequence that

$$\lim_{n\to\infty}\|f_n^{a,\mathcal{M}} - f_n^{b,\mathcal{M}}\|_{\ell_1} \to 0,$$

and then by (ii) we have that for *all* $a, b \in \mathcal{M}$

$$\lim_{m\to\infty}(\text{limsup}_{n\to\infty}\|f_{n,m}^{a,\mathcal{M}} - f_{n,m}^{b,\mathcal{M}}\|_{\ell_1} \to 0.$$

We massage this into the form required by §2.5 [7] by recalling that each $\mathcal{M} \in \mathcal{C}$ has $\mathbb{N} = \{1, 2, 3, ...\}$ as its underlying set, and letting

$$f_{n,m}^{\mathcal{M}} = f_{n,m}^{1,\mathcal{M}},$$

and thereby obtain that $\mathcal{C}$ is 2-amenable.

As discussed in [9], it follows from CH and Modobodki's theorem that $\mathcal{C}$ is amenable in our sense above.

The class $\mathcal{D}$ arises by expanding the structures in $\mathcal{C}$ through the addition of fresh unary predicates, $(U_n)_{n\in\mathbb{N}}$. For this purpose let us fix countable Borel separating family $(B_n)_{n\in\mathbb{N}}$ for Y. To each $\mathcal{M} \in \mathcal{C}$ and and function

$$\sigma : \mathcal{M} \to Y$$

we associate the structure $\mathcal{M}^\sigma$ which includes the language of $\mathcal{C}$, on which it behaves just like $\mathcal{M}$, but interprets the fresh unary predicates by obeying the dictate that

$$(\mathcal{N}^\sigma \models U_n(a)) \Leftrightarrow \sigma(a) \in B_n.$$

We let $\mathcal{D}$ be the collection $\{\mathcal{M}^\sigma | \mathcal{M} \in \mathcal{C}, \sigma : \mathcal{M} \to Y\}$. These satisfy the assumptions of 4.4 and so it suffices to show that ρ factors through $\cong|_{\mathcal{D}}$.

For each x we define a pair $n_x, \sigma_x : \mathcal{N}_x \to Y$. We will not literally have that $\mathcal{N}_x$ is in $\mathcal{C}$ because its underlying set, although countable, will not equal $\mathbb{N}$. Nonetheless we can simply choose an isomorphic copy $\mathcal{N}_x^* \in \mathcal{C}$ and argue that

$$\mathcal{M}_x = (\mathcal{N}_x^*)^{\sigma_x^*} \in \mathcal{D},$$

for some natural choice of σ_x^*, is as required.

The underlying set of $\mathcal{N}_x$ will be the set of pairs (ℓ, y) such for some $\hat{x} \in [x]_G$ we have

(a) $xR_\ell\hat{x}$;

(b) $yE_{H_1}\rho(\hat{x})$.

We set

$$(\ell, y)E^*(\bar{\ell}, \bar{y})$$

if and only if $\ell = \bar{\ell}$. We naturally enough define the $\hat{F}_h$ functions as suggested by the action of H_1, so that

$$\hat{F}_h(\ell, y) = (\ell, h \cdot y).$$

The F_ℓ functions require more care.

First fix an enumeration $(\bar{h}_i)_{i \in \mathbb{N}}$ of $H_1 \times H_2$. Then let

$$F_\ell((y, \bar{\ell})) = (\bar{h}_{i_0} \cdot y, \ell + \bar{\ell})$$

where i_0 is least such that for some $\hat{x} \in [x]_G$ we have

$$xR_{\ell+\bar{\ell}}\hat{x},$$

$$(\bar{h}_{i_0} \cdot y)E_{H_1}\rho(\hat{x}).$$

We then, in the most obvious way possible, define

$$\sigma_x : \mathcal{N}_x \to Y$$

$$(\ell, y) \mapsto y.$$

In a Borel manner it is possible to choice for each x a bijection

$$\pi_x : \mathbb{N} \cong \{(\ell, y) : \ell \in \mathbb{Z}, \exists \hat{x} \in [x]_G(yE_{H_1}\rho\hat{x}, xR_\ell\hat{x})\},$$

and we finish the definion by letting $\mathcal{N}_x^*$ be the copy on $\mathbb{N}$ of $\mathcal{N}_x$ provided by π_x and then setting

$$\mathcal{M}_x = (\mathcal{N}_x^*)^{\sigma_x \circ \pi_x}.$$

Claim(I): If $\hat{x}_1, \hat{x}_2 \in [x]_G$ with

$$xR_\ell\hat{x}_1,$$

$$xR_\ell\hat{x}_2,$$

then

$$[\rho(\hat{x}_1)]_{H_1} = [\rho(\hat{x}_2)]_{H_1}.$$

Proof of claim: Fix $g \in G$ with $g \cdot \hat{x}_1 = \hat{x}_2$.

It follows from the definition of R_ℓ that $\hat{\rho}(\hat{x}_1) = \hat{\rho}(\hat{x}_2)$, and hence

$$\beta_2(\hat{x}_1, g) = 1$$

$$\therefore \beta(\hat{x}_1, g) \in H_1 \times \{1\}$$

$$\therefore [\rho(\hat{x}_1)]_{H_1} = [\rho(\hat{x}_2)]_{H_2}.$$

(□Claim)

Claim(II): For any $x \in X$ we have $\mathcal{N}_x^*$ in $\mathcal{C}$.

Proof of claim: Appealing to claim(I) we have

$$\{y : (\ell, y) \in \mathcal{N}_x\}$$

is an H_1-equivalence class for any $\ell \in Z$. This quickly implies (i) in the definition of $\mathcal{C}$. (ii) follows from the careful choice of the F_ℓ functions. For (iii)-(v) we observe that the E^*-equivalence classes are arranged into a $\mathbb{Z}$-chain by $\{xR_\ell(\cdot) : \ell \in \mathbb{Z}\}$ and that the action of the F_ℓ functions respects that ordering. (□Claim)

Claim(III): $\mathcal{M}_{x_1} \cong \mathcal{M}_{x_2} \Rightarrow \rho(x_1)E_H\rho(x_2)$.

