Overview

Main Goal: Survey some of the recent work on Borel and $\underset{\sim}{\Sigma}^1_1$ equivalence relations We will specifically consider the $\underset{\sim}{\Sigma}^1_1$ equivalence relation which arises as the isomorphism relation on a class of countable structures.

Definition A topological space is Polish if it is separable[1] and it allows a complete metric.

Examples 1. $\mathbb{R}, \mathbb{C}$ in their usual topologies.

2. Any countable ordinal in the order topology, where the basic open sets are $\{\beta : \gamma_1 < \beta < \gamma_2\}$. (This requires some proof of course. One way to do this is to show by transfinite induction on $\alpha < \omega_1$ that for any $a < b \in \mathbb{R} \cup \{+\infty\}$ we can find a closed homeomorphic copy of α in $[a, b)$ if α is a limit ordinal or in $[a, b]$ if α is a successor.)

3. $\mathbb{N}^{\mathbb{N}} = \{f : \mathbb{N} \to \mathbb{N}\}$. Here we can take as our complete metric

$$d(f, g) = 2^{-\Delta(f,g)}$$

where $\Delta(f, g)$ is the least n at which $f(n) \neq g(n)$.

Actually for us some of the most important examples of Polish spaces will arise from model theory.

4. Let $\mathcal{L}$ be some countable language and let $\mathrm{Mod}(\mathcal{L})$ be the space of $\mathcal{L}$-structures on $\mathbb{N}$ with the topology whose basic open sets are all those of the form

$$\{\mathcal{M} | \mathcal{M} \models \varphi(n_1, n_2, .., n_k)\}$$

for $k, n_1, ..., n_k \in \mathbb{N}$ and φ a quantifier-free formula.

So as a simple case of this let us say that $\mathcal{L} = \{U, R\}$ where U is a unary function and R is a binary relation. Then we can associate to any $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ the pair

$$(U^{\mathcal{M}}, \chi_{(R^{\mathcal{M}})}) \in \mathbb{N}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}},$$

where $U^{\mathcal{M}}$ is $\mathcal{M}$'s interpretation of the unary function symbol and $\chi_{(R^{\mathcal{M}})}$ is the characteristic function of the binary relation from the point of view of $\mathcal{M}$.

We obtain as in example 3 that $\mathbb{N}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$ is Polish[2]. And our association

$$\mathcal{M} \mapsto (U^{\mathcal{M}}, \chi_{(R^{\mathcal{M}})})$$

is a homeomorphism.

5. Actually, as we will see later, the topology in which we take as basic open sets those of the form

$$\{\mathcal{M} | \mathcal{M} \models \varphi(n_1, n_2, .., n_k)\}$$

φ a *first order formula* gives rise to a Polish topology. Richer than the first, but obviously closely connected.

6. More generally, if $F \subset \mathcal{L}_{\omega_1,\omega}$ is any countable "fragment"[3] then the

[1] I.e. there is a countable dense subset

[2] For instance, $d((f_1, f_2), (g_1, g_2)) = 2^{-\Delta(f_1,g_1)} + 2^{-\Delta(f_2,g_2)}$.

[3] Again, the definition of $\mathcal{L}_{\omega_1,\omega}$ and "fragment" will be given in a couple of weeks.

topology generated by sets of the form

$$\{\mathcal{M} | \mathcal{M} \models \varphi(n_1, n_2, .., n_k)\}$$

for $\varphi \in F$ is Polish.

Definition A subset of a Polish space is *Borel* if it can be generated from the open sets by the operations of complementation, countable union, and countable intersection. A function

$$f : X \to Y$$

is *Borel* if $f^{-1}[B]$ is Borel for any Borel $B \subset Y$. (Equivalently we simply have required that the pullbacks of open sets along f are all Borel; the point is that the operations of complementation, intersection, and union behave nicely with respect to pullback.)

An equivalence relation E on a Polish space X is *Borel* if it is Borel as a subset of $X \times X$ in the product topology.

With respect to this last definition, it is not hard to see that the product of two Polish spaces is Polish; in particular $X \times X$ is Polish.

Here a theorem dating back to the early 1900's describes the possible structure of an arbitrary Borel set.

Theorem (Classical) If B is a Borel set then either

(I) $|B| \leq \aleph_0$ (B is countable), or

(II) $|B| = 2^{\aleph_0}$ (B has cardinality continuum).

Moreover any two Borel sets of the same cardinality are Borel isomorphic.

By this last part of the theorem I mean the following. If X_1, X_2 are Polish and $B_i \subset X_i$ $(i = 1, 2)$ are Borel with $|B_1| = |B_2|$ then there is a function

$$f : B_1 \to B_2$$

with

(a) f a bijection;

(b) $f^{-1}[C]$ Borel all $C \subset B_2$ Borel; and

(c) $f[C]$ Borel all $C \subset B_1$ Borel.

(Just in passing I should add that (c) above follows from the conjunction of (a) and (b) – but we probably won't need that particular fact.)

The situation for Borel equivalence relations initially appears the same.

Theorem (Silver) Let E be a Borel equivalence relation on a Polish space X. Then either

(I) $|X/E| \leq \aleph_0$ (E has only countably many equivalence classes), or

(II) $|X/E| = 2^{\aleph_0}$ (E has continuum many classes).

But there is no "moreover". The problem is with part (II), where there is an enormous range of possible types of equivalence relations with continuum many equivalence classes.

Definition Let E and F be Borel equivalence relations on Polish X and Y. We write

$$E \leq_B F,$$

and say that E is *Borel reducible* to F, if there is a Borel function

$$f : X \to Y$$

such that for all $x_1, x_2 \in X$

$$x_1 E x_2 \Leftrightarrow f(x_1) F f(x_2).$$

$E \sim_B F$ indicates that we have both

$$E \leq_B F$$

and

$$F \leq_B E.$$

Examples 1. For any Polish space X we have $\mathrm{id}(X)$ as the identity relation on X: $\{(x_1, x_2) \in X \times X : x_1 = x_2\}$.

2. E_0, eventual agreement on

$$2^{\mathbb{N}} = \{0, 1\}^{\mathbb{N}},$$

the space of infinite binary sequences. So that $f_1 E_0 f_2$ if there is some $N \in \mathbb{N}$ such that

$$\forall m > N(f_1(m) = f_2(m)).$$

3. E_v, the Vitali equivalence relation consisting of cosets of $\mathbb{Q}$ in $\mathbb{R}$. So $r_1 E_v r_2$ if

$$r_1 - r_2 \in \mathbb{Q}.$$

Actually it turns out that $E_0 \sim_B E_v$ and for any two uncountable Polish spaces X_1, X_2 one has $\mathrm{id}(X_1) \sim_B \mathrm{id}(X_2)$.

Possibility (II) from Silver's theorem represents the start of a very long story. The first major result was obtained at the end of the 1980's:

Theorem(Harrington-Kechris-Louveau) Let E be a Borel equivalence relation. Then exactly one of the following holds:

(I) $E \leq_B \mathrm{id}(2^{\mathbb{N}})$;
(II) $E_0 \leq_B E$.

It takes a non-trivial amount of descriptive set theory to prove this theorem. Instead I will probably try to work through the analogue for isomorphism relation on countable structures.

Just as a warning, for $\mathcal{L}$ a countable language we do not have that the isomorphism relation on countable $\mathcal{L}$-structures is Borel in any of the previously mentioned Polish topologies, except when $\mathcal{L}$ is composed solely of unary predicates. Thus Harrington-Kechris-Louveau will not generally apply.

However it turns out that an analogue can be proved.

Here for $\sigma \in \mathcal{L}_{\omega_1,\omega}$ I will let $\mathrm{Mod}(\sigma)$ be the subset of $\mathrm{Mod}(\mathcal{L})$ consisting of models of σ. This is a Borel subset of $\mathrm{Mod}(\mathcal{L})$ under any of our topologies, and a Polish space in its right in the topology generated by any countable fragment $F \subset \mathcal{L}_{\omega_1,\omega}$ which includes σ.

Theorem (Becker; Hjorth-Kechris) For $\mathcal{L}$ a countable language and $\sigma \in \mathcal{L}_{\omega_1,\omega}$ we have either

> (I) there is a $\underset{\sim}{\Delta}_1^{\mathrm{HC}}$ function[4] assigning elements of $2^{<\omega_1}$ as complete invariants, or
>
> (II) $E_0 \leq_B \cong|_{\mathrm{Mod}(\sigma)}$.

Beyond E_0 the picture becomes extremely complicated, and it is poorly understood. If time allows, we might be able to work in some discussion of this still largely uncharted territory. Another side topic will be the theory of continuous actions of Polish groups and some special cases of the *topological Vaught conjecture* ; this should fit naturally into the proof of "Becker; Hjorth-Kechris" above, since it too requires some discussion of the general actions of Polish groups.

[4]That is to say, some function which is $\Delta_1(x)$ over the hereditarily countable sets for some hereditarily countable x. The details of this definition are not so important; it is just that we can not use the old standby of Borel reduction for various technical reasons.

One might choose to think of the complexity class $\underset{\sim}{\Delta}_1^{\mathrm{HC}}$ as being the analogue to HC of recursive to HF (the hereditarily finite sets).

Borel sets

Definition A topological space is *Polish* if it is separable and there exists a complete metric which generates its topology.

Note then that a Polish space always has a countable basis for its topology.

Just to give some context, let me mention a classical characterization of Polish spaces. We will never use this result, so I mention it without proof.

Theorem (Hausdorff) A topological space is Polish if and only if it is homeomorphic to some subset of the Hilbert cube, $[0,1]^{\mathbb{N}}$, and it is the image of $\mathbb{N}^{\mathbb{N}}$ under some continuous, open function.

Definition The σ-algebra generated by the open sets forms the *Borel* subsets of a Polish space.

Example Let $\mathcal{L}$ be a countable language and $\mathrm{Mod}(\mathcal{L})$ the space of $\mathcal{L}$-structures on $\mathbb{N}$; if we equip this space with the topology generated by quantifier free formulas then it is naturally homeomorphic to

$$\prod_{\text{no of unary predicates}} \{0,1\}^{\mathbb{N}} \times \prod_{\text{no of binary relations}} \{0,1\}^{\mathbb{N}\times\mathbb{N}} \times$$

$$\prod_{\text{no of ternary relations}} \{0,1\}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}} \times \cdots$$

$$\prod_{\text{no of unary functions}} \mathbb{N}^{\mathbb{N}} \times \prod_{\text{no of binary functions}} \mathbb{N}^{\mathbb{N}\times\mathbb{N}} \cdots .$$

We will shortly prove that countable products of Polish spaces are Polish, and so this is as well.

Then one has for any first order formula φ and $\vec{n} \in \mathbb{N}^{<\mathbb{N}}$ that the set of

$$\{\mathcal{M} | \mathcal{M} \models \varphi(\vec{n})\}$$

is a Borel set. This is easily proved by induction on the logical complexity of φ, starting with the quantifier free formulas as the base case, and working up through the addition of existential and universal quantifiers[5] by observing that

$$\{\mathcal{M} : \mathcal{M} \models \exists x \varphi(\vec{n}, x)\} = \bigcup_{m\in\mathbb{N}} \{\mathcal{M} : \mathcal{M} \models \varphi(\vec{n}, m)\}$$

and

$$\{\mathcal{M} : \mathcal{M} \models \forall x \varphi(\vec{n}, x)\} = \bigcap_{m\in\mathbb{N}} \{\mathcal{M} : \mathcal{M} \models \varphi(\vec{n}, m)\}.$$

[5]Since every formula is provably equivalent to one with all the quantifiers on the outside and all the Boolean connectives on the inside, this is all we need to worry about.

Thus the topology of quantifier free formulas and the topology of first order logic give rise to the same Borel sets.

Definition A set X with a σ-algebra of sets $\mathcal{B}$ is said to be a *standard Borel space* if there is some Polish topology which gives rise to $\mathcal{B}$ as its Borel sets.

Now we can at least state the our temporary goals, just for this section.

1. Mod($\mathcal{L}$) in the topology of first order formulas is Polish.
2. A Borel subset of a standard Borel space is again standard Borel.
3. Any uncountable Borel set has size $2^{\aleph_0}$.

Actually in the proof of this last point we will follow the path chosen in Kechris' **Classical descriptive set theory,** reducing it to 2. and the perfect set theorem for Polish spaces.

The arguments here are all fairly elementary. One does little calculations with respect to metrics, and the typical step is to show that we can fiddle some coordinates to produce a complete metric with various desirable properties.

Ultimately we will use these little results as part of the proof the above mentioned dichotomy theorem for the isomorphism relation on classes of countable structures. The proof requires us to repeatedly modify a given space of structures, through transfinitely many steps, and at each stage it will be necessary to maintain the Polishness of the space.

The study of Polish space has emerged as a central theme in descriptive set theory. It is generally accepted that these are the spaces we can work with, applicable to a wide context but sufficiently specific to avoid certain pathologies.

Recently a few people – notably Becker and Kechris - have made some use of the more general "strong Choquet" spaces. But it is not clear that this is likely to ever unseat the current prejudice in favor of Polish.

Lemma Let X be a Polish space. Then it has as a compatible complete metric bounded by 1.

Proof Let d' be some complete metric. Let

$$d(x_1, x_2) = \frac{d'(x_1, x_2)}{1 + d'(x_1, x_2)}.$$

The transformation

$$r \mapsto \frac{r}{1+r}$$

provides an order preserving transformation of $[0, \infty)$ onto $[0, 1)$. Thus, jumping ahead to the end of the argument, once we establish that d *is* a metric then we have the same Cauchy sequences for the two metrics, and from this completeness follows for free. Similarly the two metrics have the same open balls, and hence give rise to the same topologies.

We are therefore only left with proving it to be a metric. And here the main issue is showing that the triangle inequality holds.

