Knight's model, its automorphism group, and characterizing the uncountable cardinals

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Contents

1 Introduction

This paper comes about as a response to both the recent work relating to the topological Vaught conjecture in [2], [5], and the much older work by Julia Knight on characterizing cardinals.

Definition A countable structure $\mathcal{M}$ is said to *characterize* a cardinal κ if

1. there is a model of cardinality κ satisfying exactly the same $\mathcal{L}_{\omega_1,\omega}$ sentences as $\mathcal{M}$;
2. but there is no model with cardinality greater than κ satisfying the same $\mathcal{L}_{\omega_1,\omega}$ sentences as $\mathcal{M}$.

It is well known that for a countable model $\mathcal{M}$ the corresponding *Scott sentence*, as defined in §8 below, $\varphi_{\mathcal{M}} \in \mathcal{L}_{\omega_1,\omega}$ (see [13]) has the property that for any $\mathcal{N}$ we have

$$\mathcal{N} \models \varphi_{\mathcal{M}}$$

if and only if $\mathcal{N}$ satisfies the same $\mathcal{L}_{\omega_1,\omega}$ sentences as $\mathcal{M}$. Thus we may equivalently rephrase the above definition in terms of the models $\mathcal{N}$ satisfying the Scott sentence of $\mathcal{M}$.

Here it is easy to see that there is a countable model characterizing $\aleph_0$. For instance, in the language with infinitely many unary functions $(U_n)_{n\in\omega}$, any model satisfying

$$\exists a \forall b \bigvee_{n\in\omega} (U_n(a) = b)$$

does so.

On the other hand, [15] shows that there are much more subtle examples of models characterizing $\aleph_0$, and thus the following theorem covers an interesting range of automorphism groups.

Theorem 1.1 *(Becker, Gao; see [2], [5].) If $\mathcal{M}$ characterizes $\aleph_0$, then the automorphism group of $\mathcal{M}$ satisfies Vaught's conjecture.*

Indeed here it was shown under sharps in [2] (and in just ZFC in [9]) that such an automorphism group will satisfy Vaught's conjecture even on analytic sets. This very strong conclusion, defined precisely in §7 below, to the effect that a continuous action of the group always give rise to either countably many or perfectly many orbits on $\underset{\sim}{\Sigma}^1_1$ sets, is known to fail for the infinite symmetric group.

At the level of the next cardinal it was demonstrated in the 1970's that:

Theorem 1.2 *(Knight; see [14]) There is a countable model characterizing $\aleph_1$.*

And recently:

Theorem 1.3 *(See [9]) The automorphism group of Knight's model satisfies the Vaught conjecture even on analytic sets.*

In view of these results one might hope that there would be a heirarchy of properties corresponding to whether a model characterizes a certain cardinal.

Here however we halt the direction of the above results, and thereby answer negatively a question from [9]:

Theorem 1.4 *There is a model characterizing $\aleph_1$ whose automorphism group has a closed subgroup mapping continuously and homomorphically onto S_∞ and thereby violates the topological Vaught conjecture on analytic sets.*

The model constructed here is, as one would expect, very different in flavor from Knight's model. In particular it makes possible generalizations which appear more resistant to her method:

Theorem 1.5 *For each $\alpha < \omega_1$ there is a countable model characterizing $\aleph_\alpha$.*

The question of characterizing the $\aleph_n$'s in this way is mentioned as an open problem in [14].[1] However for $\kappa \geq \aleph_{\omega_1}$ we cannot hope for the same kind of theorem in ZFC. For one thing, the Hanf number of sentances in $\mathcal{L}_{\omega_1,\omega}$ is $\beth_{\omega_1}$, and under GCH this equals $\aleph_{\omega_1}$. But even with GCH failing badly, Shelah in [16], and independently Hrushovski and Velickovic in unpublished work, have shown it is relatively consistent with $2^{\aleph_0} > \aleph_{\omega_1}$ that every sentence in $\mathcal{L}_{\omega_1,\omega}$ with a model of size $\aleph_{\omega_1}$ has a model of size $2^{\aleph_0}$.

In the final section of the paper we consider a question of Howard Becker's and obtain a near counterexample by showing that for G_K the automorphism group of Knight's countable model one has:

Theorem 1.6 *Whenever G_K acts continuously on a Polish space X with resulting equivalence relation $E^X_{G_K}$:-*

(I) we may assign in a $\utilde{\Delta}^1_2$ manner real numbers as a complete invariant for $E^X_{G_K}$; or

(II) we may continuously embed the equivalence relation of eventual agreement on infinite binary sequences into $E^X_{G_K}$.

In alternative (I) of this theorem one would expect to be able to prove that the assignment can be carried out in a *Borel* manner; thus the result is unlikely to be optimal. On the other hand, the proof – with its curious use of large cardinal ideas, and its strange forcing over the Frechet filter on $\aleph_1$ – would seem to have some independent interest.

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2 Technicalities

Let us indulge in the usual notational shortcuts.

Models will be identified with their underlying sets. Thus $\mathcal{M} \subset \mathcal{N}$ indicates that the underlying set of the structure $\mathcal{M}$ is included in $\mathcal{N}$ and that $\mathcal{M}$ is a substructure of $\mathcal{M}$. Instead of writing $\vec{b} = (b_0, b_1, ...b_{k-1})$ is a finite sequence from the underlying set, we may instead simply write

$$\vec{b} \in \mathcal{M}.$$

For Q a predicate in the language of $\mathcal{M}$, we may find it convenient to use $Q^{\mathcal{M}}$ for the set of all a in the underlying set of $\mathcal{M}$ for which

$$\mathcal{M} \models Q(a).$$

Similarly if $R(\cdot,\cdot)$ is a relation in the language of $\mathcal{M}$ and $a \in \mathcal{M}$, then $R(a,\cdot)^{\mathcal{M}}$ indicates the set

$$\{b : \mathcal{M} \models R(a,b)\}.$$

[1]Though here, as we point out in the section on atomic models, this had essentially been solved for $n = 2$ by Lakowski and Shelah.

Much of the paper deals with infinitary logic, but unless otherwise indicated *elementary substructure* means elementary with respect to finitary logic, $\mathcal{L}_{\omega,\omega}$, and will be denoted by $\prec$. I will loosely use $\mathcal{L}_{\omega_1,\omega}$ to indicate the countably infinitary logic generated by whatever language is currently of interest; strictly speaking we should distinguish, with say $(\mathcal{L}_0)_{\omega_1,\omega}$ the countably infinitary logic generated by the language $\mathcal{L}_0$ and $(\mathcal{L}_1)_{\omega_1,\omega}$ the countably infinitary logic generated by the language $\mathcal{L}_1$.

The usual kind of logical folklore is presupposed. For instance, the theory of dense linear orderings without end points is $\aleph_0$-categorical, and any countable model of this theory can be realized as the *Fraisse limit* of finite structures. A discussion of this Fraisse construction can be found in [4].

Lemma 2.1 *(Folklore) Let $\varphi \in \mathcal{L}_{\omega_1,\omega}$ be such that any two countable models of φ are isomorphic. Then any two models of φ of any size at all satisfy the same sentences in $\mathcal{L}_{\infty,\omega}$.*

Proof Let $\psi \in \mathcal{L}_{\infty,\omega}$. By induction on logical complexity one sees that whether a given structure satisfies ψ is absolute between transitive models of ZFC.

Let $\mathcal{A}, \mathcal{B}$ be two possibly uncountable models of φ. Let $V[G]$ be a generic extension in which $\mathcal{A}$ and $\mathcal{B}$ are countable. They both still satisfy φ. By Shoenfield absoluteness (see [12]) it follows that even in $V[G]$ any two countable models of φ are isomorphic, and thus

$$V[G] \models \mathcal{A} \cong \mathcal{B}$$

$$\therefore (\mathcal{A} \models \psi \Leftrightarrow \mathcal{B} \models \psi).$$

□

The necessary infinitary logic appears in [13] and [1]. We will eventually present a precise definition of Scott sentance in §8; for the first few sections below a much more primitive working definition is provided if we simply accept that for each countable model $\mathcal{M}$ there is a corresponding $\varphi_m \in \mathcal{L}_{\omega_1,\omega}$ such that for all other countable $\mathcal{N}$ we have

$$\mathcal{M} \cong \mathcal{N}$$

if and only if

$$\mathcal{N} \models \varphi_{\mathcal{M}}.$$

3 Characterizing $\aleph_1$

Definition We let $\mathcal{L}_0$ be the language generated by $\{S_n : n \in \omega\}$ and $\{R_k : k \in \omega\}$ where each S_n is binary and each R_k is $k+2$-ary.

Definition We introduce the following axioms, stated in infinitary logic:

(*1): the conjunction of:

(*1a) $\forall a, b \bigvee_{n\in\omega} S_n(a, b)$;

(*1b) $\forall a, b \bigwedge_{n\neq m}(S_n(a,b) \Rightarrow \neg S_m(a,b))$.

(*2): the conjunction of:

(*2a) for each k and $i < j < k$

$$\forall a_0, a_1, b_0, ..., b_{k-1}(R_k(a_0, a_1, b_0, ..., b_{k-1}) \Rightarrow (b_i \neq b_j \wedge a_0 \neq a_1));$$

(*2b) for each k and permutation π of k

$$\forall a_0, a_1, b_0, ..., b_{k-1}(R_k(a_0, a_1, b_0, ..., b_{k-1}) \Leftrightarrow R_k(a_1, a_0, b_{\pi(0)}, ..., b_{\pi(k-1)}));$$

(*2c) $\forall a_0, a_1, b_0, ... \bigwedge_{n\in\omega} \wedge_{i<k}(R_k(a_0, a_1, b_0....) \Rightarrow (S_n(a_0, b_i) \Leftrightarrow S_n(a_1, b_i))$;

(*2d) $\forall a_0, a_1, b_0, ...\forall c \bigwedge_{n\in\omega}[(R_k(a_0, a_1, b_0....) \wedge_{i<k} c \neq b_i) \Rightarrow$
$(S_n(a_0, c) \Rightarrow \neg S_n(a_1, c))]$.

In other words, (*1) asserts that for each a and b there is a (unique) natural number $f(a, b)$ with $S_{f(a,b)}(a, b)$. (*2) asserts that given $R_k(a_0, a_1, b_0, ..., b_{k-1})$ we have that $\{b_0, b_1, ..., b_{k-1}\}$ exactly comprises the set on which $f(a_0, \cdot)$ and $f(a_1, \cdot)$ agree.

Definition An $\mathcal{L}_0$ structure F is $\mathcal{L}_0$-*neat* if

1. F is finite;
2. $F \models (*1), (*2)$;
3. $\forall a_0, a_1 \in F(a_0 \neq a_1 \Rightarrow \exists k \exists \vec{b} \in F(R_k(a_0, a_1, \vec{b})))$.

Lemma 3.1 *Let F be an $\mathcal{L}_0$-neat structure. Let F_0, F_1 be $\mathcal{L}_0$-neat structures extending F. Then there are $\mathcal{L}_0$-neat $H_0 \supset F_0$, $H_1 \supset F_1$ and an isomorphism*

$$i : H_0 \cong H_1$$

with $i|_F$ the identity map.

Proof We may assume $F_0 \cap F_1 = F$. We then take $H = F_0 \cup F_1$ and are faced with the task of making it into a neat structure extending F_0 and F_1.

For any $a_0 \in F_0$ and $a_1 \in F_1$ we stipulate

$$H \models R_k(a_0, a_1, b_0, b_1, ...b_{k-1}),$$

where $b_0, b_1, ...b_{k-1}$ enumerates the elements b of F for which there is some m with

$$F_0 \models S_m(a_0, b_i)$$

and

$$F_1 \models S_m(a_1, b_i).$$

We can then easily find one to one

$$f : (F_0 \setminus F) \times (F_1 \setminus F) \cup (F_1 \setminus F) \times (F_0 \setminus F) \hookrightarrow \omega$$

such that for all $a_0, a_1 \in F_0$ and $b_0, b_1 \in F_1$, if $S_n(a_0, a_1)$ or $S_n(b_0, b_1)$ then n is not in the range of f. We then complete the task of making H into a neat structure with

$$H \models S_{f(c_0,c_1)}$$

for each $(c_0, c_1) \in (F_0 \setminus F) \times (F_1 \setminus F) \cup (F_1 \setminus F) \times (F_0 \setminus F)$.

Then $H_0 = H = H_1$ and $i = \mathit{identity}$ suffice.

□

Note that the proof works even for $F = \emptyset$.

Definition An $\mathcal{L}_0$-structure $\mathcal{M}$ is $\mathcal{L}_0$*-full* if

1. for all finite $A \subset \mathcal{M}$ there is $\mathcal{L}_0$-neat $F \supset A$, $F \subset \mathcal{M}$;
2. for all $\mathcal{L}_0$-neat $F \subset \mathcal{M}$ and $\mathcal{L}_0$-neat $H_0 \supset F$ there is neat $H_1 \subset \mathcal{M}$ with

   $F \subset H_1$

   there is $i : H_0 \cong H_1$ with $i|_F$ the identity.