Proof of claim: Since the functions $\sigma_{x_1} \circ \pi_{x_1}$ and $\sigma_{x_2} \circ \pi_{x_2}$ must have a point in common. (□Claim)

Claim(IV): $\rho(x_1)E_H\rho(x_2) \Rightarrow \mathcal{M}_{x_1} \cong \mathcal{M}_{x_2}$.

Proof of claim: If $x_1 E_G x_2$ then we can find some $\hat{\ell}$ with

$$x_1 R_{\hat{\ell}} x_2,$$

and hence for all ℓ

$$x_1 R_{\ell+\hat{\ell}}(\cdot) = x_2 R_\ell(\cdot).$$

We simply define an isomorphism

$$\psi : \mathcal{N}_{x_1} \cong \mathcal{N}_{x_2}$$

by

$$\psi(\ell, y) = (\ell - \bar{\ell}, y).$$

This isomorphism clealy intertwines σ_{x_1} and σ_{x_2}. (□Claim)

Thus ρ factors through $\cong |_{\mathcal{D}}$ and we finish by an appeal to 4.4.

□

Theorem 5.2 *Let $G = G_1 \times G_2 \times ...G_N \times K$ and $H = (\mathbb{F}_2)^M \times ... \times \bar{K}$ be countable groups, $K \cong \mathbb{Z}$, $\bar{K}$ abelian, $M < N$, G acting by mpts on standard Borel (X, μ) and H acting freely by measure preserving transformations on standard Borel Y, with the action of K on (X, μ) being ergodic.*[1] *Let $\rho : X \to Y$ be a homomorphism of E^X_G to E^Y_H.*

Then at some $i \le N$, ρ viewed as a homomorphism from $E^X_{G_i}$ to E^Y_H factors through a hyperfinite equivalence relation μ-a.e.

Proof We use ρ to define induced cocyles

$$\beta_j : X \times G \to H_j$$

$$\beta_{i,j} : X \times G_i \to H_j$$

$$\beta_{N+1,j} : X \times K \to H_j$$

for $i \le N$, $j \le M$ in the usual way.

Definition: We say that a coordinate $j \le M$ is *passive* for G_i is either:

(a) $\beta_{i,j}$ is cohomologous to a cocycle into an abelian subgroup of H_j; or

(b) there is a standard Borel Z_j and hyperfinite countable F_j on Z_j and a homomorphism

$$\hat{\rho}_j : X \to Z_j$$

from E_{G_i} to F_j such for all $g \in G_i$ and μ-a.e. $x \in X$

$$\hat{\rho}_j(g \cdot x) = \hat{\rho}_j(x) \Leftrightarrow \beta_{i,j}(x, g) = 1.$$

[1] Here and beyond we identify K with $\{1\} \times ... \times \{1\} \times K \triangleleft G$, we identify G_i with $\{1\} \times ... \times G_i \times \{1\} \times ...\{1\} \times \{0\} \triangleleft G$, we identify $\bar{K}$ with $\{1\} \times ... \times \{1\} \times \bar{K} \triangleleft H$, and in keeping with the notation of 3.3 we let $H_j = \{1\} \times \{1\} \times ...\mathbb{F}_2{}^{(\text{j}^{\text{th}}\text{ place})} \times \{1\}... \times \{0\} \triangleleft H$.

Naturally we want to advance this terminology and say that a coordinate is *active* if it is not passive.

Claim(I) If $j \leq M$ is active for G_i then there is a Borel assignment of measures

$$X \mapsto M^+_{1,3}(\partial H_j)$$

$$x \mapsto \mu_x$$

such that for all $k \in K$ and μ-a.e. $x \in X$

$$\mu_{k \cdot x} = \beta_{N+1,j}(x,k) \cdot \mu_x.$$

Proof of claim: Following appendix B we may certainly find a Borel assignment of measures

$$X \mapsto M^+_1(\partial H_j)$$

$$x \mapsto \mu_x$$

such that for all $k \in K$ and μ-a.e. $x \in X$ $\mu_{k \cdot x} = \beta_{N+1,j}(x,k) \cdot \mu_x$. The task is to do this so that the measures do not simply concentrate on two or fewer ends.

Suppose then that we can find no such suitably invariant assignment with $\mu_x \in M^+_{1,3}(\partial H_j)$ almost everywhere. Following the proof of a parallel step in 3.3 we may conclude that

$$\mu_{g \cdot x} = \beta_{i,j}(x,g) \cdot \mu_x$$

all $g \in G_i$ and μ-a.e. $x \in X$.

Following A.6 and the notation discussed there, we obtain a split in cases:

Case(a) Almost everywhere $\mu_x \in B_2$, the set of $\nu \in M^+_1(\partial H_j)$ with non-trivial stabilizer under the action of H_j.

Then from A.6 we have $E^{B_2}_{H_j}$ smooth, and so by appealing to the ergodicity of the K action we can find a single ν_0 such that

$$\mu_x E^{B_2}_{H_j} \nu_0$$

almost everywhere. Appealing to the selection theorem for Borel sets with countable sections, we may find a Borel function

$$\gamma : X \to H_j$$

such that

$$\gamma(x) \cdot \mu_x = \nu_0$$

almost everywhere.