Let $x_1, x_2, x_3 \in X$. We want to show

$$d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3).$$

We may assume that

$$d'(x_1, x_3) \geq d'(x_1, x_2), d'(x_2, x_3);$$

since if for instance $d'(x_1, x_3) \leq d'(x_1, x_2)$ then

$$d(x_1, x_3) \leq d(x_1, x_2),$$

since our change in coordinates is order preserving, and we are done.

But now we have

$$d(x_1, x_2) + d(x_2, x_3) = \frac{d'(x_1, x_2)}{1 + d'(x_1, x_2)} + \frac{d'(x_2, x_3)}{1 + d'(x_2, x_3)}$$

$$\geq \frac{d'(x_1, x_2) + d'(x_2, x_3)}{1 + d'(x_1, x_3)},$$

by our assumption above, which in turn, by the triangle inequality for d', is at least as large as $\frac{d'(x_1,x_3)}{1+d'(x_1,x_3)}$. □

Corollary Let $(X_i)_i$ be a sequence of Polish spaces. Then

$$\prod_{i \in \mathbb{N}} X_i$$

is Polish.

Proof Let d_i be a complete metric on X_i, $i \in \mathbb{N}$. By the lemma we may assume the metric bounded by one. Then for

$$\vec{x} = (x_0, x_1, x_2, ...), \vec{y} = (y_0, y_1, y_2, ...) \in \prod_{i \in \mathbb{N}} X_i$$

we can let

$$d(\vec{x}, \vec{y}) = \sum_{i \in \mathbb{N}} 2^{-i} d_i(x_i, y_i).$$

□

Lemma An open subset of a Polish space is Polish in the subspace topology.

Proof Let X be a Polish space, d a complete metric, $O \subset X$ open.

Let us fix a continuous

$$\pi : O \to \mathbb{R}$$

which has the property

$$\pi(x) \to +\infty$$

as $x \to X \setminus O$. For instance we can take

$$\pi(x) = \frac{1}{\inf\{d(x,y) | y \notin O\}}.$$

Then let

$$d_O(x_1, x_2) = d(x_1, x_2) + |\pi(x_1) + \pi(x_2)|.$$

Since π is continuous we do not add to the topology of O by taking this metric. For instance, if $V \subset O$ is open, $x \in V$, then we can find some ϵ such that for all y with $d(x,y) < \epsilon$ we have $y \in V$; but then from the definition of d_O we have every y with $d_O(x,y) < \epsilon$ in V, which is just what we need to show V is open with respect to d_O. Conversely if $V \subset O$ is d_O-open, $x \in V$, then we can choose some ϵ with the open ball of radius ϵ with respect to d_O included in V; and then the continuity of π enables us to find some $\delta < \frac{\epsilon}{2}$ such that for all $y \in O$ with $d(x,y) < \delta$ we have

$$|\pi(x) - \pi(y)| < \frac{\epsilon}{2},$$

and thus the ball of radius δ with respect to d is included in V.

And the various facts required for d_O to be a metric are trivial to verify. The main issue is completeness.

But if $(x_i)_i$ is a Cauchy sequence in (O, d_O) then it will necessarily be d-Cauchy, and so it will have a limit $x_\infty \in X$. We need to check $x_\infty \in O$.

But otherwise we obtain

$$\pi(x_i) \to +\infty$$

and hence for all x_N on our sequence

$$|\pi(x_N) - \pi(x_i)| \to +\infty$$

as $i \to \infty$, contradicting our assumption on the sequence $(x_i)_i$. □

Definition A subset of a Polish space is G_δ if it is given by a countable intersection of open sets.

Some people use $\underset{\sim}{\Pi}^0_2$ instead of G_δ. And in fact this is a better notation, since it forms part of a general scheme to refer to the entire Borel hierarchy, while the

$$G_\delta, F_\sigma, G_{\delta\sigma}, F_{\sigma\delta}, \ldots$$

method only extends to the finite levels.

Note that every closed set is G_δ. If $C \subset X$ is closed, then it equals

$$\bigcap_{n \in \mathbb{N}} \{x : \exists y \in C(d(x,y) \le \frac{1}{n})\}.$$

Corollary Any G_δ subset of a Polish space is Polish in the subspace topology.

Proof Let X be Polish,

$$B = \bigcap_{i \in \mathbb{N}} O_i$$

a G_δ subset, where each O_i is open, and let d_i be a complete metric on O_i for $i \in \mathbb{N}$.

We may assume each d_i is bounded by 1, and then let

$$d_B(x, y) = \sum_{i \in \mathbb{N}} 2^{-i} d_i(x, y).$$

$\square$

Lemma For $\mathcal{L}$ a countable language, the space Mod($\mathcal{L}$) equipped with the topology of first order formulas is a Polish space.

Proof Let $\{c_n | n \in \mathbb{N}\}$ be fresh constants and let $\hat{\mathcal{L}} = \mathcal{L} \cup \{c_n | n \in \mathbb{N}\}$ be the language obtained by adding them to $\mathcal{L}$. Let S be the collection of sentences in $\hat{\mathcal{L}}$. Then

$$\{0, 1\}^S$$

in the product topology is naturally homeomorphic to $\{0, 1\}^{\mathbb{N}}$, and thus Polish.

Let B be the set of $f \in \{0, 1\}^S$ such that

(a) $\{\varphi | f(\varphi) = 1\}$ is consistent, and

(b) for all φ we have

$$f(\varphi) = 0 \Leftrightarrow f(\neg\varphi) = 1,$$

and

(c) for all φ we have

$$f(\exists x \varphi(x)) \Leftrightarrow \exists n \in \mathbb{N}(f(\varphi(c_n)) = 1).$$

Claim: B is a G_δ subset of $\{0, 1\}^S$.

Proof of Claim: Since countable intersections of G_δ sets are G_δ, it suffices to show that the conditions (a), (b), and (c) all correspond to G_δ sets. This is almost trivially true for (a), since a first order theory is inconsistent if and only if it includes a finite set of set of sentences which are inconsistent; thus if $\mathcal{I} \subset [S]^{<\mathbb{N}}$ is the set of finite inconsistent subset of S then (a) corresponds to the G_δ set

$$\bigcap_{F \in \mathcal{I}} \bigcup_{\psi \in F} \{f | f(\psi) = 0\}.$$

Similarly (b) is G_δ (and as with (a), it is actually closed).

Finally, (c) corresponds to the conjunction of the sets

$$\bigcap_{\varphi \in S} \bigcap_{n \in \mathbb{N}} \{f | f(\varphi(c_n)) = 1 \Rightarrow f(\exists x \varphi(x)) = 1\}$$

and

$$\bigcap_{\varphi \in S} (\{f | f(\exists x \varphi(x)) = 0\} \cup \bigcup_{n \in \mathbb{N}} \{f | f(\varphi(c_n)) = 1\}),$$

and is thus G_δ. (Claim$\square$)

To each $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ let $T_\mathcal{M} \in \{0,1\}^S$ be given by

$$T_\mathcal{M}(\varphi(c_{n_1}, c_{n_2}, ...)) = 1 \Leftrightarrow \mathcal{M} \models \varphi(n_1, n_2, ...).$$

It is more or less immediate from the definitions that each such $T_\mathcal{M}$ is in B.

Claim: The map

$$\mathrm{Mod}(\mathcal{L}) \to B$$

$$\mathcal{M} \mapsto T_\mathcal{M}$$

is a bijection of $\mathrm{Mod}(\mathcal{L})$ onto B.

Proof of Claim: The map is obviously one-to-one, since for any $\mathcal{M}_1 \neq \mathcal{M}_2$ there will be some atomic $\psi(\vec{x})$ and $n_1, n_2, ...$ on which they disagree, whence $T_{\mathcal{M}_1}$ and $T_{\mathcal{M}_2}$ disagree on $\psi(c_{n_1}, c_{n_2}, ...)$.

As for showing it onto, fix $T \in B$. We define $\mathcal{M}$ by the requirements that for any relation $R \in \mathcal{L}$

$$\mathcal{M} \models R(n_1, n_2, ...)$$

if and only if $T(R(c_{n_1}, c_{n_2}, ...)) = 1$, and for any function symbol $F \in \mathcal{L}$

$$F^\mathcal{M}(n_1, n_2, ...) = \ell$$

if and only if $T(F(c_{n_1}, c_{n_2}, ...) = c_\ell) = 1$. (This *is* well defined. By (a) and (b) we have $T(\exists x(F(c_{n_1}, c_{n_2}, ...) = x)) = 1$, and then by (c) we have some witness c_ℓ.) Then an easy induction on logical complexity shows that for any first order ψ

$$\mathcal{M} \models \psi(n_1, n_2, ...) \Leftrightarrow T(\psi(c_{n_1}, c_{n_2}, ...)) = 1.$$

(Claim$\square$)

But this basically finishes the lemma. It follows from the definitions of the topologies that

$$\mathcal{M} \mapsto T_\mathcal{M}$$

is a homeomorphism. Thus $\mathrm{Mod}(\mathcal{L})$ in the topology generated by first order formulas is homeomorphic to B; and we know B must be Polish since it is a G_δ subset of a Polish space. $\square$

Up to now I have done the usually sloppy, convenient, but not quite correct thing, and simply identified a topological space with its underlying set. Over the next couple of lemmas we need to be more precise. Thus let us agree to write (X, τ) for a Polish space with underlying set X and topology τ.

Lemma Let (X, τ) be a Polish space and $\mathcal{O} \subset X$ open. Then there is a new topology $\hat{\tau}$ such that:

(a) $\hat{\tau} \supset \tau$;
(b) every $\hat{\tau}$-open set is τ-Borel (and hence the two topologies give rise to the same standard Borel structure);
(c) $\mathcal{O}$ is $\hat{\tau}$-clopen.

Proof: Let d_X be a compatible complete metric on (X, τ). Since $\mathcal{O}$ is open we can find a complete metric $d_{\mathcal{O}}$ on $\mathcal{O}$ which is compatible with its subspace topology from τ. By earlier lemmas we can assume both to be bounded by 1.

Now we can define d on X by cases.
(i) If $x, y \in \mathcal{O}$ then $d(x, y) = d_{\mathcal{O}}(x, y)$.
(ii) If $x, y \notin \mathcal{O}$ then $d(x, y) = d_X(x, y)$.
(iii) If $x \in \mathcal{O}, y \notin \mathcal{O}$ then $d(x, y) = 2$.
(iii) If $x \notin \mathcal{O}, y \in \mathcal{O}$ then $d(x, y) = 2$.

It is as if we have split the space up into two halves, $\mathcal{O}$ and $X \setminus \mathcal{O}$, each a distance of 2 away from the other. Any d-Cauchy sequence must eventually decide to stay in one of these two halves, and hence will be either d_X or $d_{\mathcal{O}}$ Cauchy. □

Really this proof is using the fact that the disjoint union of two Polish spaces is again Polish.

The next lemma appears to be a folklore result. I am going to give a proof I saw presented by Ramez Sami.

Lemma Let (X, τ) be a Polish space and $B \subset X$ Borel. Then there is a richer topology $\hat{\tau} \supset \tau$ such that:
(a) $\hat{\tau} \supset \tau$;
(b) every $\hat{\tau}$-open set is τ-Borel;
(c) B is $\hat{\tau}$-open.

Proof: Let $\mathcal{C}$ be the collection of all $B \subset X$ for which we can find $\hat{\tau}$ satisfying the lemma. We have previously seen that this include all open sets. Then by the structure of the definition we have that it must be closed under complementation – that is to say, if $B \in \mathcal{C}$ then $(X \setminus B) \in \mathcal{C}$.

Therefore to show it forms a σ-algebra we need only show it is closed under countable intersections. For this the following observation suffices.

Claim: Suppose for each i we have a Polish topology $\tau_i \supset \tau$. Then the union of these topologies generates a Polish topology.

Proof of Claim: Consider the space

$$\prod_{i \in \mathbb{N}} (X, \tau_i).$$

This is a product of Polish spaces, and hence Polish. As is the closed subset

$$\{\vec{x} | \forall i, j \in \mathbb{N}(x_i = x_j)\}$$

consisting of all constant sequences.

If we associate to each $x \in X$ the sequence with constant value x then we obtain a homeomorphism of X under the topology generated by $\bigcup_{i\in\mathbb{N}} \tau_i$ and $\{\vec{x} | \forall i, j \in \mathbb{N}(x_i = x_j)\}$ with the subspace topology. (Claim$\square$)

$\square$

Definition A subset P of a Polish space X is *perfect* if it is

(a) non-empty,
(b) closed, and
(c) it has no isolated points – that is to say, if $U \subset X$ is open with $|U \cap P| \geq 1$ then we actually have $|U \cap P| \geq 2$.

Lemma If X is an uncountable Polish space, then it contains a perfect set.
Proof Let $\mathcal{B} = \{U_n | n \in \mathbb{N}\}$ be a countable basis for X. Then define $\mathcal{B}_0 \subset \mathcal{B}$ to be the set of $\{U_n | U_n \cap P \text{ is countable }\}$ and define P to be the elements of X not contained in any element of $\mathcal{B}_0$:

$$P = X \setminus (\bigcup \mathcal{B}_0).$$

Since $\bigcup \mathcal{B}_0$ is a union of open sets, we certainly have that P is closed.

If P is empty, then X is a countable union of countable sets and we have a contradiction.

So instead assume $P \neq \emptyset$, $x \in P$, and we try to show x not isolated.

For this purpose fix open U containing x. After possibly shrinking U we can assume $U \in \mathcal{B}$. But then if $|U \cap P| = 1$ we would have

$$|U \setminus (\bigcup \mathcal{B}_0)| = 1$$

and hence

$$|U| \leq |\bigcup \mathcal{B}_0| + 1 \leq \aleph_0 + 1 = \aleph_0,$$

which would place U into $\mathcal{B}_0$, contradicting $x \in P$. $\square$

It is implicit in this proof that any Polish space without isolated points must contain a perfect subset.