The lemma 3.1 entails the existence of $\mathcal{L}_0$-full structures in the usual way.

Lemma 3.2 *Any two $\mathcal{L}_0$-full structures $\mathcal{M}, \mathcal{N}$ of cardinality $\aleph_0$ are isomorphic.*

In fact for any $\mathcal{L}_0$ neat $F \subset \mathcal{M}$ and $\mathcal{L}_0$ neat $H \subset \mathcal{N}$ and isomorphism $i : H \cong F$ there is an isomorphism

$$i_\infty : \mathcal{M} \cong \mathcal{N}$$

extending i.

Proof Fix $i : H \cong F$. We may repeatedly apply the fullness assumption in a back and forth argument to find sequences

$$F_0 = F \subset F_1 \subset F_2 ... \subset \mathcal{M},$$

$$H_0 = H \subset H_1 \subset H_2 ... \subset \mathcal{N},$$

$$i_0 = i \subset i_1 \subset i_2 \subset ...,$$

with each $i_n : F_n \cong H_n$ and

$$\bigcup_{n \in \omega} F_n = \mathcal{M},$$

$$\bigcup_{n \in \omega} H_n = \mathcal{N},$$

and so

$$i_\infty = \bigcup_{n \in \omega} i_n$$

is as required. □

Thus we may find $\varphi_0 \in \mathcal{L}_{\omega_1,\omega}$ such that φ_0 is the Scott sentence of any $\mathcal{L}_0$-full model.

The models of φ_0 can be viewed as arising from a version of the Fraisse construction. Literally they fall short of actually arising as a Fraisse limit of finite structures, since we only obtain the distinctive amalgamation property for the cofinal collection of *neat* structures, and while it is true that the method could be shoe horned into the usual Fraisse format by enlarging the language, it seems less natural and more complicated to do so.

Lemma 3.3 *φ_0 has no model of size $\aleph_2$.*

Proof Suppose instead that $\mathcal{N} \models \varphi_0$ is of size $\aleph_2$ and choose

$$\mathcal{N}_0 \prec \mathcal{N},$$

$$|\mathcal{N}_0| = \aleph_1,$$

noting in particular that for all $a_0, a_1 \in \mathcal{N}_0$ there is some neat $F \subset \mathcal{N}_0$ containing a_0, a_1, and thus for all $b \in \mathcal{N} \setminus \mathcal{N}_0$ we have

$$\bigwedge_{n\in\omega} S_n(a_0, b) \Rightarrow \neg S_n(a_1, b).$$

Choosing some fixed $b \notin \mathcal{N}_0$ we would in particular obtain that

$$a \mapsto \mu n(S_n(a, b))$$

provides an injection from $\mathcal{N}_0$ into ω, a contradiction. $\Box$

Definition Let $\mathcal{L}_1$ be the language obtained from $\mathcal{L}_0$ by adding unary predicates $(P_n)_{n\in\omega}$.

Definition We introduce the following infinitary axiom:
(*3): the conjunction of:
(*3a): $\forall a \bigvee_{n\in\omega} P_n(a)$;
(*3b): $\forall a \bigwedge_{n\neq m} \neg(P_n(a) \wedge P_m(a))$.

In other words, (*3) states that for each a there is a unique $g(a) \in \omega$ with $P_{g(a)}(a)$.

Definition An $\mathcal{L}_1$ structure F is $\mathcal{L}_1$-*neat* if

1. F is finite;
2. $F \models$ (*1), (*2), (*3);
3. $\forall a_0, a_1 \in F(a_0 \neq a_1 \Rightarrow \exists k \exists \vec{b} \in F(R_k(a_0, a_1, \vec{b})))$.

So we have simply bumped up the definition of neatness from $\mathcal{L}_0$ to $\mathcal{L}_1$, accounting for the new axiom.

Lemma 3.4 *Let F be an $\mathcal{L}_1$-neat structure. Let F_0, F_1 be $\mathcal{L}_1$-neat structures extending F. Then there are $\mathcal{L}_1$-neat $H_0 \supset F_0$, $H_1 \supset F_1$ and an isomorphism*

$$i : H_0 \cong H_1$$

with $i|_F$ the identity map.

Proof As in 3.1; the only minor difference is that now when we take $H = F_0 \cup F_1$ we (obviously) are required to define for each $a \in H$

$$(H \models P_n(a)) \Leftrightarrow [(a \in F_0 \wedge F_0 \models P_n(a)) \vee (a \in F_1 \wedge F_1 \models P_n(a))].$$

□

Definition An $\mathcal{L}_1$-structure $\mathcal{M}$ is $\mathcal{L}_1$*-full* if

1. for all finite $A \subset \mathcal{M}$ there is $\mathcal{L}_1$-neat $F \supset A$, $F \subset \mathcal{M}$;
2. for all $\mathcal{L}_1$-neat $F \subset \mathcal{M}$ and $\mathcal{L}_1$-neat $H_0 \supset F$ there is neat $H_1 \subset \mathcal{M}$ with
   $F \subset H_1$
   there is $i : H_0 \cong H_1$ with $i|_F$ the identity.

Lemma 3.5 *Any two $\mathcal{L}_1$-full structures $\mathcal{M}, \mathcal{N}$ of cardinality $\aleph_0$ are isomorphic.*

Proof Exactly as in the proof of 3.2. □

Lemma 3.6 *Let $\mathcal{M}$ be a countable $\mathcal{L}_1$-full structure and let $\mathcal{M}|_{\mathcal{L}_0}$ denote the reduct of $\mathcal{M}$ to the language $\mathcal{L}_0$. Then for any $\pi \in S_\infty$ there are corresponding σ in the automorphism group of $\mathcal{M}|_{\mathcal{L}_0}$ such that for all $a \in \mathcal{M}$*

$$\mathcal{M} \models P_n(a) \Leftrightarrow P_{\pi(n)}(\sigma(a)).$$

Proof We define $\mathcal{N}$ by requiring

$$\mathcal{N}|_{\mathcal{L}_0} = \mathcal{M}|_{\mathcal{L}_0},$$

but at each $n \in \omega$ and $a \in \mathcal{M}$

$$\mathcal{N} \models P_n(a) \Leftrightarrow \mathcal{M} \models P_{\pi(n)}(a).$$

Now it simply follows from the structure of the definitions that $\mathcal{N}$ must also be $\mathcal{L}_1$-full, and so we can apply 3.5 to find σ. □

Corollary 3.7 *For $\mathcal{M}_0$ a countable model of φ_0 we have that S_∞ divides* $\mathrm{Aut}(\mathcal{M}_0)$, *in the sense that there is a closed subgroup $G < \mathrm{Aut}(\mathcal{M}_0)$ and continuous onto homomorphism*

$$p : G \twoheadrightarrow S_\infty.$$

Proof Let $\mathcal{M}_1$ be an expansion of $\mathcal{M}_0$ to an $\mathcal{L}_1$-full model. Let G be the group of all automorphisms σ of $\mathcal{M}_0$ such that for all $a, b \in \mathcal{M}_0$, for all $n, m \in \omega$, if

$$\mathcal{M}_1 \models P_n(a), P_n(b), P_m(\sigma(a))$$

then

$$\mathcal{M}_1 \models P_m(\sigma(b)).$$

To each such σ we may continuously and homomorphically associate π_σ with

$$\mathcal{M}_1 \models P_n(a) \Leftrightarrow \mathcal{M}_1 \models P_{\pi_\sigma(n)}(\sigma(a)).$$

By 3.6 the map

$$\sigma \mapsto \pi_\sigma$$

is onto. □

Of course the proof of this corollary is general. Given any countable language and related notion of finite "neat" structure which allows amalgamation we may form a corresponding notion of "full" and observe S_∞ to divide the automorphism group of any countable full model.

Theorem 3.8 *(Gao) (i) If G and H are Polish groups, G allowing a compatible complete left invariant metric, and*

$$\pi : G \twoheadrightarrow H$$

is continuous, onto homomorphism, then H allows a complete left invariant metric.

(ii) A countable model $\mathcal{M}$ characterizes $\aleph_0$ if and only if Aut($\mathcal{M}$) allows a complete left invariant metric.

Corollary 3.9 *(i) For $\mathcal{M}$ a countable model of φ_0,* $\mathrm{Aut}(\mathcal{M})$ *does not have a complete left invariant metric.*

(ii) φ_0 has a model of size $\aleph_1$.

Proof It is well known that S_∞ does not have a complete left invariant metric, and so by 3.8(i) and 3.7 neither can $\mathrm{Aut}(\mathcal{M})$. Then we obtain (ii) by 3.8(ii) and the non-existence of a complete left invariant metric. □

Thus:

Theorem 3.10 *There is a countable model $\mathcal{M}$ such that:*

1. *the Scott sentence of* $\mathcal{M}$ *characterizes* $\aleph_1$*;*
2. S_∞ *divides* $\mathrm{Aut}(\mathcal{M})$*;*
3. *and hence* $\mathrm{Aut}(\mathcal{M})$ *violates Vaught's conjecture on analytic sets.*

Proof 1. and 2. above are covered by earlier lemmas; 3. is a consequence of 2. and [3] together with results from [8]. $\square$

4 Characterizing $\aleph_2$

Before becoming seriously involved in this construction, we step back a moment and note some variations on 3.5 and the notion of $\mathcal{L}_1$-full.

Definition Let $\hat{\mathcal{L}}_{\omega_1}$ be the expansion of $\mathcal{L}_0$ by unary predicates $(P_\alpha)_{\alpha\in\omega_1}$. For each $\alpha < \omega_1$ let $\hat{\mathcal{L}}_\alpha$ be the expansion of $\mathcal{L}_0$ just by $(P_\beta)_{\beta<\alpha}$.

Definition We introduce a new infinitary axion for each $\alpha \leq \omega_1$:
(*N$_\alpha$): the conjunction of:
(*N$_\alpha$a): $\forall a \bigvee_{\beta\in\alpha} P_\beta(a)$;
(*N$_\alpha$b): $\forall a \bigwedge_{\beta_1\neq\beta_2} \neg(P_{\beta_1}(a) \wedge P_{\beta_2}(a))$.

In other words, (*N$_\alpha$) states that for each a there is a unique $g(a) \in \alpha$ with $P_{g(a)}(a)$.

Definition An $\hat{\mathcal{L}}_\alpha$ structure F is $\hat{\mathcal{L}}_\alpha$-*neat* if

1. F is finite;
2. $F \models$ (*1), (*2), (*N$_\alpha$);
3. $\forall a_0, a_1 \in F(a_0 \neq a_1 \Rightarrow \exists k \exists \vec{b} \in F(R_k(a_0, a_1, \vec{b})))$.

By the same proof one obtains:

Lemma 4.1 *Let* F *be an* $\hat{\mathcal{L}}_\alpha$*-neat structure. Let* F_0, F_1 *be* $\hat{\mathcal{L}}_\alpha$*-neat structures extending* F*. Then there are* $\hat{\mathcal{L}}_\alpha$*-neat* $H_0 \supset F_0$*,* $H_1 \supset F_1$ *and an isomorphism*

$$i : H_0 \cong H_1$$

with $i|_F$ *the identity map and* $i[F_0] \cap F_1 = F$*.*

Definition An $\hat{\mathcal{L}}_\alpha$-structure $\mathcal{M}$ is $\hat{\mathcal{L}}_\alpha$-*full* if

1. for all finite $A \subset \mathcal{M}$ there is $\hat{\mathcal{L}}_\alpha$-neat $F \supset A$, $F \subset \mathcal{M}$;

2. for all $\hat{\mathcal{L}}_\alpha$-neat $F \subset \mathcal{M}$ and $\hat{\mathcal{L}}_\alpha$-neat $H_0 \supset F$ there is neat $H_1 \subset \mathcal{M}$ with
   $F \subset H_1$
   there is $i : H_0 \cong H_1$ with $i|_F$ the identity.

At least for countably infinite α we have the same situation as before.

Lemma 4.2 *For $\omega \leq \alpha < \omega_1$, any two $\hat{\mathcal{L}}_\alpha$-full structures $\mathcal{M}, \mathcal{N}$ of cardinality $\aleph_0$ are isomorphic.*

Here is the point which we will need for later:

Lemma 4.3 *There exist $\hat{\mathcal{L}}_{\omega_1}$-full structures of cardinality $\aleph_1$.*

Proof It suffices to find a chain of countable structures

$$\mathcal{M}_\omega \subset \mathcal{M}_{\omega+1} \subset ... \mathcal{M}_\alpha \subset ...$$

where each $\mathcal{M}_\alpha$ is $\hat{\mathcal{L}}_\alpha$-full and of size $\aleph_0$, since then we can simply take $\mathcal{M}_{\omega_1}$ to be the union of $\{\mathcal{M}_\alpha : \alpha < \omega_1\}$.

We build $\mathcal{M}_\alpha$ by induction on α. At limit stages we can take unions, and thus all the work happens at successor steps.