Now consider the new cocycle

$$\alpha_{i,j} : X \times G_i \to H_j$$

$$(x,g) \mapsto \gamma(g \cdot x)\beta_{i,j}(x,g)\gamma(x)^{-1}.$$

This is certainly cohomologous to $\beta_{i,j}$ and we have that almost everywhere

$$\alpha_{i,j}(x,g) \cdot \nu_0 = \gamma(g \cdot x)\beta_{i,j}(x,g)\gamma(x)^{-1} \cdot \nu_0$$

$$= \gamma(g \cdot x)\beta_{i,j}(x,g) \cdot \mu_x = \gamma(g \cdot x) \cdot \mu_{g \cdot x} = \nu_0$$

and hence

$$\alpha_{i,j}(x,g) \in (H_j)_{\nu_0},$$

the stabilizer of ν_0.

From A.6 this stabilizer is abelian, and H_j is passive for G_i.

Case(b) On a positive measure subset of X the associated μ_x is not in B_2.

Then appealing to the ergodicity of the action of K we can obtain that almost everywhere μ_x has trivial stabilizer. Thus almost everywhere we have

$$\mu_{g\cdot x} = \mu_x$$

if and only if

$$\beta_{i,j}(x,g)\cdot\mu_x = \mu_x$$

if and only if

$$\beta_{i,j}(x,g) = 1.$$

Thus if we let $Z_j = B_0 \cup B_1$, and define

$$\hat{\rho}_j : X \to Z_j$$

$$x \mapsto \mu_x$$

almost everywhere, then by A.6 we obtain after all that H_j is passive for G_i. (□Claim)

Claim(II): If $j \le M$ is active for some G_i then it is passive for every other $G_{i'}$.

Proof of claim: This parallels a similar point from 3.3.

Appealing to the last claim and A.1 we may find a Borel

$$\hat{s} : X \to H_j$$

such that for any $k \in K$ we have almost everywhere

$$\hat{s}(k\cdot x) = \beta_{N+1,j}(x,k)\cdot\hat{s}(x),$$

and thus if we replace

$$\beta_j : X \times G \to H_j$$

by the cohomologous cocycle

$$\alpha_j : X \times G \to H_j$$

$$(x,g) \mapsto \hat{s}(g\cdot x)^{-1}\beta_j(x,g)\hat{s}(x)$$

then α_j is trivial for K. Appealing to 3.2 we have a homomorphism

$$\pi : G \to H_j$$

with $\pi(g) = \alpha_j(x,g)$ almost everywhere, and we finish as in the parallel part of the proof of 3.3. (□Claim)

Thus by the pigeonhole principle we obtain some $i \le N$ such that no $j \le M$ is active for G_i. From this it follows that the cocycle

$$\beta|_{X\times G_i} : X \times G_i \to H$$

is cohomologous to some

$$\beta^* : X \times G_i \to H$$

such that each of the projections

$$\beta^*_j : X \times G_i \to H_j$$

satisfy either (a) or (b) from the preceding lemma.

Fix $\gamma^* : X \to H$ such that

$$\gamma(g \cdot x)\beta(x,g)\gamma(x)^{-1} = \beta^*(x,g)$$

almost everywhere. If we define

$$\rho^* : X \to Y$$

$$x \mapsto \gamma(x) \cdot \rho(x)$$

then ρ^* is a homomorphism from $E^X_{G_i}$ to E^Y_H and β^* is the induced cocycle.

Thus by the lemma we may find an amenable countable Borel equivalence F' on standard Borel Z' and homomorphisms

$$\rho_0^* : X \to Z'$$

$$\rho_1^* : Z' \to Y$$

of $E^X_{G_i}$ to F and of F to E^Y_H such that

$$\rho_1^* \circ \rho_0^* = \rho^*.$$

As in the proof of 3.1 we may find $\hat{Z} \subset Z'$ such that $Z'|_{\hat{Z}}$ is hyperfinite and almost everywhere $\rho_0^*(x) \in \hat{Z}$.

We then take one further step and let $Z = \hat{Z} \times H$,

$$\rho_0 : x \mapsto (\rho_0^*(x), \gamma(x)),$$

$$\rho_1 : (z,h) \mapsto h^{-1} \cdot \rho_1^*(z)$$

and set $(z,h)F(z',h')$ if and only if $zF'z'$. It is easily checked that F is hyperfinite, and for almost every x we have

$$\rho_1 \circ \rho_0(x) = \rho_1(\rho_0^*(x), \gamma(x))$$

$$= \gamma(x)^{-1} \cdot \rho_1^*(\rho_0^*(x))) = \gamma(x)^{-1} \cdot \rho^*(x) = \rho(x),$$

as required.

□

Corollary 5.3 *(After Gaboriau?) Let $G = (\mathbb{F}_2)^N \times \mathbb{Z}$ and $H = (\mathbb{F}_2)^M \times ... \times \mathbb{Z}$, $M < N$, G acting freely by mpts on standard Borel probability space (X,μ) and H acting freely by measure preserving transformations on standard Borel Y, with the action of $\mathbb{Z}$ on (X,μ) being ergodic. Then*

$$E^X_G \not\leq_B E^Y_H.$$

Proof In light of the theorem and B.2, it suffices to show that if F is a countable Borel equivalence relation on a standard Borel space Z, and

$$\rho : Z \to \{0,1\}^{\mathbb{N}}$$

is a homomorphism of F to E_0, and $\rho^{-1}(w)$ is countable all $w \in 2^{\mathbb{N}}$, then F is hyperfinite.

But this follows from the fact that among the countable Borel equivalence relations the class of hyperfinite equivalence relations are closed under downward under $\leq_B$-reducibility and inclusion; see §1 [7]. □

Corollary 5.4 *(After Adams-Kechris) There are $\leq_B$-incomparable countable Borel equivalence relations.*

Proof On the one side we consider the Bernoulli shift of $G = (\mathbb{F}_2)^3 \times \mathbb{Z}$ on $\{0,1\}^G$ equipped with the usual product measure and let X be the free part.