Lemma If P is a perfect subset of a Polish space, then $P = |2^{\aleph_0}|$, and in fact contains a homeomorphic copy of

$$\{0, 1\}^{\mathbb{N}}$$

as a closed subset.
Proof We first need an observation about the structure of open sets in our perfect set.

Claim: If U is open with $U \cap P \neq \emptyset$ and $\epsilon > 0$ then there are open V_0, V_1 with:

1. $V_0 \cap P, V_1 \cap P \neq \emptyset$;

2. $\overline{V_0}, \overline{V_1} \subset U$;[6]

3. $V_0 \cap V_1 = \emptyset$;

4. $d(V_0), d(V_1) < \epsilon$.[7]

Proof of Claim: Since P has no isolated points we may choose $x_0, x_1 \in U \cap P$ with $x_0 \neq x_1$. Then we may choose sufficiently small open neighborhoods $B_{2\delta}(x_0), B_{2\delta}(x_1)$ with:[8]

1. $B_{2\delta}(x_0), B_{2\delta}(x_1) \subset V$;

2. $B_{2\delta}(x_0) \cap B_{2\delta}(x_1) = \emptyset$;

3. $\delta < \epsilon$.

Then the sets $B_\delta(x_0) = V_0$ and $B_\delta(x_1) = V_1$ are as in the claim. (Claim $\Box$)

Claim: There is a collection of open sets indexed by *finite* binary sequences, $(U_s)_{s \in \{0,1\}^{<\mathbb{N}}}$, such that:

1. $U_s \cap P \neq \emptyset$ all $s \in \{0,1\}^{<\mathbb{N}}$;

2. $\overline{U_{s^\frown 0}}, \overline{U_{s^\frown 1}} \subset U_s$ all $s \in \{0,1\}^{<\mathbb{N}}$;[9]

3. $d(U_s) < \frac{1}{n}$ all $s \in \{0,1\}^n$;

4. $U_{s^\frown 0} \cap U_{s^\frown 1} = \emptyset$ all $s \in \{0,1\}^{<\mathbb{N}}$.

Proof of Claim: We build $(U_s)_{s \in \{0,1\}^{\leq n}}$ by induction on n. We can just begin with $U_\emptyset$ any sufficiently small open set meeting P.

Given the collection $(U_s)_{s \in \{0,1\}^n}$ we can choose for each s

$$U_{s^\frown 0}, U_{s^\frown 1} \subset U_s$$

as in the last claim for $\epsilon = \frac{1}{n+1}$. (Claim $\Box$)

In this way we have obtained a nested collection of open sets whose structure resembles the canonical basis of open sets in the space $\{0,1\}^{\mathbb{N}}$ of infinite binary sequences. For each $f : \mathbb{N} \to \{0,1\}$ and $n \in \mathbb{N}$ we let $f|_n$ be the binary sequence of length n obtained by restricting f to its first n many values.

Claim: For each $f : \mathbb{N} \to \{0,1\}$ there is $x_f \in P$ with

$$x_f \in \bigcap_{n \in \mathbb{N}} U_{f|_n}.$$

[6] $\overline{A}$ indicates the closure of a set A.

[7] Here and elsewhere I use $d(V)$ to denote the *diameter* of V, $\sup\{d(x,y) : x, y \in V\}$, where d is some complete metric on our Polish space.

[8] In general $B_r(x)$ denotes the open ball of radius r around x: $\{y : d(x,y) < r\}$.

[9] For $s : \{0, 1, ..., n-1\} \to \{0,1\}$ we use $s^\frown i$ to indicate the sequence of length $n+1$ obtained by concatenating s with the sequence $\langle i \rangle$; that is to say, $s^\frown i$ extends s, has length equal to length$(s) + 1$, and assumes i as its final value.

Proof of Claim: We may choose $z_n \in U_{f|_n} \cap P$ for each n by assumption on our nested collection of open sets. Then we obtain that $d(z_n, z_m) < \frac{1}{n}$ all $m \geq n$ since both points are in $U_{f|_n}$ and this open set has diameter less than $\frac{1}{n}$. Thus (z_n) must be a Cauchy sequence, which will necessarily converge to some point

$$z_\infty \in P \cap \bigcap_{n \in \mathbb{N}} \overline{U_{f|_n}}.$$

Since $\overline{U_{f|_{n+1}}} \subset U_{f|_n}$ we obtain $x_f = z_\infty$ is as required. (Claim $\square$)

For each such f there will be exactly one x_f as in the claim, for any other

$$x'_f \in \bigcap_{n \in \mathbb{N}} U_{f|_n}$$

would have to have $d(x_f, x'_f) < 1/n$ all n.

Thus $f \mapsto x_f$ provides some kind of map from the space of infinite binary sequences to P. It is obviously one to one, since if $f \neq f'$ then we may let $s = f \cap f'$ be the longest initial segment contained in both, and obtain that one of $x_f, x_{f'}$ is in $U_{s^\frown 0}$ and the other is in $U_{s^\frown 1}$. The main issue is continuity.

Claim: $f \mapsto x_f$ is continuous.

Proof of Claim: Let V be an open neighborhood of x_f. (We need an open W containing f with $\{x_h : h \in W\} \subset V$.) Choose n with $B_{1/n}(x_f) \subset V$. Then since any two points y_0, y_1 of $U_{f|_n}$ have $d(y_0, y_1) < 1/n$ we obtain $U_{f|_n} \subset V$, and thus for W the open set $\{h : \mathbb{N} \to \{0,1\} | h \supset f|_n\}$ we have $x_h \in V$ all $h \in W$. (Claim $\square$)

Thus

$$\{0,1\}^{\mathbb{N}} \to P$$

$$f \mapsto x_f$$

is a continuous function. Since $\{0,1\}^{\mathbb{N}}$ is compact the map sends closed sets to closed sets. Since it is injective we then have that it sends open sets to sets which are relatively open in its image, and thus it is a homeomorphism onto a closed subset of P.

P is at least as big as the number of subset of $\mathbb{N}$; thus $|P| \geq 2^{\aleph_0}$. But conversely, P is a Polish space in its own right, and so it has a countable dense subset, $\mathcal{D}$ say; and then P is no bigger than the number of Cauchy sequences from $\mathcal{D}$, which gives us the bound

$$|P| \leq |\mathcal{D}|^{\mathbb{N}} \leq |\mathcal{P}(\mathcal{D} \times \mathbb{N})| = |2^{\mathcal{D} \times \mathbb{N}}| = 2^{\aleph_0}.$$

$\square$

Corollary Any uncountable Borel set has size $2^{\aleph_0}$.

Proof Since we have previously seen that any Borel set allows a Polish topology and any uncountable Polish space contains a perfect subset and hence has cardinality continuum. $\square$

Corollary Any Polish space without isolated points has size $2^{\aleph_0}$.
Proof As remarked above, a Polish space without isolated points must contain a perfect set. $\square$

Thus in particular $\mathbb{Q}$ in the subspace topology it inherits from $\mathbb{R}$ is not a Polish space.

We are yet to see that any two Borel sets of the same cardinality are Borel isomorphic. In light of our lemmas on changing topologies we need only see that any two Polish spaces of the same cardinality are isomorphic as standard Borel spaces. The proof of this is sketched in the exercises below.

Exercises

1. Show that $\mathbb{N}^{\mathbb{N}}$ is homeomorphic to a G_δ subset of $\{0,1\}^{\mathbb{N}}$. (Namely, the set of binary sequences with infinitely many 1's).

2.(a) Show that if X is Polish then we may find a nested collection of sets $(V_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ such that

1. each V_s is non-empty and open,
2. $\overline{V_s} \subset V_t$ whenever s strictly extends t,
3. $V_\emptyset = X$,
4. $d(V_s) < \frac{1}{n}$ whenever $s : \{0,1,...,n-1\} \to \mathbb{N}$, $n > 0$,
5. each V_s equals $\bigcup_{k\in\mathbb{N}} V_{s^\frown k}$.

(b) Conclude that any Polish space is the continuous, open image of $\mathbb{N}^{\mathbb{N}}$.
(c) Show that if X is Polish then there is a Borel $B \subset \mathbb{N}^{\mathbb{N}}$ which is Borel isomorphic to X.

3. Let X_1, X_2 be Polish spaces, $B_1 \subset X_1, B_2 \subset X_2$ Borel sets, $\pi_1 : X_1 \to B_2, \pi_2 : X_2 \to B_1$ Borel isomorphisms.
Show that X_1 and X_2 are Borel isomorphic. (Hint: This is close to the proof of Schroeder-Bernstein which does *not* use choice. You can find *that* proof in Tom Jech's **Set Theory**.)

4. Conclude from 1-3 above and the earlier results about perfect sets that any two uncountable Borel sets are Borel isomorphic.

5. Show that if X is a countable Polish space then any subset is Borel. Thus any function between countable Polish spaces is Borel.

Borel equivalence relation

Definition An equivalence relation E on a Polish space X is *Borel* if it is Borel as a subset of $X \times X$ in the product topological structure. A function

$$\theta : X \to Y$$

between Polish spaces is *Borel* if $\theta^{-1}[U]$ is Borel for any open set $U \subset Y$. (Actually this is equivalent to the graph of θ being Borel as a subset of $X \times Y$, but we are unlikely to need that particular result.)

For E, F equivalence relations on X, Y we write

$$E \leq_B F,$$

E Borel reducible to F, if there is a Borel $\theta : X \to Y$ such that for all $x_1, x_2 \in X$

$$x_1 E x_2 \Leftrightarrow \theta(x_1) F \theta(x_2).$$

We can then go ahead and set $E <_B F$ if there is a Borel reduction of E to F but not the other way around, and $E \sim_B F$ if there is a Borel reduction in both directions.

Of course there are other notions of comparison we might use for Borel equivalence relations. We might ask that the reduction be one-to-one. Or that it provide a bijection with some invariant Borel subset of the target space. And so on.

Although certainly not the only notion worth investigating, $\leq_B$ is the main one studied by descriptive set theorists. It, and its variant, which allow classes of functions more general than Borel, also provides the most generous notion of reduction, and thus the non-reducibility below have the most force if we phrase them for $\leq_B$.

Notation For X a Polish space we let $\mathrm{id}(X)$ be the identity relation on X. (Some people use $\Delta(X)$ for this relation, which is suggested by it being the diagonal in $X \times X$.)

E_0 is the equivalence relation of agreement mod finite on $\{0,1\}^{\mathbb{N}}$: $f_1 E_0 f_2$ if

$$\exists N \forall m > N (f_1(m) = f_2(m)).$$

¿From now on let me just write $2^{\mathbb{N}}$ for the space $\{0,1\}^{\mathbb{N}}$. It is a little less cumbersome and formal.

Lemma $\mathrm{id}(2^{\mathbb{N}}) \leq_B E_0$.
Proof Let $(s_n)_{n \in \mathbb{N}}$ enumerate the set $\{0,1\}^{<\mathbb{N}}$ of all finite binary sequences. For each $f \in 2^{\mathbb{N}}$ we define $\theta(f)$ by

$$(\theta(f))(n) = 1 \Leftrightarrow s_n \subset f.$$

That is to say, $(\theta(f))(n) = 1$ if s_n is a substring of f.

This is continuous, since if $s_n : \{0, 1, ..., k-1\} \to \{0, 1\}$ and $(\theta(f_0))(n) = i$ ($i \in \{0, 1\}$), then for any other f in the open set $\{h : \forall j < k(h(j) = f_0(j))\}$ we have $(\theta(f))(n) = i$.

And obviously if $f_1 \operatorname{id}(2^{\mathbb{N}}) f_2$ then by definition $f_1 = f_2$ and so $\theta(f_1) = \theta(f_2)$ and they are certainly E_0 equivalent.

Conversely if $f_1(k) \neq f_2(k)$ then for all s_n with $s_n \subset f_1$ of length greater than k we have $(\theta(f_1))(n) = 1$ while $(\theta(f_2))(n) = 0$. $\square$

To show failure of reducibility in the other direction I will use Baire category techniques. The idea here is to approximate Borel functions on some suitably "large" set by continuous functions.

Definition A subset of a space X is *nowhere dense* if its intersection with any non-empty open subset of X is not dense. A subset is *meager* if it is included in the countable union of sets which are each closed and nowhere dense. The complement of a meager set is said to be *comeager*.

Fact The countable unions of meager sets are meager; countable intersections of comeager sets are comeager. The meager sets form a σ-ideal; the comeager sets form a σ-filter.

Some books only state the Baire category theorem for a few specific spaces like the unit interval. The usual proof extends to Polish spaces.

Baire category theorem Any comeager subset of a Polish space is non-empty.

Corollary If X is a Polish space without isolated points then any comeager set is uncountable.

Definition A set A has *the Baire property* if for some open $\mathcal{O}$ the symmetric difference

$$A \Delta \mathcal{O} = (A \setminus \mathcal{O}) \cup (\mathcal{O} \setminus A)$$

is meager.

Lemma Every Borel set has the Baire property.
Proof Clearly the open sets have Baire property.

If C is closed then for

$$C^o = \bigcup \{U : U \subset C, U \text{ open}\}$$

we have that C^o is the largest open set included in C and $C \Delta C^o = C \setminus C^o$ is nowhere dense. Thus in general the complement of a set with Baire property has Baire property, since if $B \Delta \mathcal{O}$ is included in a meager set M, and C is the complement of the open set $\mathcal{O}$, then

$$(X \setminus B) \Delta C^o \subset M \cup (C \setminus C^o).$$

Finally if $(B_i)_i$ is a sequence of Borel sets, $(\mathcal{O}_i)_i$ a sequence of open sets with $B_i \Delta \mathcal{O}_i$ always meager, then

$$(\bigcup_{i\in\mathbb{N}} B_i)\Delta(\bigcup_{i\in\mathbb{N}} \mathcal{O}_i)$$

is included in the countable union of meager sets, and is hence meager in its own right. □

Corollary Any Borel function between Polish spaces is continuous on a comeager set.