So suppose we have $\mathcal{M}_\alpha$. We represent

$$\mathcal{M}_\alpha = \bigcup_{n \in \omega} F_n,$$

where each F_n is $\hat{\mathcal{L}}_\alpha$-neat. At once from the definitions each F_n must be $\hat{\mathcal{L}}_{\alpha+1}$-neat, and thus by 4.1, for any $\hat{\mathcal{L}}_{\alpha+1}$-neat

$$H \supset F_n,$$

with $H \cap \mathcal{M}_\alpha = F_n$, we can find $\hat{\mathcal{L}}_{\alpha+1}$-neat

$$H' \supset H \cup F_{n+1}.$$

Therefore repeated applications of 4.1 enables us to find a sequence of $\hat{\mathcal{L}}_{\alpha+1}$-neat structures

$$H_0 \subset H_1 \subset ... H_n \subset H_{n+1} \subset ...$$

where each $F_n \subset H_n$, each $H_n \cap \mathcal{M}_\alpha = F_n$, and for any $\hat{\mathcal{L}}_{\alpha+1}$-neat

$$H \supset F_n$$

there will exist a sufficiently large m at which we have some $H' \subset H_m$ and

$$i : H \cong H',$$

with $i|_{F_n} =$ identity.

Thus we move on with $\mathcal{M}_{\alpha+1} = \bigcup H_n$. $\square$

Definition We let $\mathcal{L}_2$ be generated by $\mathcal{L}_0$, the new unary predicate Q, and binary relations E_0, E_1, and $<$.

Definition We introduce a whole collection of new axioms.

(*4) $<$ linearly orders the objects satisfying Q and is completely inert outside Q:

(*4a) $\forall a, b, c(Q(a) \wedge Q(b) \wedge Q(c) \Rightarrow$
$(a < b \vee b < a \vee a = b) \wedge (\neg(a < a)) \wedge ((a < b \wedge b < c) \Rightarrow a < c)))$;

(*4b) $\forall a, b(a < b \Rightarrow (Q(a) \wedge Q(b))$.

(*5) As a ranges over Q the sets $\{c : E_0(a, c)\}$ partition $\neg Q$ and this is all that E_0 does:

(*5a) $\forall c(Q(c) \vee \exists! a(Q(a) \wedge E_0(a, c)))$;

(*5b) $\forall a, c(E_0(a, c) \Rightarrow (Q(a) \wedge \neg Q(c)))$.

(*6) In a similar way, for each a the sets $\{c : E_1(b, c)\}$ as b ranges over $b < a$ partition $\{c : E_0(a, c)\}$:

(*6a) $\forall c, a((Q(a) \wedge E_0(a, c) \rightarrow \exists! b(E_1(b, c)))$;

(*6b) $\forall a, b, c((E_0(a, c) \wedge E_1(b, c)) \Rightarrow b < a)$;

(*6c) $\forall b, c(E_1(b, c) \Rightarrow (Q(b) \wedge \neg Q(c)))$.

(*7) For each a in Q we have that the set $\{c : E_0(a, c)\}$ viewed as an $\mathcal{L}_0$ structure satisfies (*1) and (*2) and moreover the $\mathcal{L}_0$ relations are inert except inside the sets of the form $\{c : E_0(a, c)\}$:

(*7a) $\forall d, b[(\exists a(E_0(a, b) \wedge E_0(a, d))) \Rightarrow \bigvee_{n \in \omega} S_n(d, b)]$;

(*7b) $\forall d, b \bigwedge_{n \neq m}(S_n(d, b) \Rightarrow \neg S_m(d, b))$.

(*7c) for each k and $i < j < k$

$$\forall c_0, c_1, b_0, ..., b_{k-1}(R_k(c_0, c_1, b_0, ..., b_{k-1}) \Rightarrow (b_i \neq b_j \wedge c_0 \neq c_1));$$

(*7d) for each k and permutation π of k

$$\forall c_0, c_1, b_0, ..., b_{k-1}(R_k(c_0, c_1, b_0, ..., b_{k-1}) \Leftrightarrow R_k(c_1, c_0, b_{\pi(0)}, ..., b_{\pi(k-1)});$$

(*7e) $\forall c_0, c_1, b_0, ... \bigwedge_{n \in \omega} \wedge_{i<k}(R_k(c_0, c_1, b_0 ....) \Rightarrow (S_n(c_0, b_i) \Leftrightarrow S_n(c_1, b_i))$;

(*7f) $\forall c_0, c_1, b_0, ... \forall d \bigwedge_{n \in \omega}[(R_k(c_0, c_1, b_0 ....) \wedge_{i<k} d \neq b_i) \Rightarrow$
$(S_n(a_0, d) \Rightarrow \neg S_n(a_1, d))]$;

(*7g) $\bigwedge_n \forall c_0, c_1(S_n(c_0, c_1) \Rightarrow \exists a(E_0(a, c_0) \wedge E_0(a, c_1)))$;

(*7h) $\bigwedge_n \forall c_0, c_1 \forall b_0, b_1, ... b_{k-1}(R_k(c_0, c_1, \vec{b}) \Rightarrow$
$\exists a(E_0(a, c_0) \wedge E_0(a, c_1) \wedge \bigwedge_{i<k} E_0(a, b_i)))$.

Definition An $\mathcal{L}_2$ structure F is $\mathcal{L}_2$-*neat* if

1. F is finite;
2. $F \models$ (*4)-(*7);

3. $\forall c_0, c_1 \in F \forall a((E_0(a, c_0) \wedge E_0(a, c_1) \wedge c_0 \neq c_1) \Rightarrow \exists k \exists \vec{b} \in F(R_k(c_0, c_1, \vec{b})))$.

In other words, an $\mathcal{L}_2$-neat structure is a finite structure in which the various fibres $\{c : E_0(a, c)\}$ are $\mathcal{L}_2$-neat as ranges over Q, and for each $c \in \neg Q$ there are unique $b < a$ in Q with $E_1(b, c)$ and $E_0(a, c)$.

We will prove something stronger than the next lemma shortly, but it seems helpful to at least present this simpler argument on its own before we become enmeshed in proving results at full generality.

Lemma 4.4 *Let F be an $\mathcal{L}_2$-neat structure. Let F_0, F_1 be $\mathcal{L}_2$-neat structures extending F. Then there are $\mathcal{L}_2$-neat $H_0 \supset F_0$, $H_1 \supset F_1$ and an isomorphism*

$$i : H_0 \cong H_1$$

with $i|_F$ the identity map.

Proof We may assume first of all that the substructures of F_0 and F_1 formed by

$$\{a : F_0 \models Q(a)\},$$

$$\{a : F_1 \models Q(a)\},$$

are in fact equal for the simple reason that finite linear orderings have the *strong amalgamation property*, in the sense that given any two finite linear orderings L_1 and L_2 which agree on their intersection we may find a linear ordering $L \supset L_1 \cup L_2$.

Then we can assume that for each a with $F_0 \models Q(a)$ (or equivalently given our simplifying assumption $F_1 \models Q(a)$) we have that

$$E_0(a, \cdot)^{F_0} \cap E_0(a, \cdot)^{F_1} \subset F.$$

But now we can simply go through each a with $Q(a)$ and amalgamate the sets $E_0(a, \cdot)^{F_0}$ and $E_0(a, \cdot)^{F_1}$ just as in the proof of 3.4, treating the sets

$$\{\{c : E_1(b, c)\} : b < a\}$$

as we treated the fresh predicates $\{P_n : n \in \omega\}$. □

By now the next definition comes as no surprise.

Definition An $\mathcal{L}_2$-structure $\mathcal{M}$ is $\mathcal{L}_2$-*full* if

1. for all $A \subset \mathcal{M}$ finite there is $\mathcal{L}_2$-neat $F \supset A$, $F \subset \mathcal{M}$;
2. for all $\mathcal{L}_1$-neat $F \subset \mathcal{M}$ and $\mathcal{L}_2$-neat $H_0 \supset F$ there is neat $H_1 \subset \mathcal{M}$ with
   $F \subset H_1$
   there is $i : H_0 \cong H_1$ with $i|_F$ the identity.

Again we obtain that any two countable $\mathcal{L}_2$-full models are isomorphic and thus there is $\varphi_2 \in \mathcal{L}_{\omega_1,\omega}$ which is the Scott sentence of any $\mathcal{L}_2$-full model.

Lemma 4.5 *φ_2 has no model of size $\aleph_3$.*

Proof Otherwise let $\mathcal{N}$ be such a model and note that for each a with $\mathcal{N} \models Q(a)$ we have that $E_0(a,\cdot)^{\mathcal{N}}$ has size at most $\aleph_1$ by 3.3 and the fact that this set viewed as an $\mathcal{L}_0$ structure is $\mathcal{L}_0$-full. Since for each $a \in Q^{\mathcal{N}}$ the axiom (*6) provides a surjection

$$p_a : \{c : \mathcal{N} \models E_0(a,c)\} \twoheadrightarrow \{b \in \mathcal{N} : \mathcal{N} \models b < a\}$$

we conclude that each set of the form

$$\{b \in \mathcal{N} : \mathcal{N} \models b < a\}$$

has cardinality at most $\aleph_1$.

This means that $(Q^{\mathcal{N}}; <^{\mathcal{N}})$ is a linear ordering of size $\aleph_3$ all of whose initial segments are no bigger than $\aleph_1$, which is impossible. $\square$

Lemma 4.6 *φ_2 does have a model of size $\aleph_2$.*

Proof First let us choose a linear ordering $\mathcal{A} = (A; <)$ such that

1. A has size $\aleph_2$;
2. for each $a \in A$ the set $\{b \in A : \mathcal{A} \models b < a\}$ has size $\aleph_1$;
3. $(A; <)$ is a dense linear order without endpoints.

Then for each $a \in A$ we can let

$$\pi_a : \{b \in A : \mathcal{A} \models b < a\} \to \omega_1$$

be a bijection. For each $a \in A$ we can apply 4.3 to choose a $\hat{\mathcal{L}}_{\omega_1}$-full structure $\mathcal{N}_a$ of size $\aleph_1$. We can assume that for $a \neq a'$ we have $\mathcal{N}_a$ and $\mathcal{N}_{a'}$ disjoint.

Now we define $\mathcal{M}$. The set $Q^{\mathcal{M}}$ will be A, with the linear ordering $<^{\mathcal{A}}$ above being $<^{\mathcal{M}}$. The set $\neg Q^{\mathcal{M}}$ will be the disjoint union of the underlying sets of the structures $\{\mathcal{N}_a : a \in A\}$. In the natural way we demand that $\neg Q^{\mathcal{M}}$ viewed as an $\mathcal{L}_0$ structure is just the disjoint union of the $\mathcal{L}_0$ structures $\{\mathcal{N}_a|_{\mathcal{L}_0} : a \in A\}$. For each $a \in Q^{\mathcal{M}}$ we let $\{c : \mathcal{M} \models E_0(a,c)\}$ just be the underlying set of $\mathcal{N}_a$.

We are only left with defining the relation E_1.

But now we appeal to the extra information obtained from the various $\mathcal{N}_a$'s actually being $\hat{\mathcal{L}}_{\omega_1}$-full structures. We demand that for each $a \in Q^{\mathcal{M}}$ and b with

$$\mathcal{M} \models b < a$$

and for each $c \in \mathcal{N}_a$

$$(\mathcal{M} \models E_1(b,c)) \Leftrightarrow \mathcal{N}_a \models P_{\pi_a(b)}(c).$$

It follows more or less immediately from the definitions that this structure is $\mathcal{L}_2$-full, the main point being that the various sets

$$\{c : \mathcal{M} \models E_0(a,c)\}$$

are isolated from one another with respect to the language of $\mathcal{M}$. □

5 The general construction

The next few lemmas would seem to be a slight diversion from the task at hand, but isolate the combinatorics used in the general construction.

Definition For κ an uncountable cardinal and $\gamma \leq \kappa$, let $\mathcal{L}^{\kappa,\gamma}$ be the language generated by binary relations $\{S_\beta : \beta < \kappa\}$, unary predicates $\{P_\beta : \beta < \gamma\}$, and for each $k \in \omega$ a corresponding $(k+2)$-ary relation R_k.

Definition Some new axioms for $\mathcal{L}^{\kappa,\gamma}$, fashioned after the old:

($*1^{\kappa,\gamma}$): the conjunction of:

($*1\mathrm{a}^{\kappa,\gamma}$) $\forall a, b \bigvee_{\beta\in\kappa} S_\beta(a,b)$;

($*1\mathrm{b}^{\kappa,\gamma}$) $\forall a, b \bigwedge_{\beta\neq\alpha}(S_\beta(a,b) \Rightarrow \neg S_\alpha(a,b))$.