On the other side we first being with

$$H_1 = (\mathbb{F}_2)^2 \times \mathbb{Z}$$

$$H_2 = \mathbb{Z} \times (\mathbb{F}_2)^2$$

along with their corresponding Bernoulli shifts on $\{0,1\}^{H_1}$ and $\{0,1\}^{H_2}$. We let $H = (\mathbb{F}_2)^2 \times \mathbb{Z} \times (\mathbb{F}_2)^2$. We let Y_1, Y_2 be the respectively free parts of these Bernoulli shifts and take $Y = Y_1 \times Y_2$. We let H act on Y by

$$(h_1, \ell, h_2) \cdot (y_1, y_2) = ((h_1, \ell) \cdot y_1, (h_2, \ell) \cdot y_2).$$

Since the product of mixing actions of $\mathbb{Z}$ are mixing, we obtain that the induced action of $\mathbb{Z}$ on Y is mixing and therefore certainly ergodic, and hence by the last corollary

$$E_H^Y \not\leq E_G^X.$$

For the converse, suppose

$$\rho : X \to Y$$

is a homomorphism of E_G^X to E_H^Y. We may write $\rho = (\rho_1, \rho_2)$ where

$$\rho_1 : X \to Y_1$$

$$\rho_2 : X \to Y_2$$

are homomorphisms of E_G^X ot $E_H^{Y_1}$ and $E_H^{Y_2}$ respectively.

Appealing to 3.3 we obtain $X_1 \subset X$ of measure one such that $\rho_1|_{X_1}$ has countable image, and then making the same appeal once more we have X_2 of measure one with $\rho_2|_{X_2}$ also having countable image. Thus $\rho[X_2]$ is a countable subset of Y and we are done.

□

A The boundary of the free group

We adopt the notational convention that the generators of $\mathbb{F}_2$ be a and b.

Definition We let $\partial\mathbb{F}_2$ be the *boundary of the free group*, consisting of all infinite *reduced* words in a and b – that is to say, those words with no consecutive appearances of a and a^{-1} or b and b^{-1}. We let $\mathbb{F}_2$ act in the obvious way on this collection, by concatenation and then cancellation to obtain an infinite reduced word. Note then that $\partial\mathbb{F}_2$ is a compact space on which the free group acts by homeomorphisms.

$[\partial\mathbb{F}_2]^2$ be the collection of all unordered pairs or singletons from $\mathbb{F}_2$. This is given a Borel structure in the following way: First we identify $\partial\mathbb{F}_2$ with $\{\{e\} : e \in \partial\mathbb{F}_{\}} \subset [\partial\mathbb{F}_2]^2$. We then fix a Borel linear ordering, $<$, of $\partial\mathbb{F}_2 \times \partial F_2$ and identify

$$[\partial\mathbb{F}_2]^2 \setminus \mathbb{F}_2$$

with

$$\{(e_1, e_2) \in \partial\mathbb{F}_2 \times \partial\mathbb{F}_2 : e_1 < e_2\}.$$

We extend the action of $\mathbb{F}_2$ on $\partial\mathbb{F}_2$ by letting

$$\sigma \cdot \{e, e'\} = \{\sigma \cdot e, \sigma \cdot e'\}.$$

The elements of $\partial\mathbb{F}_2$ are also sometimes called the *ends* of the free group.

Definition Let $M_1^+(\partial\mathbb{F}_2)$ be the space of probability measures on $\partial\mathbb{F}_2$. We let $M_{1,3}^+(\partial\mathbb{F}_2)$ be those probability measures which do not simply concentrate on two ends – that is to say, for all $e', e'' \in \partial\mathbb{F}_2$ we have $\mu(\{e', e''\}) < 1$.

Note that if we are given three distinct ends $e_1, e_2, e_3 \in \partial\mathbb{F}_2$ then we may select a common central branch point in a way which is firstly invariant under permutations of the sequence (e_1, e_2, e_3) and which secondly, and more importantly, commutes with the action of $\mathbb{F}_2$. For instance, if we are presented with the ends

$$e_1 = (aba^{-1}bba...)$$

$$e_2 = (aba^{-1}a^{-1}ba...)$$

$$e_3 = (aba^{-1}a^{-1}bb...)$$

then we would be led to choose $s_0(e_1, e_2, e_3) = aba^{-2}b$ as their central point. Then if we were to act on each of them with the group element $\sigma = ba^2b^{-1}a^{-1}$ we obtain

$$f_1 = \sigma \cdot e_1 = (babba...)$$

$$f_2 = \sigma \cdot e_2 = (bba..)$$

$$f_3 = \sigma \cdot e_3 = (bbb...)$$

and are then confronted with $s_0(f_1, f_2, f_3) = b^2$ as their branching point, which is indeed equal to $\sigma s_0(e_1, e_2, e_3)$.

For measures in $M^+_{1,3}(\partial\mathbb{F}_2)$ one can use integration to extend this technique to find a *center of the measure*:

Theorem A.1 *(Lyons) There is a Borel function*

$$s : M^+_{1,3}(\partial\mathbb{F}_2) \to \mathbb{F}_2$$

such that for all $\mu \in M^+_{1,3}(\partial\mathbb{F}_2)$, $\sigma \in F_2$

$$s(\sigma \cdot \mu) = \sigma s(\mu).$$

Proof See [12] or [11]. □

Lemma A.2 *The orbit equivalence relation $E^{\partial\mathbb{F}_2}_{\mathbb{F}_2}$ given by the action of $\mathbb{F}_2$ on $\partial\mathbb{F}_2$ is hyperfinite.*

Proof First consider the tail equivalence relation F on $\partial\mathbb{F}_2$. So given two ends e, e' we set

$$eFe'$$

if there is some single end f and two words w_1, w_2 with

$$e = w_1^\frown f,$$

$$e' = w_2^\frown f.$$

It is known (see [6], [7]) that these kinds of tail equivalence relations are always hyperfinite.