That is to say, if $\theta : X \to Y$ is Borel, then there is a comeager $C \subset X$ having $\theta|_C$ continuous.

Proof Let $\{U_n | n \in \mathbb{N}\}$ be a countable basis of Y. For each n choose open $\mathcal{O}_n \subset X$ with

$$(\theta^{-1}[U_n])\Delta\mathcal{O}_n \subset M_n$$

for some meager set M_n. Let

$$C = X \setminus (\bigcup_{n\in\mathbb{N}} M_n).$$

Then we have that $\theta|_C$ pulls basic open sets back to relatively open subsets of C, and thus is continuous. □

Lemma E_0 is not Borel reducible to $\mathrm{id}(2^{\mathbb{N}})$.

Proof Instead let $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be a Borel reduction. Let C be a comeager set on which θ is continuous.

For each finite binary sequence $s \in \{0,1\}^{<\mathbb{N}}$ define

$$\psi_s : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$$

by

$$(\psi_s(f))(n) = s(n) + f(n) \mod 2$$

if n is less than the length of s, and equal to $f(n)$ otherwise.[10] Each ψ_s is a homeomorphism, and thus $\psi_s(C)$ is again comeager.

Thus we can let

$$\hat{C} = \bigcup_{s\in 2^{<\mathbb{N}}} \psi_s[C]$$

and obtain a comeager set which is invariant under E_0, in the sense that it includes any equivalence class it meets, and on which θ is continuous. Now choose $f \in \hat{C}$. The E_0-equivalence class

$$[f]_{E_0}$$

[10] A way to think of this. Identify $\{0,1\}^{\mathbb{N}}$ with the product group $\mathbb{Z}_2^{\mathbb{N}}$ and consider the subgroup G consisting of elements which have finite support. To each finite binary sequence s we can associate a corresponding $g_s \in G$ and let ψ_s be translation by g_s.

of f is dense in $2^{\mathbb{N}}$ and included in $\hat{C}$; hence it is dense in $\hat{C}$. θ is constant on $[f]_{E_0}$ by the assumption it perform a reduction.

$\theta|_{\hat{C}}$ is continuous and constant on a dense subset, and therefore it is constant on $\hat{C}$. But this really does give us a contradiction, since $\hat{C}$ must be uncountable, and we can choose some

$$h \in \hat{C} \setminus [f]_{E_0}$$

with $\theta(h) = \theta(f)$. □

Exercises

1. Show that for any two uncountable Polish spaces X and Y we have $\mathrm{id}(X) \leq_B \mathrm{id}(Y)$.

2. Let $\mathcal{L}$ be a language consisting of just countably many unary predicates and let F_2 be the isomorphism relation on $\mathrm{Mod}(\mathcal{L})$. Show that $E_0 \leq_B F_2$.

Hence that $\mathrm{id}(\mathbb{R}) <_B F_2$.

The non-reducibility of F_2 to the identity relation on a Polish space even works for notions of reducibility which allow broader classes of functions. Consistently with ZFC F_2 is non-reducible to $\mathrm{id}(\mathbb{R})$ using "set recursive" or even "$\mathbb{R}$-ordinal definable" functions.[11] So although this isomorphism relation for countably many unary predicates might seem to be almost trivial from the point of view of model theory, there is already a substantial restriction on the kinds of objects one can hope to assign as complete invariants.

If you want a considerably harder exercise you might try showing that $E_0 <_B F_2$. And at the end of the course perhaps we will have time to discuss the general result that F_2 is not Borel reducible to any equivalence relation all of whose equivalence classes are countable.

[11] I only say "consistently with ZFC" since we need to avoid, for instance, certain things which can happen in Gödel's L as a result of the simply definable well order of $\mathbb{R}$. Certainly with a little bit of forcing one can guarantee all $\utilde{\Delta}^1_2$ sets have Baire property and with quite a lot of forcing one obtains the same for the $\mathbb{R}$-ordinal definable sets.

Infinitary logic

Definition Let $\mathcal{L}$ be a language. $\mathcal{L}_{\omega_1,\omega}$ is obtained by closing the atomic formulas of $\mathcal{L}$ under the usual first order operations

(a) $\psi \mapsto \exists x\psi$, $\psi \mapsto \forall x\psi$,

(b) $(\psi, \phi) \mapsto \psi \vee \phi$, $(\psi, \phi) \mapsto \psi \wedge \phi$,

(c) $\psi \mapsto \neg\psi$,

as well as infinitary conjunction and disjunction,

(d) whenever $\{\psi_i | i \in \mathbb{N}\}$ is a countable set of formulas in $\mathcal{L}_{\omega_1,\omega}$ we have

$$\bigvee_{i\in\mathbb{N}} \psi_i \in \mathcal{L}_{\omega_1,\omega},$$

$$\bigwedge_{i\in\mathbb{N}} \psi_i \in \mathcal{L}_{\omega_1,\omega}.$$

Definition If $\mathcal{M}$ is an $\mathcal{L}$-structure, and $\varphi(x_1, x_2, ..., x_k)$ is a formula of $\mathcal{L}_{\omega_1,\omega}$ with just finitely many free variables, then we can define

$$\mathcal{M} \models \varphi(a_1, ..., a_k)$$

by induction on the complexity of φ in the usual way. The only clauses which differ from first order logic and the ones corresponding to infinite conjunction and disjunction, and here we stipulate that

$$\mathcal{M} \models \bigwedge_{i\in\mathbb{N}} \psi_i$$

$$\Leftrightarrow (\forall i \in \mathbb{N}(\mathcal{M} \models \psi_i)),$$

and

$$\mathcal{M} \models \bigvee_{i\in\mathbb{N}} \psi_i$$

$$\Leftrightarrow (\exists i \in \mathbb{N}(\mathcal{M} \models \psi_i)).$$

Technical remarks

(i) The infinite conjunction $\bigvee_{i\in\mathbb{N}} \psi_i \in \mathcal{L}_{\omega_1,\omega}$ and infinite disjunction $\bigwedge_{i\in\mathbb{N}} \psi_i \in \mathcal{L}_{\omega_1,\omega}$ depend solely on the *set* $\{\psi_i | i \in \mathbb{N}\}$, and not the particular enumeration $(\psi_i)_{i\in\mathbb{N}}$.

(ii) I have allowed formulas with infinitely many free variables, but not given any role to them in the satisfaction relation. There seems to be some variation in the literature on this point.

(iii) Later we will need to perform a range of computations with languages and formulas, and for these purposes it will be helpful to adopt some conventions. Firstly that we assume $\mathcal{L}$ consists solely of symbols which are themselves hereditarily countable[12] sets; all our languages will be countable, so this is not overly restrictive. After that we can assume that $\mathcal{L}_{\omega_1,\omega}$ is a subset of the hereditarily countable sets; for instance we can have as our convention that $\bigvee$ is some fixed hereditarily countable set, such as the number 17, and that

$$\bigvee_{i\in\mathbb{N}} \psi_i = \{\{\psi_i, \bigvee\} : i \in \mathbb{N}\}.$$

(iv) I will only use countably many variable symbols. Keisler handles this point somewhat differently.

Definition $F \subset \mathcal{L}_{\omega_1,\omega}$ is a *fragment* if:

(a) it contains the atomic formulas;

(b) it is closed under the first order operations

$$\psi \mapsto \exists x\psi,$$

$$\psi \mapsto \forall x\psi,$$

$$(\psi, \phi) \mapsto \psi \vee \phi,$$

$$(\psi, \phi) \mapsto \psi \wedge \phi,$$

$$\psi \mapsto \neg\psi;$$

(c) it is closed under subformulas, in the sense that

$$\exists x\psi \in F \Rightarrow \psi \in F,$$

$$\forall x\psi \in F \Rightarrow \psi \in F,$$

$$\psi \wedge \phi \in F \Rightarrow \psi, \phi \in F,$$

$$\psi \vee \phi \in F \Rightarrow \psi, \phi \in F,$$

$$\neg\psi \in F \Rightarrow \psi \in F,$$

$$\bigwedge_{i\in\mathbb{N}} \psi \in F \Rightarrow \{\psi_i : i \in \mathbb{N}\} \subset F,$$

$$\bigvee_{i\in\mathbb{N}} \psi \in F \Rightarrow \{\psi_i : i \in \mathbb{N}\} \subset F;$$

(d) if $\varphi(x_1, x_2, x_3 ...) \in F$ and t is a term, then

$$\varphi(t, x_2, x_3, ...) \in F.$$

In this definition of fragment (d) is important for what it *does not* do. It does not license substitutions which replace infinitely many variables with terms. Otherwise all our fragments would be uncountable.

[12] A set A is *hereditarily countable* if it is countable, all its elements are countable, all their elements are countable, and so on. More precisely, HC, the *collection of hereditarily countable sets*, is the smallest set containing all its countable subsets.

In Ramez Sami's *Polish group actions and the topological Vaught conjecture* he uses a somewhat weaker notion of fragment. As it turns out this weaker notion is still sufficient to prove that countable fragments generate Polish topologies, but not quite sufficient for us to obtain a nice characterization of *atomic model relative to a given fragment.*

Definition For $F \subset \mathcal{L}_{\omega_1,\omega}$ a fragment we let τ_F be the topology on $\mathrm{Mod}(\mathcal{L})$ whose basis consists of open sets of the form

$$\{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}) : \mathcal{M} \models \varphi(n_1, n_2, ..., n_k)\},$$

where $n_1, n_2, ..., n_k \in \mathbb{N}$ and $\varphi(\vec{x}) \in F$.

Lemma If F is a countable fragment then $(\mathrm{Mod}(\mathcal{L}), \tau_F)$ is a Polish space.
Proof This follows our proof for the topology generated by first order logic. We let $\hat{\mathcal{L}}$ be the language obtained by adding fresh constants $(c_n)_{n\in\mathbb{N}}$ to $\mathcal{L}$; and then we obtain $\hat{F}$ as the fragment generated by F in $\hat{\mathcal{L}}_{\omega_1,\omega}$. And parallel to the proof from before we let $\hat{S}$ be the set of sentences in $\hat{F}$.

$$2^{\hat{S}} =_{\mathrm{df}} \{0,1\}^{\hat{S}}$$

gives us a Polish space, and we can let $A_{\hat{S}}$ be the set of $f \in 2^{\hat{S}}$ satisfying:
(a) any finite subset of $\{\psi : f(\psi) = 1\}$ is *consistent*, in the sense that it has some model;
(b) $f(\psi) = 1$ if and only if $f(\neg\psi) = 0$;
(c) $f(\exists x \psi(x)) = 1$ if and only if there is some n with $f(\psi(c_n)) = 1$;
(d) $f(\bigvee_{i\in\mathbb{N}} \psi_i) = 1$ if and only if there is some $i \in \mathbb{N}$ with $f(\psi_i) = 1$.

We had to modify (a) a bit, so that it still corresponds to a closed condition. We also had to add a condition (d), which is G_δ by the same calculation we previously used for (c). With these modifications one can show that if we assign to each $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ the characteristic function of F-theory of its canonical expansion to a model of $\hat{\mathcal{L}}$, $T_{\mathcal{M}}$ where

$$T_{\mathcal{M}}(\psi(c_{n_1}, c_{n_2}, .., c_{n_k})) = 1 \Leftrightarrow \mathcal{M} \models \psi(n_1, n_2, ..., n_k),$$

then

$$(\mathrm{Mod}(\mathcal{L}), \tau_F) \to A_{\hat{F}}$$
$$\mathcal{M} \mapsto T_{\mathcal{M}}$$

provides a homeomorphism. $\square$

Definition If $\sigma \in \mathcal{L}_{\omega_1,\omega}$ is a sentence then we let $\mathrm{Mod}(\sigma)$ be the collection of $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ satisfying σ; we let $F(\sigma)$ be the fragment generated by σ, and obtain from the last lemma that $\mathrm{Mod}(\sigma)$ is a Polish space in the subspace topology bequeathed by $\tau_{F(\sigma)}$.

If F is a countable fragment and $T \subset F$ a theory, then we let $\mathrm{Mod}(T)$ be the models of T. This is a closed subset of $(\mathrm{Mod}(\mathcal{L}), \tau_F)$, and hence a Polish space in the subspace topology.

Definition Given a countable fragment F we can define an *n-type over F* to be some subset of the F-formulas in free variables $x_1, x_2, ..., x_n$, and more generally we can define an *F-type* to be an n-type over F for some for n. A type is *F-complete* if any $\psi(\vec{x}) \in F$ in the appropriate variables is either in the type or has its negation in F. We then say that a complete type is *principal* over a complete theory $T \subset F$ if it contains some $\varphi(\vec{x})$ such that for all other and for all $\psi(\vec{x}) \in F$ we have either

$$\forall \vec{x}(\varphi(\vec{x}) \rightarrow \psi(\vec{x}))$$

or

$$\forall \vec{x}(\varphi(\vec{x}) \rightarrow \neg\psi(\vec{x})).$$

Given a model $\mathcal{M}$ and $\vec{a} \in \mathcal{M}$ we let the *F-type over* $\mathcal{M}$ be the set of $\psi(\vec{x}) : \mathcal{M} \models \psi(\vec{a})$. $\mathcal{M}$ *omits* an F-type if there is no $\vec{a}$ which has that type as its F-type over $\mathcal{M}$. For F a fragment $\mathrm{Th}_F(\mathcal{M})$ is the set of sentences in F satisfied by $\mathcal{M}$. We say that a model $\mathcal{M}$ is *F-atomic* if it realizes no non-principal types over $\mathrm{Th}_F(\mathcal{M})$.