($*2^{\kappa,\gamma}$): the conjunction of:

($*2\mathrm{a}^{\kappa,\gamma}$) for each k and $i < j < k$

$$\forall a_0, a_1, b_0, ..., b_{k-1}(R_k(a_0, a_1, b_0, ..., b_{k-1}) \Rightarrow (b_i \neq b_j \wedge a_0 \neq a_1));$$

($*2\mathrm{b}^{\kappa,\gamma}$) for each k and permutation π of k

$$\forall a_0, a_1, b_0, ..., b_{k-1}(R_k(a_0, a_1, b_0, ..., b_{k-1}) \Leftrightarrow R_k(a_1, a_0, b_{\pi(0)}, ..., b_{\pi(k-1)}));$$

($*2\mathrm{c}^{\kappa,\gamma}$) $\forall a_0, a_1, b_0, ... \bigwedge_{\beta\in\kappa} \wedge_{i<k}(R_k(a_0, a_1, b_0....) \Rightarrow$
$(S_\beta(a_0, b_i) \Leftrightarrow S_\beta(a_1, b_i))$;

($*2\mathrm{d}^{\kappa,\gamma}$) $\forall a_0, a_1, b_0, ...\forall c \bigwedge_{\beta\in\kappa}[(R_k(a_0, a_1, b_0....) \wedge_{i<k} c \neq b_i) \Rightarrow$
$(S_\beta(a_0, c) \Rightarrow \neg S_\beta(a_1, c))]$.

($*3^{\kappa,\gamma}$): the conjunction of:

($*3\mathrm{a}^{\kappa,\gamma}$): $\forall a \bigvee_{\beta\in\gamma} P_\beta(a)$;

($*3\mathrm{b}^{\kappa,\gamma}$): $\forall a \bigwedge_{\beta\neq\alpha} \neg(P_\beta(a) \wedge P_\alpha(a))$.

Definition An $\mathcal{L}^{\kappa,\gamma}$ structure F is $\mathcal{L}^{\kappa,\gamma}$-*neat* if

1. F is finite;

2. $F \models (*1^{\kappa,\gamma}), (*2^{\kappa,\gamma}), (*3^{\kappa,\gamma})$;

3. $\forall a_0, a_1 \in F(a_0 \neq a_1 \Rightarrow \exists k \exists \vec{b} \in F(R_k(a_0, a_1, \vec{b})))$.

Definition An $\mathcal{L}^{\kappa,\gamma}$ structure $\mathcal{M}$ is $\mathcal{L}^{\kappa,\gamma}$*-happy* if for any finite $A \subset \mathcal{M}$ there is $\mathcal{L}^{\kappa,\gamma}$-neat $F \supset A$ with $F \subset \mathcal{M}$.

Definition An $\mathcal{L}^{\kappa,\gamma}$ structure $\mathcal{M}$ is $\mathcal{L}^{\kappa,\gamma}$*-full* if

1. it is $\mathcal{L}^{\kappa,\gamma}$-happy;

2. for all $\mathcal{L}^{\kappa,\gamma}$-neat $F \subset \mathcal{M}$ and $\mathcal{L}^{\kappa,\gamma}$-neat $H_0 \supset F$ there is $\mathcal{L}^{\kappa,\gamma}$-neat $H_1 \subset \mathcal{M}$ with
   $F \subset H_1$
   there is $i : H_0 \cong H_1$ with $i|_F$ the identity.

Lemma 5.1 *Let $\mathcal{M}$ be an $\mathcal{L}^{\kappa,\gamma}$-happy structure of size δ, where δ is an infinite cardinal less than κ. Let*

$$F \subset \mathcal{M}$$

and

$$H \supset F$$

both be $\mathcal{L}^{\kappa,\gamma}$-neat. Then there exists $\mathcal{L}^{\kappa,\gamma}$-happy $\mathcal{N} \supset \mathcal{M}$ of size δ and $H' \subset \mathcal{N}$ and

$$i : H \cong H'$$

with $i|_F$ = identity.

Proof Fix H and assume for notational simplicity that $H \cap \mathcal{M} = F$.

Let $T \subset \kappa$ be the set of $\beta < \kappa$ such that there exist $c_0, c_1 \in \mathcal{M}$ or $c_0, c_1 \in H$ with

$$S_\beta(c_0, c_1).$$

T has size less than κ, so we may find a one-to-one function

$$f : (\mathcal{M} \setminus H) \times H \cup H \times (\mathcal{M} \setminus H) \hookrightarrow \kappa \setminus T.$$

We can then make $\mathcal{M} \cup H$ into an $\mathcal{L}^{\kappa,\gamma}$ structure by setting

$$S_{f(c_0,c_1)}$$

(and $\neg S_\beta(c_0, c_1)$ for $\beta \neq f(c_0, c_1)$) for all $c_0, c_1 \in (\mathcal{M} \setminus H) \times H \cup H \times (\mathcal{M} \setminus H)$ and stipulating that

$$R_k(c_0, c_1, b_1, ..., b_k)$$

where $b_1, b_2, ..., b_k$ enumerate the elements b of F where there is some
$\beta \in \kappa$

$$S_\beta(c_0, b_i) \wedge S_\beta(c_1, b_i).$$

It is more or less immediate that the structure thereby produced satisfies $(*1^{\kappa,\gamma})$ and $(*3^{\kappa,\gamma})$. ¿From the assumption on f, it follows that for any $c_0, c_1 \in (\mathcal{M} \setminus H) \times H \cup H \times (\mathcal{M} \setminus H)$ the set

$$\{b \in \mathcal{M} \cup H : \exists \beta (S_\beta(c_0, b) \wedge S_\beta(c_1, b))\}$$

is included in F, and thus we have $(*2^{\kappa,\gamma})$.

¿From the way that the R_k relations have been defined over $F \cup \mathcal{M}$, it follows that F' is $\mathcal{L}^{\kappa,\gamma}$-neat and

$$F \subset F' \subset \mathcal{M}$$

then $F' \cup H$ is $\mathcal{L}^{\kappa,\gamma}$-neat. □

Corollary 5.2 *There is an $\mathcal{L}^{\kappa,\gamma}$-full structure of cardinality κ.*

Proof Note that for any finite F there are up to isomorphism only κ many $\mathcal{L}^{\kappa,\gamma}$-neat $H \supset F$. Thus by 5.1 and a direct diagonalization argument we may produce an increasing chain of structures

$$\mathcal{M}_0 \subset \mathcal{M}_1 \subset ... \mathcal{M}_\alpha \subset \mathcal{M}_{\alpha+1} \subset ...$$

such that for all $\alpha < \kappa$

1. $\mathcal{M}_\alpha$ is $\mathcal{L}^{\kappa,\gamma}$-happy;
2. $\mathcal{M}_\alpha$ has cardinality at most $\alpha + \omega$;
3. $\mathcal{M}_{\alpha+1} \setminus \mathcal{M}_\alpha$ is finite;
4. for each $\mathcal{L}^{\kappa,\gamma}$-neat $F \subset \mathcal{M}_\alpha$ and $\mathcal{L}^{\kappa,\gamma}$-neat $H \supset F$ there is some $\beta > \alpha$ and $\mathcal{L}^{\kappa,\gamma}$-neat $H' \supset F$, $H' \subset \mathcal{M}_\beta$, and

$$i : H \cong H'$$

with $i|_F = identity$.

Then

$$\mathcal{M} = \bigcup_{\alpha<\kappa} \mathcal{M}_\alpha$$

is as required. □

Lemma 5.3 *There is no $\mathcal{L}^{\kappa,\gamma}$-full structure of cardinality greater than κ^+.*

Proof This is as in 3.3. We assume $\mathcal{N}$ is a $\mathcal{L}^{\kappa,\gamma}$-full structure of cardinality greater than κ^+ and choose a substructure $\mathcal{N}_0 \prec \mathcal{N}$ of size κ^+ that is elementary with respect to first order formulas, so that for all $a_0, a_1 \in \mathcal{N}_0$ there is some $\vec{b} \in \mathcal{N}_0$ with $R_k(a_0, a_1, \vec{b})$. Thus for any $b \in \mathcal{N} \setminus \mathcal{N}_0$ we have an injection

$$\pi : \mathcal{N}_0 \hookrightarrow \kappa$$

$$a \mapsto \mu\beta(S_\beta(a,b)).$$

□

Of course $\mathcal{L}^{\kappa,\gamma}$ is an uncountable language, and the afore mentioned axioms and notions of fullness go far beyond what we can say in $\mathcal{L}_{\omega_1,\omega}$. The purpose of the last few lemmas is to give us some grip on the situation where we try to bootstrap up, using a Scott sentence φ_α which characterizes $\aleph_\alpha$ to find one for $\aleph_{\alpha+1}$. Instead of literally using κ and δ above, we instead will use some model of φ_α whose size is at most κ to give us a kind of surrogate cardinal.

Definition Let $\mathcal{M}$ and $\mathcal{N}$ be structures with languages $\mathcal{L}_\mathcal{M}$ and $\mathcal{L}_\mathcal{N}$, which we will assume for notational convenience to have no non-logical symbols in common and to be entirely relational. Let S be a ternary relation not appearing in $\mathcal{L}_\mathcal{M} \cup \mathcal{L}_\mathcal{N}$ and let P be a binary relation not appearing in $\mathcal{L}_\mathcal{M} \cup \mathcal{L}_\mathcal{N}$. For each k let T_k be a fresh $(k+2)$-ary relation. Let $\mathcal{L}(\mathcal{L}_\mathcal{M}, \mathcal{L}_\mathcal{N})$ be the language generated by $\mathcal{L}_\mathcal{M}$, $\mathcal{L}_\mathcal{N}$, S, P, and $\{T_k : k \in \omega\}$.

Definition An $\mathcal{L}(\mathcal{L}_\mathcal{M}, \mathcal{L}_\mathcal{N})$ structure $\mathcal{A}$ including $\mathcal{M}$ and $\mathcal{N}$ is $(\mathcal{M},\mathcal{N})$-*neat* if

1. $\mathcal{A} \setminus (\mathcal{M} \cup \mathcal{N})$ is finite;
2. for any relation symbol R in $\mathcal{L}_\mathcal{M}$ and $\vec{a} \in \mathcal{A}$, if
$$\mathcal{A} \models R(\vec{a})$$
then $\vec{a} \in \mathcal{M}$;
3. similarly for any relation symbol R in $\mathcal{L}_\mathcal{N}$ and $\vec{a} \in \mathcal{A}$, if
$$\mathcal{A} \models R(\vec{a})$$
then $\vec{a} \in \mathcal{N}$;
4. if $\mathcal{A} \models T_k(a_0, a_1, \vec{b})$ then $a_0, a_1, \vec{b} \in \mathcal{A} \setminus (\mathcal{M} \cup \mathcal{N})$;
5. if $\mathcal{A} \models P(a, c)$ then $a \in \mathcal{N}$ and $c \notin (\mathcal{M} \cup \mathcal{N})$;
6. if $\mathcal{A} \models S(a, c_0, c_1)$ then $a \in \mathcal{M}$ and $c_0, c_1 \notin (\mathcal{M} \cup \mathcal{N})$;

   (*$1^{\mathcal{M},\mathcal{N}}$): $\mathcal{A}$ satisfies the conjunction of:

   (*$1a^{\mathcal{M},\mathcal{N}}$) $\forall c_0, c_1 \bigvee_{a \in \mathcal{M}} S(a, c_0, c_1)$;

   (*$1b^{\mathcal{M},\mathcal{N}}$) $\forall c_0, c_1 \bigwedge_{a_0 \neq a_1} (S(a_0, c_0, c_1) \Rightarrow \neg S(a_1, c_0, c_1))$;

   (*$2^{\mathcal{M},\mathcal{N}}$): and $\mathcal{A}$ satisfies the conjunction of:

   (*$2a^{\mathcal{M},\mathcal{N}}$) for each k and $i < j < k$

$$\forall c_0, c_1, b_0, ..., b_{k-1}(T_k(c_0, c_1, b_0, ..., b_{k-1}) \Rightarrow (b_i \neq b_j \wedge c_0 \neq c_1));$$

(*2b$^{\mathcal{M},\mathcal{N}}$) for each k and permutation π of k

$$\forall c_0, c_1, b_0, ..., b_{k-1}(T_k(c_0, c_1, b_0, ..., b_{k-1}) \Leftrightarrow T_k(c_1, c_0, b_{\pi(0)}, ..., b_{\pi(k-1)}));$$

(*2c$^{\mathcal{M},\mathcal{N}}$) $\forall c_0, c_1, b_0, ... \bigwedge_{a\in\mathcal{M}} \wedge_{i<k}(T_k(c_0, c_1, b_0....) \Rightarrow$
$(S(a, c_0, b_i) \Leftrightarrow S(a, c_1, b_i));$

(*2d$^{\mathcal{M},\mathcal{N}}$) $\forall c_0, c_1, b_0, ...\forall d \bigwedge_{a\in\mathcal{M}}[(T_k(c_0, c_1, b_0....) \wedge_{i<k} d \neq b_i) \Rightarrow$
$(S(a, c_0, d) \Rightarrow \neg S(a, c_1, d))];$

(*3$^{\mathcal{M},\mathcal{N}}$): and $\mathcal{A}$ satisfies the conjunction of:

(*3a$^{\mathcal{M},\mathcal{N}}$): $\forall c \bigvee_{a\in\mathcal{N}} P(a, c);$

(*3b$^{\mathcal{M},\mathcal{N}}$): $\forall c \bigwedge_{a_0\neq a_1} \neg(P(a_0, c) \wedge P(a_1, c));$

7. for all $c_0, c_1 \in \mathcal{A} \setminus (\mathcal{M} \cup \mathcal{N})$ there is $k \in \omega$ and $\vec{b} \in \mathcal{A}$ with

$$\mathcal{A} \models T_k(c_0, c_1, \vec{b}).$$

In other words, it is as if we are equipping the finite set $\mathcal{A}\setminus(\mathcal{M}\cup\mathcal{N})$ with binary relations $(S_a)_{a\in\mathcal{M}}$ and unary predicates $(P_b)_{b\in\mathcal{N}}$ and demanding that the resulting structure satisfy the analog of $\mathcal{L}^{\kappa,\gamma}$ neatness for $\kappa = |\mathcal{M}|$, $\gamma = |\mathcal{N}|$.