Then since $E^{\partial\mathbb{F}_2}_{\mathbb{F}_2} \subset F$ we obtain that it must be hyperfinite as well. □

Lemma A.3 *The orbit equivalence relation $E^{[\partial\mathbb{F}_2]^2}_{\mathbb{F}_2}$ induced by the left concatenation of $\mathbb{F}_2$ on $[\partial\mathbb{F}_2]^2$ is hyperfinite.*

Proof First view $[\partial\mathbb{F}_2]^2$ as a subset of $\partial\mathbb{F}_2 \times \partial\mathbb{F}_2$ in the way indicated at the start of this section and consider the orbit equivalence relation $E^{\partial\mathbb{F}_2\times\partial\mathbb{F}_2}_{\mathbb{F}_2}$ given by the diagonal action of $\mathbb{F}_2$ on this product space. If we think of $\partial\mathbb{F}_2 \times \partial\mathbb{F}_2$ as being a subset of infinite strings taken from $\mathbb{F}_2 \times \mathbb{F}_2$ then the argument from A.2 indeed shows $E^{\partial\mathbb{F}_2\times\partial\mathbb{F}_2}_{\mathbb{F}_2}$ to be hyperfinite.

We clearly therefore have that the restricted equivalence relation $E^{\partial\mathbb{F}_2\times\partial\mathbb{F}_2}_{\mathbb{F}_2}|_{[\partial\mathbb{F}_2]^2}$ is hyperfinite. Now since $E^{[\partial\mathbb{F}_2]^2}_{\mathbb{F}_2}$ has finite index over the hyperfinite equivalence relation $E^{\partial\mathbb{F}_2\times\partial\mathbb{F}_2}_{\mathbb{F}_2}|_{[\partial\mathbb{F}_2]^2}$ we finish by [7]. □

Lemma A.4 *For any $e \in \partial\mathbb{F}_2$, the stabilizer of e in $\partial\mathbb{F}_2$, $(\mathbb{F}_2)_e$, is abelian.*

Proof Note that if e has non-trivial stabilizer then it must be eventually periodic. That is to say, viewed as infinite word it will have the form

$$e = w^\frown\sigma^\frown\sigma^\frown\sigma...$$

for some $\sigma, w \in \mathbb{F}_2$. Let us assume we have represented e in this way and taken σ to be as short a word as possible – which is basically to say that σ does not have the form $\tau^\frown\tau^\frown...\tau$ for any other $\tau \in \mathbb{F}_2$.

After possibly replacing e by $w^{-1} \cdot e$ and conjugating $(\mathbb{F}_2)_e$ by w, we may as well assume w is the empty word and that e actually has the form $\sigma^\frown\sigma^\frown\sigma^\frown....$ Then it is easily seen that $(\mathbb{F}_2)_e = \{\sigma^\ell : \ell \in \mathbb{Z}\}$. □

Lemma A.5 *For any $\{e, e'\} \in [\partial\mathbb{F}_2]^2$, the stabilizer, $(\mathbb{F}_2)_{\{e,e'\}}$, of $\{e, e'\}$ in $\mathbb{F}_2$ is abelian.*

Proof Following the proof of A.4 we may assume

$$e = \sigma^\frown\sigma^\frown\sigma...$$

where $\sigma \neq u^k$ for any $u \in \mathbb{F}_2$, $k > 1$. Observe in particular that any string of the form $\sigma^\frown\sigma^\frown\sigma...\sigma$ is already a reduced word and in particular σ does not have the form

$$\sigma = v\sigma_0 v^{-1}$$

for any non-trival $v \in \mathbb{F}_2$.

For any $\tau \in (\mathbb{F}_2)_{\{e,e'\}}$ we must either have $\tau \cdot e = e'$ and $\tau \cdot e' = e$ or $\tau \cdot e = e$ and $\tau \cdot e' = e'$. In either case we end up discovering that $\tau^2 \cdot e = e$.

Thus by the proof of the last lemma we in fact have to have $\tau^2 = \sigma^m$ some m. It then follows by the observations regarding σ observed above that τ^2 and hence τ cannot have the form $v\tau_0 v^{-1}$ for any non-trivial $v \in \mathbb{F}_2$.

Thus $\tau^\frown\tau$ is a reduced word and equal to τ^2. From this it follows that m is even and $\tau = \sigma^{\frac{m}{2}}$. □

The somewhat convoluted formulation of the next lemma is intentional. The statement given helps organize the proof.

Lemma A.6 *We may divide $M_1^+(\partial\mathbb{F}_2)$ into three invariant pieces, B_0, B_1, B_2, such that:*

(a) the orbit equivalence relation of $\mathbb{F}_2$ on B_0 is smooth (i.e. $\leq_B id(\mathbb{R})$) and the action is free (i.e. each μ has trivial stabilizer);

(b) the orbit equivalence relation of $\mathbb{F}_2$ on B_1 is hyperfinite and the action is free;

(c) the orbit equivalence relation of $\mathbb{F}_2$ on B_2 is smooth and each $\mu \in B_2$ has abelian stabilizer.

Proof We begin with $B_0 = M_{1,3}^+(\partial\mathbb{F}_2)$. A.1 certainly implies that the action on B_0 is free, and we see that the equivalence relation is smooth by taking

$$A_0 = \{\mu \in M_{1,3}^+(\partial\mathbb{F}_2) : s(\mu) = 1\}$$

as our selector.

We move on and let C be the set of linear combinations of the form

$$\mu = \alpha_1\delta_{e_1} + \alpha_2\delta_{e_2}$$

where $e_1, e_2 \in \partial\mathbb{F}_2$, δ_{e_i} is the Dirac point mass measure on e_i, $\alpha_1 + \alpha_2 = 1$, and $\alpha_1, \alpha_2 \geq 0$.We subdivide C into $(C^r)_{r\in[1/2,1]}$, where $\alpha_1\delta_{e_1} + \alpha_2\delta_{e_2} \in C^r$ if $\max\{\alpha_1, \alpha_2\} = r$.