These definitions out of the way, all the usual facts for first order logic generalize easily.

Lemma Let F be a countable fragment, $T \subset F$ a complete theory. Any two countable F-atomic models of T are isomorphic.

Lemma Let F be a countable fragment. Let $\Sigma(x_1, x_2, ..., x_n)$ be a complete non-principal type over F and let $k_1, k_2, ..., k_n \in \mathbb{N}$. Let $T \subset F$ be a complete theory. Then the set of $\mathcal{M} \in \mathrm{Mod}(T)$ with

$$\mathcal{M} \models \Sigma(k_1, k_2, ..., k_n)$$

is closed nowhere dense (with respect to τ_F).

Corollary (Omitting types theorem) F, T, $\Sigma(\vec{x})$ as above. The set of $\mathcal{M}$ in $(\mathrm{Mod}(T), F)$ omitting $\Sigma(\vec{x})$ is comeager (in $(\mathrm{Mod}(T), F)$).

Lemma Let F be a countable fragment, $T \subset F$ a complete theory, $\mathcal{M}_0 \in \mathrm{Mod}(T)$. The set of $\mathcal{N} \in (\mathrm{Mod}(T), F)$ isomorphic to $\mathcal{M}_0$ is comeager if and only if $\mathcal{M}_0$ is atomic.
Proof First suppose that $\mathcal{M}_0$ is atomic. Then:

Claim: For each $\vec{n} \in \mathbb{N}^{<\mathbb{N}}$ the set of $\mathcal{N} \in (\mathrm{Mod}(T), F)$ with $\vec{n}$ realizing a principal type is open dense.

Proof of Claim: The set is immediately open by the definition of the topology. To see density, let us consider some non-empty basic open set of the form

$$V = \{\mathcal{N} : \mathcal{N} \models \psi(\vec{n}, \vec{m})\}$$

where $\vec{m} \in \mathbb{N}^{<\mathbb{N}}$ has been chosen to be disjoint from $\vec{n}$ and $\psi(\vec{x}, \vec{y}) \in F$. Then we can consider the formula

$$\exists \vec{y} \psi(\vec{x}, \vec{y}) \wedge \bigwedge_{i<lh(\vec{x})} \bigwedge_{j<lh(\vec{y})} x_i \neq y_j.$$

Since this is realized by some tuple in some model in our space, and since $\mathcal{M}_0$ satisfies the complete theory T, we may find some $\vec{a} \in \mathcal{M}_0$ with

$$\mathcal{M}_0 \models \exists \vec{y} \psi(\vec{a}, \vec{y}) \wedge \bigwedge_{i<lh(\vec{x})} \bigwedge_{j<lh(\vec{y})} a_i \neq y_j.$$

We then find some $\varphi(\vec{x})$ with

$$\mathcal{M}_0 \models \varphi(\vec{a})$$

witnessing that $\vec{a}$ realizes some principal type. But then the set

$$U = \{\mathcal{N} : \mathcal{N} \models \psi(\vec{n}, \vec{m}) \wedge \varphi(\vec{n}) \wedge \bigwedge_{i<lh(\vec{x})} \bigwedge_{j<lh(\vec{y})} n_i \neq m_j\}$$

is an open subset of V all of whose elements have $\vec{n}$ realizing a principal type; to see that this set is non-empty we choose some $\vec{b} \in \mathcal{M}_0$ disjoint from $\vec{a}$ with $\mathcal{M}_0 \models \varphi(\vec{a}, \vec{b})$ and some isomorphic copy of $\mathcal{M}_0$ where $\vec{n}$ goes to $\vec{a}$ and $\vec{m}$ goes to $\vec{b}$. (Claim□)

Thus we have that the set of atomic models is comeager, and since any two comeager models are isomorphic we are done.

Conversely, if $\mathcal{M}_0$ is *not* atomic, then we can consider some principal complete type $\Sigma(\vec{x})$ realized by $\mathcal{M}_0$. By the omitting types theorem the collection of models omitting $\Sigma(\vec{x})$ is comeager. By the Baire category theorem we have that this set intersects any comeager set. □

Lemma Let F be a countable fragment, $T \subset F$ a complete theory, $\mathcal{M}_0 \in \mathrm{Mod}(T)$. If there is no F-atomic model of T then the isomorphism relation is meager as a subset of

$$(\mathrm{Mod}(T), \tau_F) \times (\mathrm{Mod}(T), \tau_F).$$

Proof It is easily seen that the non-existence of an atomic model is equivalent to the principal types not being dense in the space of n-types for some n. So let us choose some $\psi(\vec{x})$ which extends to no principal type. Then we have that for any $\vec{n}, \vec{m}$ the collection $U_{\vec{m}\vec{n}}$ of $(\mathcal{M}, \mathcal{N})$ with

$$\mathcal{M} \models \psi(\vec{m}),$$

$$\mathcal{N} \models \psi(\vec{n}),$$

is relatively clopen in the product space; but the subset $C_{\vec{m},\vec{n}}$ consisting of $(\mathcal{M}, \mathcal{N})$ over which $\vec{m}$ and $\vec{n}$ respectively realize the same type is a closed nowhere dense subset of $U_{\vec{m},\vec{n}}$.

But then we have for any $(\mathcal{M},\mathcal{N})$ in the comeager set

$$\bigcap_{\vec{m},\vec{n}}(X \setminus C_{\vec{m},\vec{n}})$$

there must, by completeness of T, be *some* $\vec{k},\vec{\ell}$ with

$$\mathcal{M} \models \psi(\vec{k}),$$

$$\mathcal{N} \models \psi(\vec{\ell}).$$

□

The above lemma is the main fact we will need about infinitary logic. There is one other result worth mentioning just for comparison. This further result requires another definition.

Definition S_∞ is the group of all permutations of the natural numbers with topology of pointwise convergence. Thus a basic open set consists of all bijections $\pi : \mathbb{N} \to \mathbb{N}$ with

$$\forall i < k(\pi(i) = n_i),$$

for some $k, n_1, n_2, ..., n_k \in \mathbb{N}$.

S_∞ is a topological group, since the group operations are all continuous; it is also G_δ as a subset of $\mathbb{N}^\mathbb{N}$, and hence Polish as a space.

Note that S_∞ acts continuously and in a canonical fashion on any $(\mathrm{Mod}(\mathcal{L}), \mathcal{L})$; and moreover its orbit equivalence relation is the isomorphism relation.

Theorem (Becker-Kechris) Let X be a standard Borel space on which S_∞ acts in a Borel manner[13]. Then there is some countable $\mathcal{L}$ and sentence $\sigma \in \mathcal{L}_{\omega_1,\omega}$ and Borel isomorphism

$$\theta : X \to \mathrm{Mod}(\sigma)$$

intertwining the actions – that is to say, for all $\pi \in S_\infty$ and $x \in X$

$$\pi \cdot \theta(x) = \theta(\pi \cdot x).$$

Exercises

1. Show that if T is a complete theory for the countable fragment F then every isomorphism type in $\mathrm{Mod}(T)$ is dense. That is to say, if $\mathcal{M} \in \mathrm{Mod}(T)$, then the set of $\mathcal{N} \in \mathrm{Mod}(T)$ with

$$\mathcal{M} \cong \mathcal{N}$$

[13] I.e. the map

$$(\pi, x) \mapsto \pi \cdot x,$$
$$S_\infty \times X \to X$$

is Borel

is dense relative to τ_F.

2. T, F as above. Show that if the isomorphism type of $\mathcal{M}$ is G_δ, then it is comeager; and moreover if $\mathcal{M}$ is atomic then its isomorphism type is G_δ.

3. T, F as above. Show that canonical action of S_∞ on $(\mathrm{Mod}(T), \tau_F)$ is continuous.

4. T, F as above. For $\mathcal{M} \in (\mathrm{Mod}(T), \tau_F)$ we have that the induced map

$$S_\infty \to \{\mathcal{N} \in \mathrm{Mod}(T) : \mathcal{N} \cong \mathcal{M}\}$$

$$\pi \mapsto \pi \cdot \mathcal{M}$$

is open if and only if $\mathcal{M}$ is F-atomic.

5. Conclude that for T, F as above the following are equivalent for $\mathcal{M} \in (\mathrm{Mod}(T), \tau_F)$:
(a) $\mathcal{M}$ is atomic;
(b) the isomorphism type of $\mathcal{M}$ is comeager;
(c) the isomorphism type of $\mathcal{M}$ is G_δ;
(d) the map $\pi \mapsto \pi \cdot \mathcal{M}$ is open as a map to the isomorphism type of $\mathcal{M}$.

For an arbitrary continuous action of a Polish group G on a Polish space Ed Effros has shown that an orbit $[x]$ is G_δ if and only if it is comeager in its closure, if and only if the natural map $g \mapsto g \cdot x$ is open.

Scott's analysis

Definition For $\mathcal{M}$ a model and $\vec{a}$ a finite sequence from $\mathcal{M}$ we define

$$\varphi_\alpha^{\vec{a},\mathcal{M}}$$

by induction on the ordinal α.

$$\varphi_0^{\vec{a},\mathcal{M}} = \bigwedge\{\psi(\vec{x}) : \psi(\vec{x}) \text{ quantifier free, } \mathcal{M} \models \psi(\vec{a})\}$$

$$\varphi_{\alpha+1}^{\vec{a},\mathcal{M}} = \varphi_\alpha^{\vec{a},\mathcal{M}} \bigwedge\{\exists \vec{y} \varphi_\alpha^{\vec{a}\vec{b},\mathcal{M}}(\vec{x},\vec{y}) : \vec{b} \in \mathcal{M}^{<\mathbb{N}}\} \wedge$$

$$\bigwedge_{n\in\mathbb{N}} \forall y_0, y_1, ..., y_n \bigvee\{\varphi_\alpha^{\vec{a}\vec{b},\mathcal{M}}(\vec{x},\vec{y}) : \vec{b} \in \mathcal{M}^{n+1}\}.$$

For λ a limit we set

$$\varphi_\lambda^{\vec{a},\mathcal{M}} = \bigwedge_{\alpha<\lambda} \varphi_\alpha^{\vec{a},\mathcal{M}}.$$

Note then that if $\mathcal{M}$ is a countable model of a countable language $\mathcal{L}$ then we have for each $\alpha < \omega_1$ and $\vec{a} \in \mathcal{M}$

$$\varphi_\alpha^{\vec{a},\mathcal{M}} \in \mathcal{L}_{\omega_1,\omega}.$$

For $\alpha < \beta$

$$\mathcal{M} \models \varphi_\beta^{\vec{a},\mathcal{M}} = \varphi_\beta^{\vec{b},\mathcal{M}} \Rightarrow \varphi_\alpha^{\vec{a},\mathcal{M}} = \varphi_\alpha^{\vec{b},\mathcal{M}}.$$

Thus:

Lemma: For $\mathcal{M}, \mathcal{L}$ countable there will be an ordinal $\delta < \omega_1$ such that for all $\vec{a}, \vec{b} \in \mathcal{M}$

$$\exists \gamma < \omega_1(\varphi_\gamma^{\vec{a},\mathcal{M}} \neq \varphi_\gamma^{\vec{b},\mathcal{M}}) \Leftrightarrow \varphi_\delta^{\vec{a},\mathcal{M}} \neq \varphi_\delta^{\vec{b},\mathcal{M}}.$$

In general such an ordinal δ with the property that

$$\forall \vec{a}, \vec{b} \in \mathcal{M}(\exists \gamma(\varphi_\gamma^{\vec{a},\mathcal{M}} \neq \varphi_\gamma^{\vec{b},\mathcal{M}}) \Leftrightarrow (\varphi_\delta^{\vec{a},\mathcal{M}} \neq \varphi_\delta^{\vec{b},\mathcal{M}}))$$

will be guaranteed to exist by replacement, as we may take the supremum over $\vec{a}, \vec{b} \in \mathcal{M}$ of the least α (when it exists) with

$$\varphi_\alpha^{\vec{a},\mathcal{M}} \neq \varphi_\alpha^{\vec{b},\mathcal{M}}.$$

However it may be uncountable. For instance taking the usual ordering on ordinals, the Scott height of the structure

$$(\omega_1, <)$$

is itself ω_1.

Definition For $\mathcal{M}$ a countable model the least δ as in the last lemma is the *Scott height of* $\mathcal{M}$, and will be denoted by $\alpha(\mathcal{M})$.

$$\varphi^{\emptyset,\mathcal{M}}_{\alpha(\mathcal{M})+2} =_{\mathrm{df}} \varphi^{\mathcal{M}}$$

is the (canonical) *Scott sentence of* $\mathcal{M}$;

$$\varphi^{\vec{a},\mathcal{M}}_{\alpha(\mathcal{M})+2} =_{\mathrm{df}} \varphi^{\vec{a},\mathcal{M}}$$

(or just $\varphi^{\vec{a}}$ when context indicates $\mathcal{M}$) is the *Scott sentence of* $\vec{a}$ (in $\mathcal{M}$); each $\varphi^{\vec{a},\mathcal{M}}_{\alpha}$ is the *αth approximation of the Scott sentence for* $\vec{a}$ or *the canonical α-type of* $\vec{a}$.