Definition An $\mathcal{L}(\mathcal{L}_\mathcal{M}, \mathcal{L}_\mathcal{N})$ structure $\mathcal{A}$ including $\mathcal{M}$ and $\mathcal{N}$ is $(\mathcal{M}, \mathcal{N})$-*happy* if for all finite $A \subset \mathcal{A}$ there is some $(\mathcal{M}, \mathcal{N})$-neat substructure $\mathcal{B}$ of $\mathcal{A}$ with $\mathcal{B} \supset A$.

Lemma 5.4 *Let $\gamma \leq \kappa$ be cardinals, κ infinite. Let $\mathcal{M}$ be a structure of size κ and $\mathcal{N}$ a structure of size γ. Let $\mathcal{A} \supset (\mathcal{M} \cup \mathcal{N})$ be $(\mathcal{M}, \mathcal{N})$-happy. Let*

$$F \subset \mathcal{A}$$

and

$$H \supset F$$

both be $(\mathcal{M}, \mathcal{N})$-neat. Then there is $(\mathcal{M}, \mathcal{N})$-happy $\mathcal{B} \supset \mathcal{A}$ and $H' \subset \mathcal{N}$ and

$$i : H \cong H'$$

with $i|_F =$ identity.

Proof With the obvious changes, this follows exactly as in 5.1. The obvious changes are that $\{S_\beta(\cdot, \cdot) : \beta < \kappa\}$ is replaced by $\{S(a, \cdot, \cdot) : a \in \mathcal{M}\}$, $\{P_\beta(\cdot) : \beta < \gamma\}$ by $\{P(a, \cdot) : a \in \mathcal{N}\}$, and each R_k by T_k. $\square$

Corollary 5.5 *Under the same assumptions on $\mathcal{M}$ and $\mathcal{N}$, there is an $(\mathcal{M}, \mathcal{N})$-full structure of cardinality κ.*

Proof Again this has a proof "isomorphic" to 5.2. □

Lemma 5.6 *Under the same assumptions on $\mathcal{M}$ and $\mathcal{N}$, there is no $(\mathcal{M},\mathcal{N})$-full structure of size greater than κ^+.*

Proof As in the proof of 5.3. □

Lemma 5.7 *Let $|\mathcal{M}| = \aleph_0$, $|\mathcal{N}| \leq \aleph_0$, and let $\mathcal{A}$, $\mathcal{B}$ be $(\mathcal{M},\mathcal{N})$-full structures of size $\aleph_0$. Then*

$$\mathcal{A} \cong \mathcal{B},$$

and in fact there is an isomorphism $i : \mathcal{A} \cong \mathcal{B}$ with $i|_{\mathcal{M}}$ and $i|_{\mathcal{N}}$ both equal to the identity.

Proof As in the isomorphism proof in 3.2 for $\mathcal{L}_0$-full structures. □

Thus given countable $\mathcal{M}, \mathcal{N}$ with Scott sentences $\varphi_{\mathcal{M}}$, $\varphi_{\mathcal{N}}$ we may find a countable model $\mathcal{A}$ with Scott sentence $\varphi_{\mathcal{A}}$ such that for any other $\mathcal{B}$ in the same language

$$\mathcal{B} \models \varphi_{\mathcal{A}}$$

if and only if $\{(Q_i)^{\mathcal{B}} : i < 3\}$ partition $\mathcal{B}$, there are $\hat{\mathcal{M}} \models \varphi_{\mathcal{M}}$, $\hat{\mathcal{N}} \models \varphi_{\mathcal{N}}$ with underlying sets $(Q_0)^{\mathcal{B}}$ and $(Q_2)^{\mathcal{B}}$, and $\mathcal{B}|_{\mathcal{L}(\mathcal{L}_{\mathcal{M}},\mathcal{L}_{\mathcal{N}})}$ is an $(\hat{\mathcal{M}},\hat{\mathcal{N}})$-full structure.

Theorem 5.8 *For each countable ordinal α there is a countable model $\mathcal{M}_\alpha$ whose Scott sentence characterizes $\aleph_\alpha$.*

Proof By induction on α, with the passage through limit stages being trivial.

Thus let us suppose that φ_α is the Scott sentence for a countable model $\mathcal{M}_\alpha$ which characterizes $\aleph_\alpha$.

Let $\mathcal{N}_0$ be the trivial one point model whose only relation is $=$. Then following 5.5 we may find a countable model $\hat{\mathcal{M}}_\alpha$ of size $\aleph_0$ which is $(\mathcal{M}_\alpha,\mathcal{N}_0)$-full. Let $\hat{\varphi}_\alpha$ be its Scott sentence. Any model of $\hat{\varphi}_\alpha$ will be $(\hat{\mathcal{M}},\mathcal{N}_0)$-full for some $\hat{\mathcal{M}} \models \varphi_\alpha$, and thus will have size at most $\aleph_\alpha^+$ by 5.3 and the assumption on φ_α. If there is indeed a model of size $\aleph_{\alpha+1}$ then we are done, otherwise we need to go one more round.

Let us then introduce fresh unary predicates Q_0, Q_1, Q_2 and fresh binary relations E_0 and $<$, and let $\varphi_{\alpha+1}$ be the Scott sentence of a countable model $\mathcal{M}_{\alpha+1}$ where:

1. the language of $\mathcal{M}_{\alpha+1}$ is generated by $\mathcal{L}(\mathcal{L}_{\mathcal{M}_\alpha}, \{<\})$ and $\{Q_0, Q_1, Q_2, E_0\}$;
2. $Q_0^{\mathcal{M}_{\alpha+1}}|_{\mathcal{L}_{\mathcal{M}_\alpha}}$ is a model of φ_α;

3. $\langle Q_2^{\mathcal{M}_{\alpha+1}}, <^{\mathcal{M}_{\alpha+1}}\rangle$ is a dense linear ordering without end points; moreover, $<$ is elswhere vacuous, in that
$$\mathcal{M}_{\alpha+1} \models \forall a, b(a < b \Rightarrow (Q_2(a) \wedge Q_2(b)));$$
4. for each $a \in Q_2^{\mathcal{M}_{\alpha+1}}$ the set
$$Q_0^{\mathcal{M}_{\alpha+1}} \cup E_0(a, \cdot)^{\mathcal{M}_{\alpha+1}} \cup \{b : \mathcal{M}_{\alpha+1} \models b < a\}$$
restricted to the appropriate language is a $((Q_0)^{\mathcal{M}_{\alpha+1}}|_{\mathcal{L}_{\mathcal{M}_\alpha}}, \langle\{b : \mathcal{M}_{\alpha+1} \models b < a\}, =\rangle)$-full structure;
5. the sets $\{E_0(a, \cdot)^{\mathcal{M}_{\alpha+1}} : a \in Q_2^{\mathcal{M}_{\alpha+1}}\}$ partition $Q_1^{\mathcal{M}_{\alpha+1}}$;
6. $\mathcal{M}_{\alpha+1} \models \forall a, c(E_0(a, c) \Rightarrow (Q_2(a) \wedge Q_1(c)))$;
7. the various relations $\{P, S, T_0, T_1, ...\}$ are trivial in all cases not covered by above:
   for all a, b, c, we have $\mathcal{M}_{\alpha+1} \models (E_0(a, c) \wedge P(b, c)) \Rightarrow b < a$;
   for all c_0, c_1, d we have $\mathcal{M}_{\alpha+1} \models S(d, c_0, c_1) \Rightarrow \exists a(E_0(a, c_0) \wedge E_0(a, c_1))$;
   for all $c_0, c_1, b_0, b_1, ..., b_{k-1}$, all $k \in \omega$, we have
$$\mathcal{M}_{\alpha+1} \models T_k(c_0, c_1, \vec{b}) \Rightarrow \exists a(E_0(a, c_0) \wedge E_0(a, c_1) \wedge_{i<k} E_0(a, b_i)).$$

In other words, the model $\mathcal{M}_{\alpha+1}$ can be viewed as a disjoint union of a dense linear ordering, a model of φ_α, and a disjoint sequence of fibres, one associated with each strict initial segment of the linear ordering, each of which is appropriately full.

Let $\varphi_{\alpha+1}$ be the Scott sentence of $\mathcal{M}_{\alpha+1}$.

Claim(1): If $\mathcal{M}$ is a model of $\varphi_{\alpha+1}$ and $a \in Q_2^{\mathcal{M}}$ we have that $E_0(a, \cdot)^{\mathcal{M}}$ is of size at most $\aleph_\alpha$.

Proof of claim: Since we may naturally view $E_0(a, \cdot)^{\mathcal{M}}$ as a
$((Q_0)^{\mathcal{M}}|_{\mathcal{M}_\alpha}, \mathcal{N}_0)$-full
structure, and hence a model of $\hat{\varphi}_\alpha$, the claim follows by assumption on $\hat{\varphi}_\alpha$. (Claim(1)$\square$)

Claim(2): If $\mathcal{M}$ is a model of $\varphi_{\alpha+1}$ and $a \in Q_2^{\mathcal{M}}$ we have that $\{b : \mathcal{M} \models b < a\}$ has size at most $\aleph_\alpha$.

Proof of claim: This follows from the previous claim, since the assumption of
$((Q_0)^{\mathcal{M}}|_{\mathcal{M}_\alpha}, \langle\{b : \mathcal{M} \models b < a\}, =\rangle)$-fullness
yields a map from $E_0(a, \cdot)^{\mathcal{M}}$ onto $\{b : \mathcal{M} \models b < a\}$. (Claim(2)$\square$)

Claim(3): No model of $\varphi_{\alpha+1}$ has size bigger than $\aleph_{\alpha+1}$.

Proof of claim: For $\mathcal{M}$ such a model, we have that $\langle Q_2^{\mathcal{M}}, <^{\mathcal{M}}\rangle$ is by claim(2) a linear ordering all of whose initial segements have size at most $\aleph_\alpha$, and thus $Q_2^{\mathcal{M}}$ can have size at most $\aleph_{\alpha+1}$. Then claim(1) gives that $Q_1^{\mathcal{M}}$ has size at most $\aleph_{\alpha+1}$, and the inductive assumption on φ_α restricts $Q_0^{\mathcal{M}}$ to size at most $\aleph_\alpha$. (Claim(3)$\square$)

Claim(4): $\varphi_{\alpha+1}$ does have a model of size $\aleph_{\alpha+1}$.

Proof of claim: Let $\mathcal{D} = \langle D, < \rangle$ be a dense linear ordering without end points of size $\aleph_{\alpha+1}$ all of whose initial segments have size $\aleph_\alpha$. Let $\mathcal{A}_\alpha$ be a model of φ_α of size $\aleph_\alpha$.

For each $a \in D$ we can choose a

$$(\mathcal{A}_\alpha, \langle \{b : \mathcal{D} \models b < a\}, = \rangle)\text{-full}$$

model $\mathcal{N}_a$ of size $\aleph_\alpha$ and we may assume that for $a_0 < a_1$ we have

$$\mathcal{N}_{a_0} \cap \mathcal{N}_{a_1} = \{b : \mathcal{D} \models d < a_0\} \cup \mathcal{A}_\alpha.$$

Then we can obtain a model $\mathcal{A}_{\alpha+1}$ of size $\aleph_{\alpha+1}$ as the union of the $\{\mathcal{N}_a : a \in D\}$.