We then let $B_1^r = \{\alpha_1\delta_{e_1} + \alpha_2\delta_{e_2} \in C^r$: one of e_1, e_2 has non-trivial stabilizer in the action of $\mathbb{F}_2$ on $\partial\mathbb{F}_2\}$. We let B_1 be the union of these B_1^r's. It follows from the argument of A.5 that the action of $\mathbb{F}_2$ on each B_1^r is free. We are left with trying to show that the orbit equivalence relation on such a B_1^r is hyperfinite.

In the case that $r = \frac{1}{2}$ this directly reduces to A.3; in the case that $r > \frac{1}{2}$ we have an invariant way to select from the pair $\{e_1, e_2\}$ by chosing the end with highest weight, and it reduces to A.2.

Finally we take $B_2^r\{\alpha_1\delta_{e_1} + \alpha_2\delta_{e_2} \in C^r$: both e_1, e_2 have non-trivial stabilizer in the action of $\mathbb{F}_2$ on $\partial\mathbb{F}_2\}$; we are then forced to let B_2 be the union of these B_2^r's. Note that the analysis of A.4 shows that there are only countably many possibilities for e_1 and e_2; and so each B_2^r is the union of countably many orbits, and hence almost trivially smooth. □

In fact we can sharpen A.6 by observing that the stabilizers will be isomorphic to $\mathbb{Z}$ when non-trivial.

B Skew invariant measures

Lemma B.1 *Let T be a measure preserving transformation of a standard Borel probability space (X, μ) and let $\mathbb{Z}$ act on X by*

$$k \cdot x = T^k(x).$$

Let $\alpha : X \times \mathbb{Z} \to \mathbb{F}_2$ be a cocycle.

Then we may find a measurable assignment

$$X \to M_1^+(\partial\mathbb{F}_2)$$

$$x \mapsto \mu_x$$

such that for all $k \in \mathbb{Z}$ and almost all $x \in X$

$$\mu_{k\cdot x} = \alpha(x, k) \cdot \mu_x.$$

Proof (Sketch only; chapter 4 of [17] contains much more general results.)

Let $\ell_1^\infty(X, M_1^+(\partial\mathbb{F}_2))$ be the space of all measurable assignments

$$\gamma : X \to M_1^+(\partial\mathbb{F}_2).$$

We equip this space with the topology provided by the semi-norms of the form

$$\gamma \mapsto \int_X (x \mapsto \int_{\partial\mathbb{F}_2} f_x d\gamma(x)) d\mu$$

where $x \mapsto f_x$ is some measurable function from X to the functions in $C(\partial\mathbb{F}_2)$ of norm ≤ 1.

In this topology $\ell_1^\infty(X, M_1^+(\partial\mathbb{F}_2))$ becomes a compact convex set.

We let $\mathbb{Z}$ act on $\ell_1^\infty(X, M_1^+(\partial\mathbb{F}_2))$ by

$$(k \cdot \gamma)(x) = \alpha(-k \cdot x, k) \cdot \gamma(-k \cdot x).$$

If we let

$$T_N : \ell_1^\infty(X, M_1^+(\partial\mathbb{F}_2)) \to \ell_1^\infty(X, M_1^+(\partial\mathbb{F}_2))$$

$$\gamma \mapsto \frac{1}{N+1} \sum_{k \leq N} k \cdot \gamma$$

and take any point in the intersection of the closures of the images of the operators T_N then we have a fixed point as required. □

This lemma can be used to give a neat proof that free actions of $\mathbb{F}_2$ on probability spaces are never hyperfinite, and that neat proof can in turn be used to show that the above lemma cannot be proved in the Borel, or even Baire measurable, category.

Corollary B.2 *If $\mathbb{F}_2$ acts freely by mpts on standard Borel probability space (X,μ), then $E^X_{\mathbb{F}_2}$ is not hyperfinite.*

Proof Appealing to the ergodic decomposition theorem, we may assume without loss of generality that $\mathbb{F}_2$ acts ergodically. Write $\mathbb{F}_2 = \langle a,b\rangle$, so that a and b are the generators. We suppose for a contradiction that there is a Borel action of $\mathbb{Z}$ on X with $E^X_{\mathbb{Z}} = E^X_{\mathbb{F}_2}$ off of a null set.

We have a Borel cocycle

$$\alpha : X \times \mathbb{F}_2 \to \mathbb{Z}$$

given by

$$\alpha(x,\sigma) = k$$

if $k \cdot x = \sigma \cdot x$. Appealing the last lemma we may find a measure one set $X_0 \subset X$ and a Borel assignment

$$x \mapsto \mu_x$$

$$X_0 \to M^+_1(\partial\mathbb{F}_2)$$

such that

$$\mu_{\ell\cdot x} = \alpha(x,\ell)\cdot\mu_x$$

all $x \in X_0$, $\ell \in \mathbb{Z}$. Observing that for each x there will be ℓ^x_a, ℓ^x_b with

$$\alpha(x,\ell^x_a) = a,$$

$$\alpha(x,\ell^x_b) = b,$$

we obtain

$$\mu_{a\cdot x} = a\cdot\mu_x,$$

$$\mu_{b\cdot x} = b\cdot\mu_x,$$

all $x \in X_0$.

Claim: $\mu_x \notin \partial M^+_{1,3}(\partial\mathbb{F}_2)$ a.e. x.