Note that if $\vec{a}, \vec{b} \in \mathcal{M}$ with

$$\varphi^{\vec{a},\mathcal{M}}_{\alpha(\mathcal{M})} = \varphi^{\vec{b},\mathcal{M}}_{\alpha(\mathcal{M})}$$

then it follows from the definition of $\alpha(\mathcal{M})$ that in particular we must have

$$\varphi^{\vec{a},\mathcal{M}}_{\alpha(\mathcal{M})+1} = \varphi^{\vec{b},\mathcal{M}}_{\alpha(\mathcal{M})+1}.$$

Lemma If $\varphi^{\vec{a},\mathcal{M}}_{\beta} \neq \varphi^{\vec{b},\mathcal{M}}_{\beta}$ then in any other model $\mathcal{N}$

$$\mathcal{N} \models \forall \vec{x}(\varphi^{\vec{a},\mathcal{M}}_{\beta}(\vec{x}) \Rightarrow \neg\varphi^{\vec{b},\mathcal{M}}_{\beta}(\vec{x})).$$

Proof We prove this by induction on β. It is obvious for $\beta = 0$ and the passage through limits follows trivially from the structure of the definition.

For the inductive step, assume the lemma is true at β and without loss that

$$\mathcal{N} \models \varphi^{\vec{a},\mathcal{M}}_{\beta+1}(\vec{c})$$

and that there is some $\vec{d} \in \mathcal{M}$ with

$$\forall \vec{e} \in \mathcal{M}(\varphi^{\vec{a}\vec{d},\mathcal{M}}_{\beta} \neq \varphi^{\vec{b}\vec{e},\mathcal{M}}_{\beta}).$$

Then we may find some $\vec{f} \in \mathcal{N}$ with

$$\mathcal{N} \models \varphi^{\vec{a}\vec{d}}_{\beta}(\vec{c}, \vec{f}),$$

and hence for all $\vec{e} \in \mathcal{M}$ we have

$$\mathcal{N} \models \neg\varphi^{\vec{b}\vec{e},\mathcal{M}}_{\beta}(\vec{c}, \vec{f}).$$

□

Theorem (Scott) Any two countable models satisfying the same Scott sentence are isomorphic.
Proof Let $\mathcal{M}$,$\mathcal{N}$ be countable models with

$$\mathcal{N} \models \varphi^{\mathcal{M}}.$$

Fix some ordinal $\alpha = \alpha(\mathcal{M})$ as the Scott height of $\mathcal{M}$.

Claim: For all $\vec{a} \in \mathcal{N}$, $\vec{b} \in \mathcal{M}$

$$\mathcal{N} \models \varphi_{\alpha}^{\vec{b},\mathcal{M}}(\vec{a})$$

implies that

$$\mathcal{N} \models \varphi_{\alpha+1}^{\vec{b},\mathcal{M}}(\vec{a}).$$

Proof of Claim: Otherwise from the assumption that $\mathcal{N}$ satisfies the Scott sentence of $\mathcal{M}$ we have

$$\mathcal{N} \models \forall \vec{x} \bigvee_{\vec{c} \in \mathcal{M}} \varphi_{\alpha+1}^{\vec{c},\mathcal{M}}(\vec{x})$$

we may find some $\vec{c} \in \mathcal{M}$ with

$$\mathcal{N} \models \varphi_{\alpha+1}^{\vec{c},\mathcal{M}}(\vec{a})$$

$$\therefore \mathcal{N} \models \varphi_{\alpha}^{\vec{c},\mathcal{M}}(\vec{a})$$

and so by the last lemma we must have

$$\varphi_{\alpha}^{\vec{c},m} = \varphi_{\alpha}^{\vec{b},\mathcal{M}}$$

$$\therefore \mathcal{M} \models \varphi_{\alpha}^{\vec{c},\mathcal{M}}(\vec{b})$$

but by the assumption the claim fails

$$\mathcal{M} \models \neg\varphi_{\alpha+1}^{\vec{c},\mathcal{M}}(\vec{b}),$$

with a contradiction to α being the Scott height of $\mathcal{M}$. (Claim$\Box$)

But now we can just go ahead the usual back and forth kind of argument. We can define a sequence

$$a_0, a_1, a_2, ... \in \mathcal{M},$$

$$b_0, b_1, b_2, ... \in \mathcal{N},$$

with at each k

$$\mathcal{N} \models \varphi_{\alpha}^{\langle a_0, a_1, ..., a_k \rangle, \mathcal{M}}(b_0, b_1, ..., b_k)$$

and under some fixed enumerations of the two models we have that the i^{th} element of $\mathcal{M}$ appears among $\{a_0, a_1, ..., a_{2i}\}$ and the i^{th} element of $\mathcal{N}$ appears among the $\{b_0, b_1, ..., b_{2i+1}\}$. We are able to propagate this construction since the claim above in particular implies that if $\mathcal{M} \models \varphi_{\alpha}^{\vec{b},\mathcal{N}}(\vec{a})$ and $d \in \mathcal{N}$ then

$$\mathcal{M} \models \exists y \varphi_{\alpha}^{\vec{b}^\frown d,\mathcal{N}}(\vec{a}, y),$$

which is the same as saying there is some $d' \in \mathcal{M}$ with

$$\mathcal{M} \models \varphi_\alpha^{\vec{b}^\frown d,\mathcal{N}}(\vec{a}, d').$$

$\square$

This was all proved because Dana Scott wanted to show:

Corollary The isomorphism type of any countable model is Borel in $\mathrm{Mod}(\mathcal{L})$.

It has previously been shown that the isomomorphism type of any *countable ordinal* is Borel in $\mathrm{Mod}(\{<\})$; indeed it is a nice exercise to prove this by induction on the ordinal, without any appeal to the Scott analysis. With this result in mind Kuratowski had asked whether the isomorphism type of any countable structure was Borel, which in turn was explicitly answered as above.

Exercises

1. F a countable fragment and T an F-complete theory. The isomorphism type of any $\mathcal{M} \in (\mathrm{Mod}(T), \tau_F)$ has Baire property and is non-meager if and only if it is comeager.

2. Let $\mathcal{L}$ be a countable language and $(F_\alpha)_{\alpha \in \omega_1}$ be an increasing sequence of countable fragments in $\mathcal{L}_{\omega_1,\omega}$. Suppose $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ is fortunate enough that for each α and $\vec{a} \in \mathcal{M}$ there is some $\psi(\vec{x}) \in F_{\alpha+1}$ such that

$$\mathcal{M} \models \psi(\vec{a})$$

$$\mathcal{M} \models \forall \vec{x} \forall \vec{y} \bigwedge_{\varphi \in F_\alpha} (\psi(\vec{x}) \wedge \psi(\vec{y}) \Rightarrow (\varphi(\vec{x}) \Leftrightarrow \varphi(\vec{y})));$$

in other words, for each $\vec{a}$ there is some formula in $F_{\alpha+1}$ satisfied by $\vec{a}$ which "isolates its type over F_α".

Then show that there is some countable $\gamma < \omega_1$ with $\mathcal{M}$ atomic relative to the fragment F_γ and the theory $\mathrm{Th}_{F_\gamma}(\mathcal{M})$.

3. Show that if δ is the Scott height of a countable model then we have *for all* ordinals γ

$$\varphi_\gamma^{\vec{a},\mathcal{M}} \neq \varphi_\gamma^{\vec{b},\mathcal{M}} \Rightarrow \varphi_\delta^{\vec{a},\mathcal{M}} \neq \varphi_\delta^{\vec{b},\mathcal{M}}.$$

4. (a) Show that if $T \subset F$ is a complete theory then

$$\{\varphi_0^{\vec{a},\mathcal{M}} : \mathcal{M} \models T, \vec{a} \in \mathcal{M}\}$$

is a Borel set in some standard Borel structure; and hence has size continuum or $\leq \aleph_0$.

(b) Show that if σ is sentence appearing in some fragment F then the collection of countable complete "consistent"[14] $T \subset F$ containing σ is a Borel set.

(c) Conclude that for any $\sigma \in \mathcal{L}_{\omega_1,\omega}$

$$\{\varphi_0^{\vec{a},\mathcal{M}} : \mathcal{M} \models \sigma, \vec{a} \in \mathcal{M}\}$$

is either countable or has cardinality $2^{\aleph_0}$.

(d) Iterating the argument of (a)-(c) show that for any $\sigma \in \mathcal{L}_{\omega_1,\omega}$ and $\alpha < \omega_1$

$$\{\varphi_\alpha^{\vec{a},\mathcal{M}} : \mathcal{M} \models \sigma, \vec{a} \in \mathcal{M}\}$$

is either countable or has cardinality $2^{\aleph_0}$.

(e) ("Morley's lemma") Show that if σ has less than continuum many non-isomorphic countable models then it has $\leq \aleph_1$ many non-isomorphic countable models.

Actually the proof of the main theorem we are heading towards parallels the above argument due to Morley. There again is a kind of transfinite induction, where the failure at each level for their being "lots of models" gives us the ability to perform one further step of the analysis.

In Morley's lemma "lots of models" corresponds to $2^{\aleph_0}$ many countable models; and the technical tool used at each step is the perfect set theorem for Borel sets. In our setting the "lots of models" case will correspond to an embedding of E_0 and the central technical device is a version of Glimm-Effros due to Becker and Kechris.

[14] Of course I have been trying to avoid use of this word in the context of infinitary model theory. Here I mean that the theory has some model.

Glimm-Effros

Theorem (Becker-Kechris) Let E be an equivalence relation on a Polish space X and let G be a group acting by homeomorphisms on X. Then assuming

(a) E is meager as a subset of $X \times X$, and

(b) for all $g \in G, x \in X$ we have

$$g \cdot xEx$$

(that is to say, the orbit equivalence relation induced by G is included in E), and

(c) there is some $x_0 \in X$ with $[x_0]_G =_{\mathrm{df}} \{g \cdot x_0 | g \in G\}$ dense,

then

$$E_0 \leq_B E.$$

Proof Fix a complete metric d on X and a point $x_0 \in X$ whose orbit under the action of G is dense. Let $(F_n)_{n \in \mathbb{N}}$ be a countable sequence of closed nowhere dense subsets of $X \times X$ with

$$E \subset \bigcup_{n \in \mathbb{N}} F_n.$$

The theorem is proved by a construction. We build a sequence of non-empty open sets $(V_s)_{s \in 2^{<\mathbb{N}}}$ in X and an array of group elements $\{(g_{s,t})_{s,t \in \{0,1\}^n} : n \in \mathbb{N}\}$ such that:

(i) for $s : \{0, 1..., n-1\} \to \{0, 1\}$ we have

$$d(V_s) < \frac{1}{n};$$

(ii)

$$V_s \subset V_t$$

for $s \supset t$ and moreover

$$\overline{V_s} \subset V_t$$

if $t \subsetneq s$;

(iiia) for $s, t \in 2^n$ and $u \in 2^{<\mathbb{N}}$ we have

$$g_{s,t} \cdot V_{s^\frown u} = V_{t^\frown u};$$

(iiib) and in fact if $\vec{0}$ is the sequence of all zeroes of length n and $s, t \in 2^n$ then we have $g_{s,t} = g_{\vec{0},t} \circ g_{\vec{0},s}^{-1}$;

(iiic) for $s, t \in 2^n$ and $u \in 2^{<\mathbb{N}}$ we have

$$g_{s,t} = g_{s^\frown u, t^\frown u};$$

(iv) for $s, t \in 2^n$ with $s \neq t$ we have

$$V_s \times V_t \cap (\bigcup_{m \leq n} F_m) = \emptyset.$$

If we succeed in doing all this, then the theorem follows quickly. We define for each $f \in 2^{\mathbb{N}}$ a corresponding

$$x_f \in \bigcup_{n \in \mathbb{N}} V_{f|_n}.$$

There will be a unique such point x_f in the infinite intersection by (i) and (ii); and then by the nature of the construction we have

$$f \mapsto x_f$$

as a Borel, in fact continuous, map. ¿From (iiia) we have that if $N \in \mathbb{N}$ is the largest integer with $f_1(N-1) \neq f_2(N-1)$ then

$$g_{f_1|_N, f_2|_N} \cdot x_{f_1} = x_{f_2}$$

and thus

$$x_{f_1} E_0 x_{f_2}.$$

(iv) moreover yields that if there are infinitely many n with $f_1(n) \neq f_2(n)$ then there are infinitely many n with

$$(x_{f_1}, x_{f_2}) \notin \bigcup_{m \leq n} F_n,$$

and so

$$(x_{f_1}, x_{f_2}) \notin E \subset \bigcup_{n \in \mathbb{N}} F_n.$$

So let us now concentrate on showing that some such sequence of open sets and array of group elements has been produced. We do it by induction on $lh(s)$, and for this purpose assume that $(V_s)_{s \in 2^{\leq n}}$ and $\{(g_{s,t})_{s,t \in \{0,1\}^m} : m \leq n\}$ have the properties indicated above. We will begin by refining the open set $V_{\vec{0}}$, where $\vec{0}$ is the sequence of length n which takes zero as its value on each coordinate.

Remember that the group G is acting by homeomorphisms, and so

$$\{(x, y) : (g \cdot x, h \cdot y) \in F\}$$

is closed nowhere dense for any $g, h \in G$ and closed nowhere dense $F \subset X \times X$. Moreover the finite union of closed nowhere dense sets is again closed nowhere dense.

Thus we may chose some open non-empty $W, W' \subset V_{\vec{0}}$ such that for all $g, h \in \{(g_{s,t})_{s,t \in \{0,1\}^m} : m \leq n\}$

$$g \cdot W \times h \cdot W' \cap (\bigcup_{m \leq n+1} F_m) = \emptyset.$$

After further refinement we may assume that

$$d(g \cdot W), d(g \cdot W') < \frac{1}{n+1}$$

and

$$\overline{g_{\vec{0},s} \cdot W}, \overline{g_{\vec{0},s} \cdot W'} \subset V_s$$

all $s \in 2^n$.