The absence of relations between the various fibres $E_0(a_0, \cdot)^{\mathcal{A}_{\alpha+1}}$ and $E_0(a_1, \cdot)^{\mathcal{A}_{\alpha+1}}$ for $a_0 \neq a_1$ ensures that $\mathcal{A}_{\alpha+1} \models \varphi_{\alpha+1}$. (Claim(4)□) □

6 Atomic models

It should be commented that there is a close connection between characterizing a cardinal in the sense of Knight and finding a theory $T \subset \mathcal{L}_{\omega,\omega}$ which posses an atomic model at that cardinal but no higher. Since there is an infinitary sentance $\psi \in \mathcal{L}_{\omega_1,\omega}$ which holds in $\mathcal{M}$ if and only if $\mathcal{M}$ is atomic (with respect to formulas in $\mathcal{L}_{\omega,\omega}$), one easily sees that:

Lemma 6.1 *If $T \subset \mathcal{L}_{\omega,\omega}$ is a countable complete theory with atomic models of size κ but no larger, then the Scott sentance of the countable atomic model of T is realized by models of size κ but no models of larger cardinality.*

The converse is also true:

Lemma 6.2 *If the Scott sentance of $\mathcal{M}$, $\varphi_{\mathcal{M}} \in \mathcal{L}_{\omega_1,\omega}$, has a model of size κ but no larger, then there a corresponding countable first order theory $T_{\mathcal{M}}$ which has atomic models of size κ but no larger.*

Proof Let F be the countable fragment of $\mathcal{L}_{\omega_1,\omega}$ generate by $\varphi_{\mathcal{M}}$. Let $\mathcal{L}^*$ be the extension of $\mathcal{L}$ otbained by adding an n-ary relation $R_{\psi(x_1,...,x_n)}(\cdot, ..., \cdot)$ for each formula $\psi(x_1, ..., x_n) \in F$ with n free variables. Let $E(F, \mathcal{M})$ be the expansion of $\mathcal{M}$ obtained by

$$E(F, \mathcal{M}) \models R_{\psi(x_1,...,x_n)}(a_1, ..., a_n)$$

if and only if

$$\mathcal{M} \models \psi(a_1, ..., a_n).$$

Let $T_{\mathcal{M}} \subset \mathcal{L}^*_{\omega,\omega}$ be the complete (finitary) theory of the resulting model $E(F, \mathcal{M})$.

It is not hard to show that any atomic model of $T_{\mathcal{M}}$ will satisfy the Scott sentance of $\mathcal{M}$. We leave the details to the reader. □

In this way, we have the equivalence of Knight's problem with that of finding theories whose atomic models reach a given $\aleph_n$.

Here Chris Lakowski has informed me that he and Saharon Shelah many years previously constructed a countable complete theory T with atomic models of size $\aleph_2$ but no higher. Thus the solution of Knight's problem for characterizing $\aleph_2$ should be credited to them.

7 Equivalence relations

Definition For E and F equivalence relations on Polish spaces X and Y, we say that E is *Borel reducible* to F, written

$$E \leq_B F,$$

if there is a Borel function $f : X \to Y$ such that for all $x_1, x_2 \in X$ we have

$$x_1 E x_2 \Leftrightarrow f(x_1) F f(x_2).$$

We use $E \leq_c F$ to indicate that there is a continuous reduction, $E \sqsubseteq_B F$ and $E \sqsubseteq_c F$ to indicate that the reduction may respectively chosen Borel and one to one or continuous and one to one, and $E <_B F$ if $E \leq_B F$ holds and $F \leq_B E$ fails.

Of course one can consider reductions that use functions which are not Borel but still have some claim on being "reasonably nice". A discussion of these more general classes – such as absolutely $\underset{\sim}{\Delta}^1_2$ – can be found in [7] and [10].

Definition If G is a Polish group acting continuously on a Polish space X we use E^X_G for the orbit equivalence relation given by

$$x_1 E^X_G \Leftrightarrow \exists g \in G (g \cdot x_1 = x_2).$$

Among the Borel equivalence relations are: $\mathrm{id}(X)$, the identity relation on a Polish space X; E_0, the equivalence relation of eventual agreement on infinite binary sequences: thus for $x, y \in 2^{\mathbb{N}}$ we have

$$x E_0 y$$

if there exists some $N \in \mathbb{N}$ such that

$$\forall n > N (x(n) = y(n)).$$

It is known from [6] that $\mathrm{id}(\mathbb{R})$ and E_0 occupy a privileged position among the Borel equivalence relations:

Theorem 7.1 *(Harrington, Kechris, Louveau) If E is a Borel equivalence relation, then either:*
(I) $E \leq_B \mathrm{id}(\mathbb{R})$; or
(II) $E_0 \sqsubseteq_c E$.

An important class of equivalence relations arise as those of the form E_G^X, for G a Polish group acting continuously on a Polish space; in general these equivalence relations are not Borel. Thus one is not able to directly apply the Harrington-Kechris-Louveau theorem, and it is known (see [7]) that in general counterexamples do exist.

Nonetheless for many interesting classes of Polish groups one can prove the conclusion that whenever G acts continuously on a Polish space X we have either:

(I) $E_G^X \leq_B \mathrm{id}(\mathbb{R})$; or

(II) $E_0 \sqsubseteq_c E_G^X$.

Here a sequence of papers, most of which are discussed in [10], have culminated in a result of Howard Becker's, who showed that this dichotomy theorem holds for all Polish groups allowing a complete left invariant metric.

The language of equivalence relations has also led to the following reformulation of the topological Vaught conjecture. While not equivalent to the original Vaught conjecture – which stated that for every Polish group G acting continuously on a Polish space X, there are either $\geq 2^{\aleph_0}$ many orbits or $\leq \aleph_0$ many orbits – it is known to be *equiprovable* over ZFC. We can prove one in ZFC if and only if we can prove the other.

Conjecture The topological Vaught conjecture (strong form). Let G be a Polish group acting continuously on a Polish space X. Then either

(I) $E_G^X \leq_B \mathrm{id}(\mathbb{N})$ (where we view $\mathbb{N}$ as a Polish space by giving it the discrete metric); or

(II) $\mathrm{id}(\mathbb{R}) \sqsubseteq_c E_G^X$.

The conclusion of the Harrington-Kechris-Louveau theorem, when it can be proved for an orbit equivalence relation, implies even this strong form of the topological Vaught conjecture. One has that if $E_0 \sqsubseteq_c E_G^X$ then $\mathrm{id}(\mathbb{R}) \sqsubseteq_c E_G^X$, since $\mathrm{id}(\mathbb{R}) \sqsubseteq_c E_0$; alternatively if $E_G^X \leq_B \mathrm{id}(\mathbb{R})$ then the perfect set theorem for $\underset{\sim}{\Sigma}_1^1$ sets entails that either there are just countably many orbits, all of which are Borel, or there is a perfect set of orbit inequivalent points.

A discussion of these and related facts can be found in [3] and [2]; [11] also surveys recent work on the topological Vaught conjecture.

8 An amusing footnote to a theorem of Becker's

This section is an extended footnote to the afore mentioned theorem from [2] and consequent question:

Theorem 8.1 *(Becker) Let G be a Polish group admitting a complete left invariant metric. Then for any Polish G-space X, either*

(I) $E_0 \sqsubseteq_c E_G^X$ *or*

(II) $E_G^X \leq_B \mathrm{id}(\mathbb{R})$.

Question (Becker) Suppose G is a Polish group satisfying the conclusion of the above theorem. Must it have a compatible complete left invariant metric?

As a partial answer we prove here:

Theorem 8.2 *There is a Polish group G without a complete left invariant metric such that whenever X is a Polish G-space: either*
(I) $E_0 \sqsubseteq_c E_G^X$ or
(II) E_G^X is reducible to $\mathrm{id}(\mathbb{R})$ *by an absolutely $\underset{\sim}{\Delta}_2^1$ function.*

What is amusing and curious about the proof of this main theorem is its left handed use of large cardinal ideas. The key step is to force with the Frechet filter on ω_1 of some inner model and consider the combinatorial consequences of the generic ultrapower. Of course this ideal of countable subsets of ω_1 is never precipitous – the ultrapower is *never* well founded – and yet this generic embedding is precisely matched for the needs of the proof.

The reader might expect that this is somehow parallel to the proof of Silver's theorem in cardinal arithmetic, where the proof is smoother in the event of there being access to a precipitous ideal on ω_1, but can be made to work in ZFC alone using less. Here the opposite is true. The argument could not fare at all with a precipitous ideal, or even one which gives rise to $\omega_1 + 1$ being included in the well founded part of the ultrapower. There is a strong homogeneity property for $\mathcal{P}(\omega_1)/$Countable which lies diametrically opposite to any trace of well foundedness and is essential to the method.

Of course, this is not to say that the result above is optimal. In (II) one would prefer $\leq_B$ $\mathrm{id}(\mathbb{R})$, and so this is a curious footnote, rather than a finished answer.

We need the following detailed restatement of Knight's result:

Theorem 8.3 *(Knight) There is a countable structure $\mathcal{M}$ such that*
(i) there is a model $\hat{\mathcal{M}}$ of size $\aleph_1$ which is $\mathcal{L}_{\omega_1,\omega}$ equivalent to $\mathcal{M}$;
(ii) $\mathcal{M} = (M; <, (g_n)_{n\in\mathbb{N}})$ such that
(a) $<$ totally orders $\mathcal{M}$;
(b) for all $a, b \in \mathcal{M}$

$$a < b \Rightarrow \exists n(g_n(b) = a).$$

Without further addressing the details of this construction – details which we will never need – let G_K denote the automorphism group of Knight's model $\mathcal{M}$, $\mathrm{Aut}(\mathcal{M})$. This group does not admit a complete left invariant metric in light of 3.8

Theorem 8.4 *(Becker-Kechris, see [3]) Let $G =$*$\mathrm{Aut}(\mathcal{N})$ *be a closed subgroup of S_∞; let $\mathcal{L}$ be the language of $\mathcal{N}$. Let X be a Polish G-space. Then there is a language $\mathcal{L}' \supset \mathcal{L}$ extending the language of $\mathcal{N}$ and $\sigma \in \mathcal{L}'_{\omega_1\omega}$ such that $\sigma \models \varphi_{\mathcal{N}}$ (the Scott sentence of $\mathcal{N}$) and $|X/G|$ is Borel equivalent to* $\mathrm{Mod}(\sigma)$*, in the sense that there are $\theta : X \to$*$\mathrm{Mod}(\sigma)$ *and $\rho :$* $\mathrm{Mod}(\sigma) \to X$ *such that:*
(i) θ witnesses $E_G^X \leq_B \cong |\mathrm{Mod}(\sigma)$;
(ii) ρ witnesses $\cong | \mathrm{Mod}(\sigma) \leq_B E_G^X$; and
(iii) these are orbit inverses to one another in the sense that for all $x \in X$, $x E_G^X (\rho \circ \theta(x))$.

Theorem 8.5 *(Becker; Hjorth-Kechris, [7], [10]) Let G be a Polish group and X a Polish G-space; then either*
(I) $E_0 \sqsubseteq_c E_G^X$ or
(II) E_G^X is reducible to $\mathrm{id}(2^{<\omega_1})$ *by a function that is absolutely $\underset{\sim}{\Delta}^1_2$ in the codes.*

We also need to fix some notation and terminology for talking about the Scott sentence of a structure.

Definition For $\mathcal{A}$ a model and $\vec{a}$ a finite sequence from $\mathcal{A}$ we define

$$\varphi_\alpha^{\vec{a},\mathcal{A}}$$

by induction on the ordinal α.

$$\varphi_0^{\vec{a},\mathcal{A}} = \bigwedge\{\psi(\vec{x}) : \psi(\vec{x}) \text{ quantifier free, } \vec{a} \models \psi(\vec{a})\}$$

$$\varphi_{\alpha+1}^{\vec{a},\mathcal{A}} = \bigwedge\{\exists \vec{y}\varphi_{\alpha+1}^{\vec{a}\vec{b},\mathcal{A}}(\vec{x},\vec{y}) : \vec{b} \in \mathcal{A}^{<\mathbb{N}}\} \wedge \forall \vec{y} \bigvee\{\varphi_{\alpha+1}^{\vec{a}\vec{b},\mathcal{A}}(\vec{x},\vec{y}) : \vec{b} \in \mathcal{A}^{<\mathbb{N}}\}.$$

For λ a limit we set

$$\varphi_\lambda^{\vec{a},\mathcal{A}} = \bigwedge_{\alpha<\lambda} \varphi_\alpha^{\vec{a},\mathcal{A}}.$$

As shown in [13], we will come to an ordinal δ such that for all $\vec{a},\vec{b} \in \mathcal{A}$

$$\exists \gamma(\varphi_\gamma^{\vec{a},\mathcal{A}} \neq \varphi_\gamma^{\vec{b},\mathcal{A}}) \Leftrightarrow \varphi_\delta^{\vec{a},\mathcal{A}} \neq \varphi_\delta^{\vec{b},\mathcal{A}}.$$

The least such δ is the *Scott height of* $\mathcal{A}$ and is denoted by $\alpha(\mathcal{A})$.