Proof of claim: Otherwise ergodicity gives almost every $\mu_x \in M^+_{1,3}(\partial\mathbb{F}_2)$ and appealing to A.1 we may find a Borel function

$$s^* : X_0 \to \mathbb{F}_2$$

such that for all $\sigma \in \mathbb{F}_2$ and a.e. x

$$s^*(\sigma\cdot x) = \sigma\cdot s^*(x).$$

$\sigma_0 \in \mathbb{F}_2$ with Letting $A_\sigma = \{x \in X : s^*(x) = \sigma\}$ we have

$$(A_\sigma)_{\sigma\in\mathbb{F}_2}$$

partitioning a conull subset of X into $\aleph_0$ many disjoint sets of equal measure, with a contradiction to $\mu(X) = 1$. (□Proof of claim)

So we actually may assume there is a Borel function

$$X_0 \to [\partial\mathbb{F}_2]^2$$

$$x \mapsto \{e_x, e'_x\}$$

such that

$$a \cdot \{e_x, e'_x\} = \{e_{a\cdot x}, e'_{a\cdot x}\},$$
$$b \cdot \{e_x, e'_x\} = \{e_{b\cdot x}, e'_{b\cdot x}\},$$

almost every x.

Let A_0 be the set of $x \in X_0$ such that neither e_x nor e'_x begin with a or a^{-1}. For each $x \in X_0$ we may find some $n \in \mathbb{N}$ with

$$b^n \cdot x \in A_0.$$

Thus $\mu(A_0) = 0$.

But considering that for $n_1 \neq n_2 \in \mathbb{N}$

$$a^{n_1} \cdot A_0 \cap a^{n_2} \cdot A_0 = \emptyset$$

we have our contradiction. □

Counterexample B.3 *There is a Borel action of $\mathbb{Z}$ on a Polish space X and a Borel cocycle $\alpha : X \times \mathbb{Z} \to \mathbb{F}_2$ such that there is* no *comeager X_0 and Borel assignment*

$$X_0 \to M_1^+(\partial\mathbb{F}_2)$$

$$x \mapsto \mu_x$$

with for all $x \in X_0$, $n \in \mathbb{Z}$

$$\mu_{n\cdot x} = \alpha(x, n) \cdot \mu_x.$$

Proof One constructs the example in this way. Again letting a and b be the generators of $\mathbb{F}_2$ we start with the free part of the shift action of $\mathbb{F}_2$ on $\{0,1\}^{\mathbb{F}_2}$; notice that this action is *generically ergodic*, in the sense that every invariant Borel set is either meager or comeager. Following [15] we may find a comeager subset X of this free part on which

(a) $E_{\mathbb{F}_2}|X$ is hyperfinite;

after some possible further contraction of X we may additionally suppose

(b) X is $\mathbb{F}_2$-invariant;

(c) $\{a^\ell \cdot x : \ell \in Z\}$ is dense in X for all $x \in X$;

(d) $\{b^\ell \cdot x : \ell \in Z\}$ is dense in X for all $x \in X$.

As indicated by say §3.1[4], it follows easily from (c) and (d) that the orbit equivalence relations $E^X_{\langle a\rangle}$ and $E^X_{\langle b\rangle}$, induced by the respective cyclic subgroups $\langle a\rangle$ and $\langle b\rangle$, are *properly generically ergodic*, and hence non-smooth, and in fact non-smooth even when restricted to any comeager subset of X. From (a) we get a Borel action of $\mathbb{Z}$ with

$$E^X_{\mathbb{Z}} = E^X_{\mathbb{F}_2}.$$

Now for a contradiction suppose that on a comeager subset $X_1 \subset X$ we can find an assignment

$$x \mapsto \mu_x$$

which is suitably skew invariant under the cocyle from $\mathbb{Z}$ to $\mathbb{F}_2$. As in the corollary above we would obtain

$$\mu_{a\cdot x} = a \cdot \mu_x,$$

$$\mu_{b\cdot x} = b \cdot \mu_x$$

through out X_1. We may further assume that X_1 is invariant under the action of $\mathbb{F}_2$.

Claim: $\mu_x \notin \partial M^+_{1,3}(\partial\mathbb{F}_2)$ for a comeager set of $x \in X_1$.

Proof of claim: Otherwise generic ergodicity gives almost every $\mu_x \in M^+_{1,3}(\partial\mathbb{F}_2)$ and, as before, we may find a Borel function

$$s^* : X_1 \to \mathbb{F}_2$$

such that for all $\sigma \in \mathbb{F}_2$ and a comeager set of X

$$s^*(\sigma \cdot x) = \sigma \cdot s^*(x).$$

Letting $A_1 = \{x \in X : s^*(x) = 1\}$ we have that A_1 meets all the orbits on a comeager subset X_2 of X_1 in exactly one point. Then we may find appeal to the uniformization theorem for Borel sets with countable sections to see that the function

$$f : X_2 \to A_1$$

which assigns to each $x \in X_2$ the unique $f(x)E_{\mathbb{F}_2}x$ with $f(x) \in A_1$ will be $\mathbb{F}_2$ invariant, and hence constant on a comeager subset by generic ergodicity, with a contradiction to every orbit being meager. (□Proof of claim)

So as before may on a comeager subset X' of X_1 assign a pair of ends $\{e_x, e'_x\}$ with

$$\{e_{\sigma\cdot x}, e'_{\sigma\cdot x}\} = \sigma \cdot \{e_x, e'_x\}$$

all $\sigma \in \mathbb{F}_2$, $x \in X'$. We can again let A_0 be the set of x for which neither e_x nor e'_x begins with an a or an a^{-1}. By considering the action of $\langle b\rangle$ we analogously obtain that it will be non-meager. Thus by generic ergodicity of the action of $\langle a\rangle$ we obtain that for a comeager collection of x there will be some $\ell \in \mathbb{Z}$ with $a^\ell \cdot x \in A_0$. Considering the definition of A_0 and the invariance assumption on the $\{e_x, e'_x\}_{x\in X'}$ we obtain that there can be at most one $\ell \in Z$ with $a^\ell \cdot x \in A_0$, and thus A_0 is a Borel selector for a comeager subset of X, contradiction. □

C E_0-ergodicity

Lemma C.1 *(Folklore; see [7].) A countable equivalence relation E is hyperfinite if and only if $E \leq_B E_0$.*

Theorem C.2 *(Jones, Schmidt; see [8]) Let G be a countable group acting ergodically by mpts on a standard Borel probability space (X, μ). Then E^X_G is E_0 ergodic if and only if there is no sequence of sets $(A_n)_{n\in\mathbb{N}}$, with each*

$$\mu(A_k) = \frac{1}{2},$$

and for all $g \in G$

$$\mu(g \cdot A_k \Delta A_k) \to 0$$

as $k \to \infty$.