Then since $[x_0]_G$ is dense we may choose

$$x \in W \cap [x_0]_G,$$

$$x' \in W' \cap [x_0]_G.$$

In particular there will be some $h_0 \in G$ with

$$h_0 \cdot x = x'.$$

Since G acts by homeomorphisms we may find $V_{\vec{0}^\frown 0} \subset W$, $V_{\vec{0}^\frown 1} \subset W'$ with

$$h_0 \cdot V_{\vec{0}^\frown 0} = V_{\vec{0}^\frown 1}.$$

At this stage we can finish up with

$$V_{s^\frown 0} = g_{\vec{0},s} \cdot V_{\vec{0}^\frown 0},$$

$$V_{s^\frown 1} = g_{\vec{0},s} h_0 \cdot V_{\vec{0}^\frown 0} = g_{\vec{0},s} \cdot V_{\vec{0}^\frown 1},$$

$$g_{\vec{0}^\frown 0, s^\frown 1} = g_{\vec{0},s} \circ h_0,$$

for $s \in 2^n$. The decisions

$$g_{\vec{0}^\frown 0, s^\frown 0} = g_{\vec{0},s}$$

$$g_{s^\frown i, t^\frown i} = g_{s,t}$$

all $s, t \in 2^n$, $i \in \{0, 1\}$ are forced on us by (iiib) and (iiic). Taking

$$g_{s^\frown 0, t^\frown 1} = g_{\vec{0},t} \circ h_0 \circ g^{-1}_{\vec{0},s}$$

and

$$g_{s^\frown 1, t^\frown 0} = g^{-1}_{t^\frown 0, s^\frown 1}$$

provides us with the full armory of group elements to go up to $n+1$. □

Corollary Let $F \subset \mathcal{L}_{\omega_1,\omega}$ be a countable fragment, $T \subset F$ a complete theory. Then either

$$E_0 \leq_B \cong |_{\mathrm{Mod}(T)}$$

or there is an F-atomic model.

Proof Completeness of the theory gives that every isomorphism type is dense in $(\mathrm{Mod}(T), \tau_F)$. The orbit equivalence relation is induced by a continuous action of the infinite symmetric group, S_∞, and so we can use the previous lemma to conclude that either

$$E_0 \leq_B \cong |_{\mathrm{Mod}(T)}$$

or the orbit equivalence relation is not meager; and in this later case we saw two sections ago that we must have an atomic model. $\square$

Definition A topological group[15] is a *Polish group* if it is Polish as a topological space.

Exercises

1. Let G be a Polish group acting continuously on a Polish space X. Show that there is a Borel function

$$\theta : X \to \mathbb{R}$$

such that $\theta(x_1) = \theta(x_2)$ if and only if the orbits of x_1 and x_2 have the same closure.

2. Let G be a locally compact Polish group acting continuously on a Polish space X with orbit equivalence relation E_G.
(a) Show that E_G is F_σ as a subset of $X \times X$.
(b) Show that if there is dense orbit then it must contain an open set.
(c) Show that in general, even without a dense orbit, we have either

$$E_G \leq \mathrm{id}(\mathbb{R})$$

or

$$E_0 \leq_B E_G.$$

[15]That is to say, a group equipped with a topology under which the group operations of multiplication and inversion are continuous.

Final theorem

Definition HC denotes the collection of hereditarily countable sets. A formula $\psi(\vec{x})$ in the language of set theory is Σ_0 if the only quantifiers appearing are the bounded quantifiers

$$\forall x \in y$$

$$\exists x \in y.$$

A subset $A \subset \text{HC}$ is said to be $\underset{\sim}{\Sigma}_1^{\text{HC}}$ if there is some Σ_0 formula ψ and parameter $a \in \text{HC}$ such that A equals the set of $b \in \text{HC}$ for which

$$(HC, \in) \models \exists x \psi(a, b, x).$$

$D \subset \text{HC}$ is $\underset{\sim}{\Delta}_1^{\text{HC}}$ if both it and its complement are $\underset{\sim}{\Sigma}_1^{\text{HC}}$. I will say that a function is $\underset{\sim}{\Delta}_1^{\text{HC}}$ if its graph and domain are both $\underset{\sim}{\Delta}_1^{\text{HC}}$.

It will be convenient to phrase the next result in terms of $\underset{\sim}{\Delta}_1^{\text{HC}}$ reductions; but I will skip over any detailed verifications about certain functions and sets appearing in the definition being $\underset{\sim}{\Delta}_1^{\text{HC}}$. Instead here are some facts without proof. For many of these facts there are similar facts with similar proofs holding for the recursive functions.

Facts

1. Ordinal addition, multiplication, and exponentiation are $\underset{\sim}{\Delta}_1^{\text{HC}}$.
2. If

$$G : \omega_1 \times \text{HC} \to \text{HC}$$

$$H : \text{HC} \to \text{HC}$$

$\underset{\sim}{\Delta}_1^{\text{HC}}$, then so is the function defined recursively by

$$F(\alpha + 1) = H(G(\alpha, F(\alpha)))$$

and

$$F(\lambda) = \bigcup\{F(\alpha) : \alpha < \lambda\}$$

when λ a limit.

3. For $a \in \text{HC}$

$$\alpha \mapsto L_\alpha(a)$$

is $\underset{\sim}{\Delta}_1^{\text{HC}}$, where we here use $L_\alpha(a)$ to denote the α^{th} level of the constructible hierarchy built over a.

4. The $\underset{\sim}{\Delta}_1^{\text{HC}}$ recursive functions are closed under μ-recursion with respect to the ordinals. Thus if for each $z \in \text{HC}$ there exists $\alpha < \omega_1$ with

$$L_\alpha(z, a) \models \psi(a, z),$$

then the function assigning to each z the least such α is $\underset{\sim}{\Delta}_1^{\text{HC}}$.

5. Given $\mathcal{L}$ a countable language, there is a $\underset{\sim}{\Delta}_1^{\mathrm{HC}}$ function assigning to each $\mathcal{M} \in \mathrm{Mod}(T)$ and countable fragment $F \subset \mathcal{L}_{\omega_1,\omega}$ the theory of $\mathcal{M}$ in that fragment.

6. Every *set recursive function* (in the sense of Garvin Melles) is $\underset{\sim}{\Delta}_1^{\mathrm{HC}}$; and in fact the theorem below could equally be phrased in terms of set recursive functions instead of $\underset{\sim}{\Delta}_1^{\mathrm{HC}}$.

Theorem Let $\mathcal{L}$ be a countable language and $\sigma \in \mathcal{L}_{\omega_1,\omega}$. Then either

(I) $E_0 \leq_B \cong|_{\mathrm{Mod}(\sigma)}$; or

(II) there is a $\underset{\sim}{\Delta}_1^{\mathrm{HC}}$ function

$$\theta : \mathrm{Mod}(\sigma) \to 2^{<\omega_1}$$

such that for all $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(\sigma)$

$$\mathcal{M} \cong \mathcal{N} \Leftrightarrow \theta(\mathcal{M}) = \theta(\mathcal{N}).$$

In other words, $\cong|_{\mathrm{Mod}(\sigma)}$ is either as complicated as E_0 or allows a $\underset{\sim}{\Delta}_1^{\mathrm{HC}}$ assignment of bounded subset of $\aleph_1$ as complete invariants.

Proof Assume E_0 is not Borel reducible to $\cong|_{\mathrm{Mod}(\sigma)}$; by the corollary to Becker-Kechris this will ensure that we have many atomic models, no matter which fragment we choose to consider.

First fix a fragment F_0 containing σ. For each $\alpha < \omega_1$ and positive integer n let $P_{\alpha,n}$ be a new n-ary relation; we can do this so that

$$(\alpha, n) \mapsto P_{\alpha,n}$$

is a $\underset{\sim}{\Delta}_1^{\mathrm{HC}}$ function. We let $\mathcal{L}^\alpha$ be the language generated by $\mathcal{L}$ and $\{P_{\beta,n} : n > 0, \beta < \alpha\}$ and we let $F_\alpha \subset \mathcal{L}^\alpha_{\omega_1,\omega}$ be the fragment generated by F_0 and all formulas of the form

$$\bigvee_{\{\beta : \gamma \leq \beta < \gamma'\}} P_{\beta,n}(\vec{x})$$

for $\gamma < \gamma' \leq \alpha$. We may simultaneously fix a $\underset{\sim}{\Delta}_1^{\mathrm{HC}}$ increasing sequence of ordinals,

$$(\gamma_\alpha)_{\alpha \in \omega_1}$$

and bijections

$$\pi_\alpha : \gamma_\alpha \to F_\alpha$$

which are uniformly $\underset{\sim}{\Delta}_1^{\mathrm{HC}}$, in the sense that

$$(\alpha, \beta) \mapsto \pi_\alpha(\beta)$$

is a $\underset{\sim}{\Delta}_1^{\mathrm{HC}}$ function.

Fix $\mathcal{M} \in \mathrm{Mod}(\sigma)$. We describe a a sequence of expansions

$$\mathcal{M}_0 = \mathcal{M}, \mathcal{M}_1, \cdots, \mathcal{M}_\alpha \cdots$$

out through the ordinals, the isomorphism type of each $\mathcal{M}_\alpha$ depending only on the isomorphism type of $\mathcal{M}$ and each $\mathcal{M}_\alpha$ an $\mathcal{L}^\alpha$-structure; at each α we let $T^{\mathcal{M}}_\alpha$ be the theory of $\mathcal{M}_\alpha$ with respect to the fragment F_α.

Thus in particular we begin with

$$T^{\mathcal{M}}_0 = \mathrm{Th}_F(\mathcal{M}),$$

the theory of $\mathcal{M}$ with respect to F.

Claim: $T^{\mathcal{M}}_0$ has an atomic model.

Proof of Claim: Or else Becker-Kechris gives

$$E_0 \leq_B \cong|_{\mathrm{Mod}(T^{\mathcal{M}}_0)},$$

$$\therefore E_0 \leq_B \cong|_{\mathrm{Mod}(\sigma)}.$$

($\square$Claim)

We then define $\mathcal{M}_1$ to be the expansion of $\mathcal{M}$ to $\mathcal{L}^1$ as follows. For each $a_1, a_2, ..., a_n \in \mathcal{M}$ we let

$$\mathcal{M}_1 \models P_{0,n}(\vec{a})$$

if and only if $\vec{a}$ realizes a principal type relative to the theory $T^0_{\mathcal{M}}$. Then we let $T^{\mathcal{M}}_1$ be the theory of $\mathcal{M}_1$ relative to the fragment F_1.

And we can keep going in the obvious way. We let

$$T^{\mathcal{M}}_\alpha = \mathrm{Th}_{F_\alpha}(\mathcal{M}_\alpha);$$

$\mathcal{M}_{\alpha+1}$ is the expansion to $\mathcal{L}^{\alpha+1}$ obtained by letting

$$\mathcal{M}_{\alpha+1} \models P_{\alpha,n}(\vec{a})$$

if $lh(\vec{a}) = n$ and $\vec{a}$ realizes a principal type over the theory $T^{\mathcal{M}}_\alpha$. At limit stages we can let $\mathcal{M}_\lambda$ be the unique model in the language

$$\mathcal{L}^\lambda = \bigcup_{\alpha<\lambda} \mathcal{L}^\alpha$$

which expands the all the preceding

$$\{\mathcal{M}_\alpha : \alpha < \lambda\}.$$

One possibility is that regardless of our particular choice of $\mathcal{M}$ we always arrive at some F_κ with $\mathcal{M}_\kappa$ the $T^{\mathcal{M}}_\kappa$-atomic model for F_κ. Then we are done. We can just let $\kappa(\mathcal{M})$ be the first such κ for $\mathcal{M}$ and let

$$\theta(\mathcal{M}) = \{\beta < \gamma_{\kappa(\mathcal{M})} : \mathcal{M}_{\kappa(\mathcal{M})} \models \pi_{\kappa(\mathcal{M})}(\beta)\} \cup \{\gamma_{\kappa(\mathcal{M})}\}$$

to obtain an invariant which indicates to us $\gamma_{\kappa(\mathcal{M})}$, and hence $\kappa(\mathcal{M})$, and hence the fragment $F_{\kappa(\mathcal{M})}$, as well as the theory $T^{\mathcal{M}}_{\kappa(\mathcal{M})}$ and the fact that $\mathcal{M}_{\kappa(\mathcal{M})}$ is atomic with respect to this theory; since each complete countable theory has at most one atomic model up to isomorphism, the invariant indicates the isomorphism type of $\mathcal{M}_{\kappa(\mathcal{M})}$; since $\mathcal{M}_{\kappa(\mathcal{M})}$ is up to isomorphism an invariant of $\mathcal{M}$, we obtain that $\theta(\mathcal{M})$ is a complete invariant for the isomorphism type of $\mathcal{M}$. And finally in this case it would be routine to establish that the association

$$\mathcal{M} \mapsto \theta(\mathcal{M})$$

is $\underset{\sim}{\Delta}_1^{\mathrm{HC}}$.

So instead let us try to show that some such ordinal $\kappa(\mathcal{M})$ always exists. And for this let us assume that $\mathcal{M}$ is a counterexample and try to derive a contradiction.