This ordinal will have the same cardinality as $\mathcal{A}$. $\varphi_{\alpha(\mathcal{A})+2}^{\emptyset,\mathcal{A}} =_{df} \varphi^{\mathcal{A}}$ is the (canonical) *Scott sentence of* $\mathcal{A}$; $\varphi_{\alpha(\mathcal{A})+2}^{\vec{a},\mathcal{A}} =_{df} \varphi^{\vec{a},\mathcal{A}}$ (or just $\varphi^{\vec{a}}$ when context indicates $\mathcal{A}$) is the *Scott sentence of* $\vec{a}$ (in $\mathcal{A}$); each $\varphi_\alpha^{\vec{a},\mathcal{A}}$ is the αth approximation of the Scott sentence for $\vec{a}$. In the text below we will also call it the *canonical α-type of $\vec{a}$.*

Proof of Theorem 8.2

Letting X be a Polish G_K space, we can apply the result above due to Becker and Kechris, and let $\sigma \in \mathcal{L}'_{\omega_1\omega}$ be the infinitary sentence whose countable models correspond naturally to the orbits of G_K in X. Assuming for the moment that $E_0 \sqsubseteq_c E_G^X$ fails, we apply the result of Becker, Hjorth, and Kechris to find

$$\theta : \mathrm{Mod}(\sigma) \to 2^{<\omega_1},$$

absolutely $\underset{\sim}{\Delta}^1_2$ in the codes such that for all $\mathcal{N}_1, \mathcal{N}_2 \in \mathrm{Mod}(\sigma)$ we have

$$\mathcal{N}_1 \cong \mathcal{N}_2 \Leftrightarrow \theta(\mathcal{N}_1) \cong \theta(\mathcal{N}_2).$$

Let z code θ and σ. We wish to show that we can somehow massage θ into an absolutely $\underset{\sim}{\Delta}^1_2$ injection from $\mathrm{Mod}(\sigma)/\cong$ into $2^{\mathbb{N}}$.

Claim(1) It suffices to show that there is always (for any $\mathcal{N}$ appearing in any generic extension of $\mathbb{V}$) $\mathcal{N}_0 \in L[\theta(\mathcal{N}), z] \cap \mathrm{Mod}(\sigma)$ (so in particular it has $\mathbb{N}$ as its underlying set) with

$$\mathcal{N}_0 \cong \mathcal{N}.$$

Proof of claim: If so, choose $\mathcal{N}_0$ least such in the canonically definable in $\theta(\mathcal{N}), z$ well order of $L[\theta(\mathcal{N}), z]$. (This we *may* do since $\theta(\mathcal{N})$ is a subset of the ordinals.) Then we choose $x_{\mathcal{N}}$ to be the first (in $L[\theta(\mathcal{N}), z]$) element of, say, $2^{\mathbb{N}}$ coding $\mathcal{N}_0$. Since $\theta(\mathcal{N})$ depends only on $[\theta(\mathcal{N})]_{\cong}$ (the isomorphism type of $\mathcal{N}$)

$$\mathcal{N} \mapsto x_{\mathcal{N}}$$

is an invariant; since $x_{\mathcal{N}}$ codes a model isomorphic to $\mathcal{N}$ it is a complete invariant, and we do indeed have reduction to $\mathrm{id}(2^{\mathbb{N}})$, and hence $\mathrm{id}(\mathbb{R})$.

The reduction is clearly $\underset{\sim}{\Delta}{}^1_2$; by appealing to the absoluteness of θ in the Becker/Hjorth-Kechris theorem, we have that it is absolutely $\underset{\sim}{\Delta}{}^1_2$. (Claim $\square$)

Claim(2) In any generic extension $L[\theta(\mathcal{N}), z][G]$ of $L[\theta(\mathcal{N}), z]$ in which $\theta(\mathcal{N})$ is countable we have

$$L[\theta(\mathcal{N}), z][G] \models \exists \mathcal{N}_0 \in \mathrm{Mod}(\sigma)(\theta(\mathcal{N}_0) = \theta(\mathcal{N})).$$

Proof of Claim: Since for w some real coding $\theta(\mathcal{N})$ in $L[\theta(\mathcal{N}), z][G]$ the existence of some such $\mathcal{N}_0$ as indicated is $\Sigma^1_2(z, w)$, and so we may apply Shoenfield absoluteness. (Claim $\square$)

Claim(3) For any partial order $\mathbb{P}$ in $L[\theta(\mathcal{N}), z]$ and $\mathbb{P}$-term τ with

$$L[\theta(\mathcal{N}), z] \models (\mathbb{P} \Vdash \tau(\dot{G}) \in \mathrm{Mod}(\sigma) \wedge \theta(\tau(\dot{G})) = \theta(\mathcal{N})))$$

we have

$$L[\theta(\mathcal{N}), z] \models \mathbb{P} \times \mathbb{P} \Vdash \tau(\dot{G}_l) \cong \tau(\dot{G}_r).$$

Proof of Claim. Otherwise choose $(p_1, p_2) \in \mathbb{P} \times \mathbb{P}$ forcing otherwise, and then $G_l \times G_r \subset \mathbb{P} \times \mathbb{P}$ below this condition which is $\mathbb{V}$-generic. By Shoenfield, $\tau(G_l) \cong \tau(G_r)$ is absolute to $L[\theta(\mathcal{N}), z][G_l \times G_r]$, as is the calculation of $\theta(\tau(G_l))$ and $\theta(\tau(G_r))$, and so for either $\mathcal{N}_0 = \tau(G_l)$ or $\mathcal{N}_0 = \tau(G_r)$ we have a generic extension of $\mathbb{V}$ containing some $\mathcal{N}_0 \in \mathrm{Mod}(\sigma)$ with

$$\theta(\mathcal{N}_0) = \theta(\mathcal{N}),$$
$$\mathcal{N}_0 \ncong \mathcal{N}.$$

Once more by Shoenfield, existence of such an $\mathcal{N}_0$ is absolute to $\mathbb{V}$, and this contradicts the assumption on θ. (Claim $\square$)

Claim(4) There is $\varphi \in (\mathcal{L}'_{\infty,\omega})^{L[\theta(\mathcal{N}),z]}$ such that $L[\theta(\mathcal{N}), z]$ believes that for $\mathbb{P} = Coll(\omega, |\theta(\mathcal{N})|)$ we have

$\mathbb{P} \Vdash \varphi$ is the Scott sentence of some $\mathcal{N}_0$ with $\theta(\mathcal{N}_0) = \theta(\mathcal{N})$.

Proof of Claim. Let τ be a $\mathbb{P}$-term for some model in the generic extension $L[\theta(\mathcal{N}), z]^{\mathbb{P}}$ with $\theta(\tau(\dot{G})) = \theta(\mathcal{N})$; this term exists by Claim(1). Then the Scott sentence φ does not depend on the choice on G by the last claim. Hence, by standard facts about forcing (namely: whenever $\mathbb{P}_1 \times \mathbb{P}_2 \Vdash \tau_1(\dot{G}_l) = \tau_2(\dot{G}_r)$ one may prove by induction on the rank of the terms τ_1 and τ_2 that there is some a in the ground model with $\mathbb{P}_1 \times \mathbb{P}_2 \Vdash \tau_1(\dot{G}_l) = \tau_2(\dot{G}_r) = a$) we can find $\varphi \in L[\theta(\mathcal{N}), z]$ as required. (Claim $\Box$)

Now let α be the Scott height of any model of φ and let $A = (\psi_\beta)_{\beta\in\kappa}$ enumerate the canonical α-types realized in any model of φ. (This *can* be calculated in $L[\theta(\mathcal{N}), z]$.) We need to introduce a prelinear ordering $\leq^*$ on these canonical α-types. For ψ, ψ' appearing in the sequence A, let us define

$$\psi \leq^* \psi'$$

to hold if in any model $\mathcal{N}_0$ of φ there exist $x, y \in \mathcal{N}_0$ with

$$\mathcal{N}_0 \models \psi(x),$$

$$\mathcal{N}_0 \models \psi'(y),$$

and

$$\mathcal{N}_0 \models x \leq y;$$

equivalently:

$\exists x, y \in \mathcal{N}_0, n \in \mathbb{N}(\mathcal{N}_0 \models \psi(x), \mathcal{N}_0 \models \psi'(y), g_n(y) = x)$;

equivalently since ψ' determines every $\mathcal{L}'_{\omega_1,\omega}$ fact about a tuple satisfying it:

$\exists n \in \mathbb{N}$ such that for all $y \in \mathcal{N}_0(\mathcal{N}_0 \models \psi'(y) \Rightarrow \psi(g_n(y)))$.

Thus $\leq^*$ is a prelinear order on A with every initial segment of the form $\{a \in A : a \leq^* b\}$ countable. If A were itself countable, then we would necessarily have that φ is countable, and we would happily sign off with Claim(1). So suppose instead that A is uncountable in $L[\theta(\mathcal{N}), z]$, and we prepare to force to make A larger.

Here it should be pointed out that many of the elements of A diverge even before $\omega_1^{L[\theta(\mathcal{N}),z]}$ – and so that in this sense only a countable ordinal is needed to verify that they are distinct:

Claim(5) There is in $L[\theta(\mathcal{N}), z]$ some $L[\theta(\mathcal{N}), z]$-uncountable $Z \subset \kappa$ and $\gamma \leq \omega_1^{L[\theta(\mathcal{N}),z]}$ such that

(i) for all $\beta_1 \neq \beta_2$ in Z we have

$$\psi|_{\beta_1} \neq \psi|_{\beta_2};$$

(ii) each $\psi_\beta|_\gamma$ is either in $\mathrm{HC}^{L[\theta(\mathcal{N}),z]}$ or of the form

$$\bigwedge_{\xi\in\omega_1^{L[\theta(\mathcal{N}),z]}} \psi_{\beta,\xi},$$

with each $\psi_{\beta,\xi} \in\mathrm{HC}^{L[\theta(\mathcal{N}),z]}$.

Proof of Claim: Since if the Scott height of a model $\mathcal{N}_0$ is greater than or equal to some ordinal γ then for $\beta < \gamma$ there are $\vec{a}, \vec{b} \in \mathcal{N}_0^{<\mathbb{N}}$ such that

$$\varphi_\beta^{\vec{a},\mathcal{N}_0} = \varphi_\beta^{\vec{b},\mathcal{N}_0}$$

(the canonical β-types are the same) but

$$\varphi_{\beta+1}^{\vec{a},\mathcal{N}_0} \neq \varphi_{\beta+1}^{\vec{b},\mathcal{N}_0}$$

(at the next stage of the Scott analysis they diverge). (Claim $\Box$)

¿From now until the finish let $\mathbb{P}$ denote the forcing

$$(\mathcal{P}(\omega_1)/\text{Countable})^{L[\theta(\mathcal{N}),z]}$$

consisting of $L[\theta(\mathcal{N}), z]$'s uncountable subsets of $\omega_1^{L[\theta(\mathcal{N}),z]}$ considered up to (its version) of the countable symmetric difference. For $G \subset \mathbb{P}$ a $L[\theta(\mathcal{N}), z]$-generic object, let U_G indicate the generic ultrafilter (of all those $Y \subset \omega_1^{L[\theta(\mathcal{N}),z]}$ with $[Y] \in G$). We then define the generic embedding

$$j_G : L[\theta(\mathcal{N}), z] \to \text{Ult}(U_G, L[\theta(\mathcal{N}), z])$$

of $L[\theta(\mathcal{N}), z]$ into the generic ultrapower.

Let $\mathbb{Q}$ denote the forcing $\mathbb{P} * \text{Coll}(\omega, |j_{\dot{G}}(\theta(\mathcal{N}))|^{\text{Ult}(U_{\dot{G}}, L[\theta(\mathcal{N}),z])})$ – where we force intitially with $\mathbb{P}$, and then in the illfounded model $\text{Ult}(U_{\dot{G}}, L[\theta(\mathcal{N}), z])$ we collapse its $|j_{\dot{G}}(\theta(\mathcal{N}))$. Using elementarity and claim(2) we can let $\dot{\tau}$ be a $\mathbb{Q}$-term for a model that satisfies in $\text{Ult}(U_{\dot{G}}, L[\theta(\mathcal{N}), z])$ the (from its point of view) infinitary sentence $j_{\dot{G}}(\varphi)$.

The key point in all these steps is the next one:

Claim(6) $\mathbb{Q} \times \mathbb{Q} \Vdash \tau(\dot{G}_l) \ncong \tau(\dot{G}_r)$.