Without going into the depths of the proof, we may first tackle the *if* direction by supposing $\theta : X \to \{0,1\}^{\mathbb{N}}$ is a homomorphism of E^X_G to E_0. If $[\theta(x)]_{E_0}$ is not almost everywhere constant, then ergodicity implies that $\mu(\theta^{-1}[\{p\}])$, the measure of the pullback of the point p, is null for each $p \in \{0,1\}^{\mathbb{N}}$. Then it is not hard to show that for each k there exists a set $B_k \subset \{0,1\}$ such that $\mu(\theta^{-1}[B_k])$ always equals a half and that for each $f, g \in \{0,1\}^{\mathbb{N}}$ with $f(n) = g(n)$ all $n > k$ we have

$$f \in B_k \Leftrightarrow g \in B_k.$$

Then $A_k = \theta^{-1}[B_k]$ gives us the required sequence.

For the converse, suppose that (A_k) is our asympotically invariant sequence of sets of measure one half. After possibly refining the sequence we may assume that for all $g \in G$ there is a measure one set of x for which

$$\exists N \forall k > N(x \in A_k \Leftrightarrow g \cdot x \in A_k).$$

Then we devise a measurable homomorphism by assigning to $x \in X$

$$f_x : \mathbb{N} \to \{0,1\}$$

with

$$f_x(k) = 1 \Leftrightarrow x \in A_k.$$

Theorem C.3 *(Schmidt; see [14]) Let S be a countable set and equip $\{0,1\}^{\mathbb{F}_2 \times S}$ with the product measure μ. Then the shift action of $\mathbb{F}_2$ is E_0 ergodic.*

This result, which actually extends to any non-amenable countable group in the place of $\mathbb{F}_2$, can be proved in various ways. It suffices to show that we can write the induced representation of $\mathbb{F}_2$ on $L^2(\{0,1\}^{\mathbb{F}_2 \times S}, \mu)$ as a product of the trivial representation and $\aleph_0$ many copies of the regular left representation,

$$L^2(\{0,1\}^{\mathbb{F}_2 \times S}, \mu) \cong 1 \otimes \bigotimes_{\aleph_0} \ell^2(\mathbb{F}_2).$$

For this it suffices to show that the square integrable functions orthogonal to the constant function, that is to say those $f \in L^2(\{0,1\}^{\mathbb{F}_2 \times S}, \mu)$ with

$$\int f\mu = 0,$$

admit an orthonormal basis which is set wise invariant under the action of $\mathbb{F}_2$.

Towards this goal we define for each finite $F \subset \mathbb{F}_2 \times S$ the function

$$A_F : \{0,1\}^{\mathbb{F}_2 \times S} \to \{1,-1\}$$

by

$$A_F(x) = -1$$

if the cardinality of $\{(\sigma, s) \in F : x(\sigma, s) = 1\}$ is odd, and

$$A_F(x) = 1$$

if the cardinality is even.

Obviously this collection is set wise invariant. The Stone-Weierstrass theorem immediately implies that they span the functions orthogonal to the constant functions. We omit the routine calculation that they are orthogonal.

Things would certainly be nicer if mixing actions of $\mathbb{F}_2$ were necessarily E_0-ergodic. However it is known from say [16] that this is untrue. We give an elementary counterexample.

Counterexample We first consider the shift action of $\mathbb{F}_2$ on $\{0,1\}^{\mathbb{F}_2}$ and define a sequence of measures, $(\mu_n)_{n \in \mathbb{N}}$. We will make sure that these measures are $\mathbb{F}_2$-invariant and that they are becoming increasingly "sticky", with sets of measure a half which are nearly invariant for ever larger subset of $\mathbb{F}_2$. We finish with the diagonal action of $\mathbb{F}_2$ on

$$\prod_{\mathbb{N}} (\{0,1\}^{\mathbb{F}_2 \times S}, \mu_n).$$

It suffices to define the measure μ_n on sets of the form

$$C_s = \{x \in \{0,1\}^{\mathbb{F}_2} : x|_T = s\}$$

for $s : T \to \{0,1\}$ and $T \subset \mathbb{F}_2$ a finite set which forms a tree under the usual Cayley graph for $\mathbb{F}_2$.

For such an s and T we let $(e_i)_{i=1}^{i=|T|-1}$ enumerate the edges in T. We let K_0 be the number of edges for which s takes the same value at both vertices at either end and we let K_1 be the number of edges for which s assumes differing values at the two ends. We set

$$\mu_n(C_s) = \frac{1}{2}(\frac{n-1}{n})^{K_0}(\frac{1}{n})^{K_1}.$$

We omit the routine verification that this will indeed define a measure, and simply observe that for each $g \in \mathbb{F}_2$

$$\mu_n(\{x \in \{0,1\}^{\mathbb{F}_2}\}\Delta g \cdot \{x \in \{0,1\}^{\mathbb{F}_2}\})$$

gravitates towards zero as n marches towards infinity. Thus by C.2 the diagonal action of $\mathbb{F}_2$ on the product space $\prod_{\mathbb{N}}(\{0,1\}^{\mathbb{F}_2 \times S}, \mu_n)$ fails to be E_0-ergodic. Since each individual action of $\mathbb{F}_2$ on some individual $(\{0,1\}^{\mathbb{F}_2 \times S}, \mu_n)$ is mixing, so too the overall diagonal action of $\mathbb{F}_2$ on the infinite product will be mixing.

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