Claim: For each $n > 0$ and $\delta < \omega_1$ there is a larger countable ordinal $\delta' > \delta$ such that for each $a_1, a_2, ..., a_n \in \mathcal{M}$

$$\mathcal{M}_{\delta'} \models \bigvee_{\alpha \in [\delta, \delta')} P_{\alpha,n}(\vec{a}).$$

Proof of Claim: Instead we obtain that each δ'

$$\mathcal{M}_{\delta'} \models \exists \vec{x} \bigwedge_{\alpha \in [\delta, \delta')} \neg P_{\alpha,n}(\vec{x}).$$

Since $T_{\delta}^{\mathcal{M}}$ has an atomic model, by Becker-Kechris, we obtain some principal type refining the type

$$\bigwedge_{\alpha \in [\delta, \delta')} \neg P_{\alpha,n}(\vec{x}),$$

and thus by the nature of our construction some $\vec{a}^{\,\delta'}$ with

$$\mathcal{M}_{\delta'+1} \models P_{\delta'}(\vec{a}^{\,\delta'}) \bigwedge_{\alpha \in [\delta, \delta')} \neg P_{\alpha,n}(\vec{a}^{\,\delta'});$$

but then

$$\delta' \mapsto \vec{a}^{\,\delta'}$$

gives us $\aleph_1$ distinct n-tuples in $\mathcal{M}$, with an obvious contradiction to its countability. (Claim$\square$)

Thus by repeated application of this claim we can find an increasing sequence of ordinals,

$$(\delta_\alpha)_{\alpha < \omega_1},$$

such that for each $\alpha \in \omega_1$, $n > 0$, $\vec{a} \in \mathcal{M}^n$ there is some

$$\beta > \bigcup_{\alpha' < \alpha} \delta_{\alpha'}$$

with

$$\beta \leq \delta_\alpha$$

and

$$\mathcal{M}_{\delta_\alpha} \models P_{\beta,n}(\vec{a}).$$

We may also fix for each $\beta < \omega_1$ and $n > 0$ a sequence of formulas $(\psi^m_{\beta,n})_m$ such that

$$\mathcal{M}_{\beta+1} \models \forall x_1, x_2, ..., x_n (P_{\beta,n}(\vec{x}) \Leftrightarrow \bigvee_{m \in \mathbb{N}} \psi^m_{\beta,n}(\vec{x}))$$

and each $\psi^m_{\beta,n}(\vec{x}) \in F_\beta$ defines a principal type over $T^{\mathcal{M}}_\beta$. Unfortunately the disjunction $\bigvee_{m\in\mathbb{N}} \psi^m_{\beta,n}(\vec{x})$ has not been placed in any of our fragments, so we need to observe that the equivalence

$$\mathcal{N} \models \forall x_1, x_2, ..., x_n (P_{\beta,n}(\vec{x}) \Leftrightarrow \bigvee_{m \in \mathbb{N}} \psi^m_{\beta,n}(\vec{x}))$$

holds for any sufficiently "generic" $\mathcal{N}$ in the space of $T^{\mathcal{M}}_\beta$ models, for any $\alpha > \beta$.

Claim: For each $n > 0$, $\vec{a} \in \mathbb{N}^n$ and $\beta < \alpha$, the set of

$$\mathcal{N} \in (\mathrm{Mod}(T^{\mathcal{M}}_\alpha), \tau_{F^{\mathcal{M}}_\alpha})$$

with

$$\mathcal{N} \models P_{\beta,n}(\vec{a}) \Leftrightarrow \bigvee_{m \in \mathbb{N}} \psi^m_{\beta,n}(\vec{a})$$

is open dense.

Proof of Claim: It is a straight calculation to determine that the set of such $\mathcal{N}$ is open. For density we use that the isomorphism type of $\mathcal{M}_\alpha$ is dense in $(\mathrm{Mod}(T^{\mathcal{M}}_\alpha, \tau_{F^{\mathcal{M}}_\alpha})$ and

$$\mathcal{M}_\beta \models \forall x_1, x_2, ..., x_n (P_{\beta,n}(\vec{x}) \Leftrightarrow \bigvee_{m \in \mathbb{N}} \psi^m_{\beta,n}(\vec{x})).$$

(Claim$\square$)

Thus for each α there is a comeager set

$$C_\alpha \subset (\mathrm{Mod}(T^{\mathcal{M}}_\alpha), \tau_{F^{\mathcal{M}}_\alpha})$$

such that for all $\beta < \alpha, n > 0, \mathcal{N} \in C_\alpha$

$$\mathcal{N} \models \forall x_1, x_2, ..., x_n (P_{\beta,n}(\vec{x}) \Leftrightarrow \bigvee_{m \in \mathbb{N}} \psi^m_{\beta,n}(\vec{x})).$$

In the next claim I will use $\mathcal{N}|_{\mathcal{L}}$ to indicate the reduction of some $\mathcal{L}^\alpha$ model to our base language $\mathcal{L}$. As usual,

$$\varphi^{\vec{\ell},\mathcal{N}|_{\mathcal{L}}}_\kappa$$

will indicate the κ^{th} approximation to the Scott sentence of $\vec{\ell}$ over $\mathcal{N}|_{\mathcal{L}}$.

Claim: For each $\mathcal{N} \in C_\alpha$, $\delta_\kappa \leq \beta < \alpha < \omega_1$, $n > 0$, $m \in \mathbb{N}$, $\vec{\ell}, \vec{k} \in \mathbb{N}^n$ we have that if

$$\mathcal{M}_\alpha \models \psi^m_{\beta,n}(\vec{k})$$

and

$$\mathcal{N} \models \psi^m_{\beta,n}(\vec{\ell})$$

then

$$\varphi_\kappa^{\vec{\ell},\mathcal{N}|\mathcal{L}} = \varphi_\kappa^{\vec{k},\mathcal{M}}.$$

Proof of Claim: We prove this by induction on κ. It should be clear for $\kappa = 0$, since $\psi^m_{\beta,n} \in F_0$ defines a principal type over F_0, and thus is sufficient to decide the quantifier free type, even if $\mathcal{N}$ does not belong to our comeager set C_α. The limit steps follow almost vacuously from the structure of the Scott analysis and its requirement that we take conjunctions at limit stages.

For successor steps, let us suppose $\delta_{\kappa+1} \leq \beta < \alpha < \omega_1$,

$$\mathcal{M}_\alpha \models \psi^m_{\beta,n}(\vec{k}),$$

and

$$\mathcal{N} \models \psi^m_{\beta,n}(\vec{\ell}).$$

Choose some $\vec{a} \in \mathcal{N}$; we need to show that there is some other $\vec{b} \in \mathcal{M}$ with

$$\varphi_\kappa^{\vec{\ell}\vec{a},\mathcal{N}|\mathcal{L}} = \varphi_\kappa^{\vec{k}\vec{b},\mathcal{M}}.$$

But since $\mathcal{N}$ and $\mathcal{M}_\beta$ realize the same the theory it must be the case that there

$$\mathcal{N} \models \forall x_1, ..., x_n \bigvee_{\{\beta' : \gamma_\kappa \leq \beta' < \gamma_{\kappa+1}\}} P_{\beta',n}(\vec{x}),$$

and thus in particular for some $\beta' \in [\gamma_\kappa, \gamma_{\kappa+1})$ and $n' = lh(\vec{\ell}) + lh(\vec{a})$ we have

$$\mathcal{N} \models P_{\beta',n'}(\vec{\ell}, \vec{a});$$

then by assumption on $\mathcal{N} \in C_\alpha$ we have some m' with

$$\mathcal{N} \models \psi^{m'}_{\beta',n'}(\vec{\ell}, \vec{a});$$

then since $\psi^m_{\beta,n}$ decides the type of $\vec{k}$ over the fragment $F_{\beta'}$ we have

$$\mathcal{M}_\alpha \models \exists \vec{y} \psi^{m'}_{\beta',n'}(\vec{k}, \vec{y});$$

and so for some $\vec{b}$

$$\mathcal{M}_\alpha \models \exists \vec{y} \psi^{m'}_{\beta',n'}(\vec{k}, \vec{k})$$

and so by the inductive assumption

$$\varphi_\kappa^{\vec{\ell}\vec{a},\mathcal{N}|\mathcal{L}} = \varphi_\kappa^{\vec{k}\vec{b},\mathcal{M}}.$$

The converse direction is that for all $\vec{b} \in \mathcal{M}$ there is $\vec{a} \in \mathcal{N}$ with $\varphi_\kappa^{\vec{\ell}\vec{a},\mathcal{N}\lceil_\mathcal{L}} = \varphi_\kappa^{\vec{k}\vec{b},\mathcal{M}}$. This is similar, but only easier. (Claim$\square$)

Thus we obtain in particular that for any $\alpha > \alpha(\mathcal{M}) + 2$ and $\mathcal{N} \in C_\alpha$

$$\varphi_{\alpha(\mathcal{M})+2}^{\emptyset,\mathcal{N}\lceil_\mathcal{L}} = \varphi_{\alpha(\mathcal{M})+2}^{\emptyset,\mathcal{M}},$$

and thus by Scott's theorem

$$\mathcal{N}\lceil_\mathcal{L} \cong \mathcal{M};$$

but then it follows from the definition of the various $\mathcal{M}_\beta$'s that the isomorphism from $\mathcal{N}\lceil_\mathcal{L}$ to $\mathcal{M}$ lifts to one from $\mathcal{N}$ to $\mathcal{M}_\beta$ for each $\beta < \alpha$.

Thus the isomorphism type of $\mathcal{M}_\alpha$ *will* be comeager in some

$$(\mathrm{Mod}(T_\alpha^\mathcal{M}), \tau_{F_\alpha^\mathcal{M}});$$

and thus $\mathcal{M}_\alpha$ *will* be atomic, and so the process must have terminated at some stage after all. $\square$.

In general this theorem can be improved by slightly sharpening the reduction to one that is *provably* or *absolutely* $\underset{\sim}{\Delta}_1^{\mathrm{HC}}$; these complexity classes are a little technical to define[16], but have the advantage of being just inside the region for which ZFC alone can prove regularity properties, such as any *absolutely* $\underset{\sim}{\Delta}_1^{\mathrm{HC}}$ function between Polish spaces is continuous on a comeager set. In this sharper form the theorem would then become a dichotomy theorem: (I) and a suitably amended version of (II) would be incompatible.

A respect in which the theorem cannot be sharpened is with regards to the *kinds of* invariants we obtain in (II). For instance we cannot ask that we have say elements of $2^\mathbb{N}$ being assigned as complete invariants. The isomorphism relation on abelian p-groups provides a counterexample under suitable set theoretical assumptions for general $\underset{\sim}{\Delta}_1^{\mathrm{HC}}$ functions and outright in ZFC for the absolutely $\underset{\sim}{\Delta}_1^{\mathrm{HC}}$.

However this class is not first order definable, so I don't know whether one might hope to prove that for any first order theory T we have either have

(I) $E_0 \leq_B \cong|_{\mathrm{Mod}(T)}$, or
(II') $\cong|_{\mathrm{Mod}(T)} \leq_B \mathrm{id}(2^\mathbb{N})$.

Even showing this with (II') replaced by

(II") there is a $\underset{\sim}{\Delta}_1^{\mathrm{HC}}$ assignment of elements of $2^\mathbb{N}$ as complete invariants.

would be extremely interesting, and sufficient to prove Vaught's conjecture under large cardinal assumptions, or even prove Vaught's conjecture outright in ZFC if, as most likely, one could obtain

(II*) there is an absolutely $\underset{\sim}{\Delta}_1^{\mathrm{HC}}$ assignment of elements of $2^\mathbb{N}$ as complete invariants.

[16] See e.g. **Classification and orbit equivalence relations**, G. Hjorth, AMS, Rhode Island, 2000 for the precise definitions

More reading

There is always more to read, but in this case especially there are a number of issues we only touched on which could have been discussed at length.

The canonical references for the descriptive set theory of group actions connections with the isomorphism relation on countable models are

> H. Becker, A.S. Kechris, **The descriptive set theory of Polish group actions**, London Mathematical Society Lecture Notes Series, 232, Cambridge University Press, Cambridge, 1996

and

> R. Sami, *The topological Vaught conjecture*, **Transactions of the American Mathematical Society**, vol. 341(1994), pp. 335-353.

The serious mathematical discussion of dichotomy theorems for Borel equivalence relations was initiated in

> L.A. Harrington, A.S. Kechris, A. Louveau, *A Glimm-Effros dichotomy for Borel equivalence relations,* **Journal of the American Mathematical Society,** vol. 3(1990), pp. 903-928.

> A more recent survey is given by

> G. Hjorth, A.S. Kechris, *New dichotomies for Borel equivalence relations*, **Bulletin of Symbolic Logic**, vol. 3(1997), pp. 329–346.

A continued discussion of some of the material above, in much the same elementary style, is given by

> G. Hjorth, **Classification and orbit equivalence relations,** Mathematical Surveys and Monographs, 75, American Mathematical Society, Providence, RI, 2000.

This includes some easy proofs of some things we didn't quite get to, such as F_2 not being Borel reducible to any of the countable Borel equivalence relations, such as E_0.

The main theorem, which we finished with, was first presented in

> G. Hjorth, A.S. Kechris, *Analytic equivalence relations and Ulm-type classifications,* **Journal of Symbolic Logic,** vol. 60(1995), pp. 1273-1300,

and is also independently due to Howard Becker. The proof given there is rather different, in that it is derived as a corollary of Harrington-Kechris-Louveau; and indeed in this paper the main battle was to extend the result to general $\underset{\sim}{\Sigma}^1_1$ equivalence relations, while above we have been primarily concerned with developing an argument that uses no non-trivial descriptive set theory.

If there had been more time we might have discusses special versions of this dichotomy theorem which can be proved for specific classes of Polish groups. Then one often can hope to replace (II) by

(II') $\cong|_{\mathrm{Mod}(T)} \leq_B \mathrm{id}(2^{\mathbb{N}})$.

In

G. Hjorth, S. Solecki, *Vaught's conjecture and the Glimm-Effros property for Polish transformation groups,* **Transactions of the American Mathematical Society,** vol. 351(1999), pp. 2623-2641

this was done for the orbit equivalence relations induced by continuous actions of nilpotent and invariantly metrizable Polish groups, while

H. Becker, *The topological Vaught's conjecture and minimal counterexamples,* **Journal of Symbolic Logic,** vol. 59(1994), pp. 757-784

obtained more general results for solvable or under the assumption of a complete left invariant metric.

A rather different approach to some of these questions can be found in

G. Melles, *One cannot show from ZFC that there is an Ulm-type classification of the torsion-free abelian groups,* pp. 293-309, **MSRI Publication 26,** Springer-Verlag, 11992, New-York.