Proof of Claim: Otherwise (after perhaps refining to some $(q_1, q_2) \in \mathbb{Q} \times \mathbb{Q}$) we can apply the argument of Claim(4) and obtain some $q \in \mathbb{Q}$ and $\varphi' \in L[\theta(\mathcal{N}), z]$ such that

$$q \Vdash_{\mathbb{Q}} \tau \text{ has } \varphi' \text{ as its Scott sentence.}$$

Let α' be the Scott height of any model of φ' and let $A' = (\psi'_\beta)_{\beta \in \kappa'}$ enumerate the canonical α'-types of any model of φ'. By analogy with the preceding definition, for ψ, ψ' appearing in A'

$$\psi \leq^* \psi'$$

if in any model $\mathcal{N}_0$ of φ' there exist $x, y \in \mathcal{N}_0$ with

$$\mathcal{N}_0 \models \psi(x),$$

$$\mathcal{N}_0 \models \psi'(y),$$

and

$$\mathcal{N}_0 \models x \leq y.$$

Let $G \subset \mathbb{P}$ be $L[\theta(\mathcal{N}), z]$-generic, and let $B = (\varphi_\beta)_{\beta\in\zeta}$ be $j_G(A)$ in $\mathrm{Ult}(U_{\dot{G}}, L[\theta(\mathcal{N}), z])$. (Note that ζ will only be an ordinal from the point of view of $\mathrm{Ult}(U_{\dot{G}}, L[\theta(\mathcal{N}), z])$, and not genuinely wellfounded in $\mathbb{V}$; similarly I have no real right to use notation indicating that B is only a sequence from the perspective of this highly non-transitive – not even wellfounded – model $\mathrm{Ult}(U_{\dot{G}}, L[\theta(\mathcal{N}), z])$.) Now let ζ' be an ordinal of $\mathrm{Ult}(U_{\dot{G}}, L[\theta(\mathcal{N}), z])$ not in the image of j_G; in other words, let us choose

$$\zeta' \in \mathrm{Ordinals}^{\mathrm{Ult}(U_{\dot{G}}, L[\theta(\mathcal{N}), z])} \setminus j_G[\mathrm{Ordinals}].$$

Remembering the prelinear ordering $\leq^*$ of A we have

$$\forall\alpha < \kappa(\mathrm{Ult}(U_{\dot{G}}, L[\theta(\mathcal{N}), z]) \models j_G(\psi_\alpha) \leq^* \varphi_{\zeta'})$$

since j_G fixes $\mathbb{N}$ and each element of A appears in a countable $\leq^*$ initial segment. Thus for $G_1 \subset \mathrm{Coll}(\omega, |j_{\dot{G}}(\theta(\mathcal{N}))|^{\mathrm{Ult}(U_{\dot{G}}, L[\theta(\mathcal{N}), z])})$ an $L[\theta(\mathcal{N}), z][G]$-generic we have some $\mathcal{N}_0 \in \mathrm{Mod}(\sigma) \cap \mathrm{Ult}(U_{\dot{G}}, L[\theta(\mathcal{N}), z])[G_1]$ with

$$\mathrm{Ult}(U_{\dot{G}}, L[\theta(\mathcal{N}), z])[G_1] \models (\mathcal{N}_0 \models j_G(\varphi)).$$

Thus we may fix a which $\mathrm{Ult}(U_{\dot{G}}, L[\theta(\mathcal{N}), z])[G_1]$ considers to be in $\mathcal{N}_0$ with

$$\mathrm{Ult}(U_{\dot{G}}, L[\theta(\mathcal{N}), z])[G_1] \models (\mathcal{N}_0 \models \varphi_{\zeta'}(a)).$$

Hence for all $\beta < \kappa$ there exists $b_\beta \in^{\mathrm{Ult}(U_{\dot{G}}, L[\theta(\mathcal{N}), z])[G_1]} \mathcal{N}_0$ and $n_\beta \in \mathbb{N}$ with

$$\mathrm{Ult}(U_{\dot{G}}, L[\theta(\mathcal{N}), z])[G_1] \models (\mathcal{N}_0 \models j_G(\psi_\beta)(b_\beta) \wedge g_{n_\beta}(a) = b_\beta).$$

Now let φ^* be the true[2] canonical α'-type of a and for $\beta \in \kappa$ let φ^*_β be the true canonical α'-type of b_β. By the characteristic properties of the Scott sentence

$$\varphi' \models \forall x(\varphi^*(x) \Rightarrow \varphi^*_\beta(g_{n_\beta}(x))).$$

Thus in particular, $\varphi^* \geq^* \varphi^*_\beta$. Note that φ', φ^* and so on are all in $L[\theta(\mathcal{N}), z]$.

Now there is a split into two cases. For $\psi = \varphi^{\vec{a},\mathcal{N}_0}_\alpha$ some canonical α-type in the model $\mathcal{N}_0$, and for $\gamma < \alpha$, let $\psi|_\gamma$ denote $\varphi^{\vec{a},\mathcal{N}_0}_\gamma$ – the γ level approximation to $\psi = \varphi^{\vec{a},\mathcal{N}_0}_\alpha$. This is welldefined; it does not depend on the choice of $\mathcal{N}_0$ and $\vec{a}$.

Now recall from claim(5) that we have some sequence $(\psi_\beta)_{\beta\in Z}$ of canonical α-types, appearing in the Scott analysis of $\mathcal{N}$, and we have some $\gamma \leq \omega_1^{L[\theta(\mathcal{N}),z]}$ such that each $\psi_\beta|_\gamma$ is either in $\mathrm{HC}^{L[\theta(\mathcal{N}),z]}$ or of the form

$$\bigwedge_{\xi\in\omega_1^{L[\theta(\mathcal{N}),z]}} \psi_{\beta,\xi},$$

with each $\psi_{\beta,\xi} \in \mathrm{HC}^{L[\theta(\mathcal{N}),z]}$.

[2]As calculated by any wellfounded inner model – for instance $L[\theta(\mathcal{N}), z][G][G_1]$

Case A. $\gamma < \omega_1^{L[\theta(\mathcal{N}),z]}$.

Note then that each such $\psi_\beta|\gamma$ is in $\mathrm{HC}^{L[\theta(\mathcal{N}),z]}$ and then left fixed by j_G. Thus in particular $b_\beta \models \psi_\beta|\gamma$, and so $\varphi^*_\beta \Rightarrow \psi_\beta|\gamma$. Then we may choose in $L[\theta(\mathcal{N}),z]$ and a maximal (and necessarily uncountable by case assumption) $Z_0 \subset \kappa$ such that for all distinct $\beta_1, \beta_2 \in Z_0$ we have

$$\psi_{\beta_1}|\gamma \neq \psi_{\beta_2}|\gamma,$$
$$\therefore \psi_{\beta_1}|\gamma \Rightarrow \neg\psi_{\beta_2}|\gamma.$$

For each $\beta \in Z_0$ let $A_\beta = \{\beta' \in \kappa' : \psi'_{\beta'} \Rightarrow \psi_\beta|\gamma\}$. We have then immediately from the above that the A_β's are non-empty, disjoint as β ranges over Z_0, and for each $\beta \in \kappa$ there is some $\psi' \in A_\beta$ with

$$\psi' \leq^* \varphi^*,$$

and hence (since φ' is the Scott sentence of an expansion of Knight's model) there is some n with

$$\varphi' \models \forall x(\varphi^*(x) \Rightarrow \psi'(g_n(x))).$$

This contradicts the uncountability of Z_0.

Case B. Not case A.

But we still have $\psi_\beta|_{\gamma'}$ is in $\mathrm{HC}^{L[\theta(\mathcal{N}),z]}$ for each $\gamma' < \omega_1^{L[\theta(\mathcal{N}),z]}$ and hence invariant under j_G. Analogously to case A we have that $b_\beta \models \psi_{(\omega_1)^{L[\theta(\mathcal{N}),z]}}$, and hence $\varphi^*_\beta \Rightarrow \psi_\beta|\omega_1^{L[\theta(\mathcal{N}),z]}$. Letting $C_\beta = \{\beta' \in \kappa' : \psi'_{\beta'} \Rightarrow \psi_\beta|\omega_1^{L[\theta(\mathcal{N}),z]}\}$ we arrive at the same contradiction as in case A. (Claim $\Box$)

For $G \subset \mathbb{P}$ an $L[\theta(\mathcal{N}),z]$-generic let φ_G denote the Scott sentence in $L[\theta(\mathcal{N}),z][G]$ of any model in $\mathrm{Ult}(U_{\dot{G}}, L[\theta(\mathcal{N}),z])^{\mathrm{Coll}(\omega,\omega_1^{L[\theta(\mathcal{N}),z][G]})}$ satisfying $j_G(\varphi)$. (So we can identify the isomorphism type of $\dot{\tau}$ without proceeding on to the second step of the forcing; this uses the upwards absoluteness of $\underset{\sim}{\Sigma}^1_1$ between ω-models, and that $\mathrm{Ult}(U_{\dot{G}}, L[\theta(\mathcal{N}),z])$ itself does not believe that the isomorphism type depends on the second part of the forcing $\mathbb{Q}$.)

Claim(7) For any $p_1, p_2 \in \mathbb{P}$ there is an automorphism

$$\pi : \mathbb{P} \cong \mathbb{P}$$

$L[\theta(\mathcal{N}),z]$ with p_1 and $\pi(p_2)$ compatible, and such that if $G \subset \mathbb{P}$ is $L[\theta(\mathcal{N}),z]$-generic, then there is some

$$\hat{\pi} : \mathrm{Ult}(U_G, L[\theta(\mathcal{N}),z]) \cong \mathrm{Ult}(U_{\pi(G)}, L[\theta(\mathcal{N}),z])$$

such that

$$\hat{\pi} \circ j_G = j_{\pi(G)}.$$

Proof of Claim. Let $A_1, A_2 \subset \omega_1^{L[\theta(\mathcal{N}),z]}$ be representatives of p_1, p_2 respectively. Without loss of generality, they are both cocountable. So $L[\theta(\mathcal{N}),z]$ may find a bijection π_0 of its ω_1 with

$$\pi_0[A_2] = A_1.$$

Then π_0 naturally induces π, which in turn naturally lifts to an isomorphism of the ultrapowers since $\pi_0 \in L[\theta(\mathcal{N}), z]$. (Claim $\Box$)

Now we can complete the Glimm-Effros/Harrington-Kechris-Louveau type of argument of [6] or [3]. One finds in [3] a theorem to the effect that if we have a Polish group action where every orbit is dense and meager, then E_0 continuously reduces to the orbit equivalence relation. Claim(6) was the analogue of every orbit meager; Claim(7) was in spirit that every orbit is dense.

Let $(\mathcal{D}_i)_{i\in\mathbb{N}}$ enumerate the open dense subsets of $\mathbb{P}\times\mathbb{P}$ existing in $L[\theta(\mathcal{N}), z]$. We build conditions $p_s \in \mathbb{P}$ for $s \in 2^{<\mathbb{N}}$ and automorphisms $\pi_n \in L[\theta(\mathcal{N}), z]$ as in Claim(7) such that:

(i) for $s \supset t$ we have $p_s \leq p_t$;

(ii) for $n \in \mathbb{N}$ and $s, t \in 2^n$ we have

$$(p_{s0}, p_{t1}), (p_{t1}, p_{s0}) \in \mathcal{D}_n \cap \mathcal{D}_{n-1} \cap \ldots \cap \mathcal{D}_0;$$

(iii) for $\vec{0}^n \in 2^n$ consisting of n zeroes, and $u \in 2^{<\mathbb{N}}$ we have

$$\pi_n(p_{\vec{0}^n 0u}) = p_{\vec{0}^n 1u}.$$

We can clearly do all this; (i) and (ii) just follow by the usual abstract nonsense, while (iii) is a direct consequence of claim(7).

Then for $w \in 2^{\mathbb{N}}$ let G_w be the filter generated by $\{p_{w|n} : n \in \mathbb{N}\}$.

Claim(8) For $w_0, w_1 \in 2^{\mathbb{N}}$, if they are E_0-inequivalent, then (G_{w_0}, G_{w_1}) is $\mathbb{P}\times\mathbb{P}$-generic.

Proof of Claim: By (ii). (Claim$\Box$)

Thus $\varphi_{G_{w_0}} = \varphi_{G_{w_1}}$ implies $w_0 E_0 w_1$ by Claim(6).

Claim(9) For $w_0, w_1 \in 2^{\mathbb{N}}$, $w_0 E_0 w_1$ implies $\varphi_{G_{w_0}} = \varphi_{G_{w_1}}$.

Proof of Claim. Since by applying (ii) finitely many times we may find some $\pi \in L[\theta(\mathcal{N}), z]$ with

$$\pi : \mathbb{P} \cong \mathbb{P}$$

lifting to some $\hat{\pi} : \mathrm{Ult}(U_{G_{w_0}}, L[\theta(\mathcal{N}), z]) \cong (U_{G_{w_1}}, L[\theta(\mathcal{N}), z])$ as in Claim(7). Then since these models are isomorphic under an isomorphism sending $j_{G_{w_0}}(\varphi)$ to $j_{G_{w_1}}(\varphi)$ it follows from the definition of these Scott sentences that $\varphi_{G_{w_0}} = \varphi_{G_{w_1}}$. (Claim$\Box$)

By passing to a generic extension in which a large enough initial segment of $L[\theta(\mathcal{N}), z]$ is countable we may then easily obtain that in some generic extension $\mathbb{V}[G_2]$ we have

$$E_0 \leq_B \cong |_{\mathrm{Mod}(\sigma)}.$$

But then it is a general fact that whenever E_0 is Borel reducible to an equivalence relation it is continuously and injectively reducible as well, therefore

$$\mathbb{V}[G_2] \models E_0 \sqsubseteq_c \cong |_{\mathrm{Mod}(\sigma)}.$$

This last fact is easily seen to be absolute between transitive class models, and hence

$$\mathbb{V} \models E_0 \sqsubseteq_c \cong |_{\mathrm{Mod}(\sigma)},$$

and we are finished. □

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