Non-smooth infinite dimensional group representations

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§0 When it's bad it's worse

It is known from [6] and [4] that the infinite dimensional irreducible representations of a countable group do not posses a *nice* Borel structure unless G is abelian-by-finite. Roughly speaking, the equivalence relation of conjugacy, or isomorphism, interprets the equivalence relation of eventual agreement on infinite sequences of integers, and thus is *non-smooth*: the irreducibles considered up to conjugacy do not form a standard Borel space, there is no Borel selector, there is no reasonable way to assign reals as complete invariants.

This note asks when the classification problem for the irreducibles is no *worse than* that of infinite sequences considered up to eventual agreement.

Surprsingly, the equivalence relation is *always* more complicated when non-smooth. Thus there is no reasonable way to assign complete invariants in sequences mod finite when G is not abelian-by-finite. Equivalently, the cosets of $\mathbb{Q}$ in $\mathbb{R}$ can not provide complete invariants – that is to say, there is no Borel function θ from the space of irreducible representations to $\mathbb{R}$ such that representations τ_1 and τ_2 are conjugate if and only if $\theta(\tau_1) - \theta(\tau_2) \in \mathbb{Q}$. Worse: there is no *countable Borel lacunary section* (a Borel subset of the space of irreducibles meeting each equivalence class in exactly $\aleph_0$ many points). We cannot assign countable sets of reals as complete invariants. And strongest of all – and this is the real theorem – for G not abelian-by-finite, we cannot complete a classification by countable discrete structures considered up to isomorphism.

§1 Some facts

From here on let G be a countable discrete group, H a separable infinite dimensional Hilbert space, $U_\infty = U(H)$ the unitary group of H – the group consisting of all bounded linear operators $T : H \rightarrow H$ with $T^* = T^{-1}$.

1.1 Definition Define $\mathrm{Irr}(G, H)$ to be the space of *irreducible representations* – that is the set of all homorphisms $\tau : G \rightarrow U_\infty$ such that there are no non-trivial topologically closed, linearly closed, $\tau[G]$- invariant subspaces of H.

For $\tau \in \mathrm{Irr}(G, H)$ and $T \in U_\infty$, define $T \cdot \tau \in \mathrm{Irr}(G, H)$ by

$$(T \cdot \tau)(g) = T \circ \tau(g) \circ T^{-1}.$$

Let $\tau_1 \approx \tau_2$ denote that there is some $T \in U_\infty$ with $T \cdot \tau_1 = \tau_2$.

Note that $\mathrm{Irr}(G, H)$ is a G_δ subset of H^G, and hence Polish (in the product topology). U_∞ is a Polish group and the action of this group on $\mathrm{Irr}(G, H)$ is continuous.

The following is certainly implicit in [3].

1.2 Lemma (Effros, Mackey) Let $\tau_1, \tau_2 \in \mathrm{Irr}(G, H)$, $T_i \in U_\infty$, $e, \hat{e} \in H$, with

$$T_i \cdot \tau_1 \rightarrow \tau_2,$$

$$\mathrm{liminf}_{j \rightarrow \infty}(\langle T_j(e), \hat{e}\rangle) \neq 0.$$

Then $\tau_1 \approx \tau_2$.

Proof. Without loss of generality there is a bounded linear operator $T \in \mathcal{B}(H)$ such that

$$T_j(e') \rightarrow T(e')$$

weakly (in the sense that $\langle T_j(e'), e''\rangle \rightarrow \langle T(e'), e''\rangle$ all $e'' \in H$) for all $e' \in H$. Note then by replacing τ_2 by suitable $T' \cdot \tau_2$ and T_i by $T'T_i$ we may assume $T_j(e) \rightarrow \delta e$ weakly for some $\delta > 0$.

Claim(1): For all $e' \in H$,

$$T_j^*(e') \rightarrow T^*(e').$$

Proof of claim. Since for all $e', e'' \in H$ we have that

$$\langle e', T^*(e'')\rangle = \langle T(e'), e''\rangle$$

$$= \lim_{i \rightarrow \infty}\langle T_i(e'), e''\rangle$$

$$= \lim_{i \rightarrow \infty}\langle e', T_i^*(e'')\rangle.$$

($\square$Claim(1))

Claim(2): For all $g \in G$

$$T \circ \tau_1(g) = \tau_2(g) \circ T.$$

Proof of claim: Since for all $e', e'' \in H$

$$\langle T \circ \tau_1(g)(e'), e'' \rangle = \lim \langle T_i \circ \tau_1(g)(e'), e'' \rangle$$

$$= \lim_{i \to \infty} \langle \tau_2(g) \circ T_i(e'), e'' \rangle = \lim_{i \to \infty} \langle T_i(e'), \tau_2(g)^*(e'') \rangle$$

$$\langle T(e'), \tau_2(g)^*(e'') \rangle$$

$$= \langle \tau_2(g) \circ T(e'), e'' \rangle.$$

($\square$Claim(2))

Claim(3): For all $g \in G$

$$T^* \circ \tau_2(g) = \tau_1(g) \circ T^*.$$

Proof of claim: Since for all $e' \in H$ and $j \in \mathbb{N}$ we have

$$T_j^* \circ \tau_2(g)(T_j(e')) = T_j^* \circ T_j \circ \tau_1(e')$$

$$= \tau_1(e') = \tau_1 \circ T_j^*(T_j(e')),$$

we can use claim(1) to just repeat the proof of claim(2). ($\square$Claim(3))

Claim(4): For all $g \in G$

$$T^* \circ T \circ \tau_1(g) = \tau_1(g) \circ T^* \circ T.$$

Proof of claim: Follows by claim(2) and claim(3). ($\square$Claim(4))

So now let $B \subset \mathcal{B}(H)$ be the C^*-algebra generated by the operator $T^* \circ T$. Since $T^* \circ T$ is self adjoint we have that the polynomials in $T^* \circ T$ are dense in B, and hence that every $\tau_1(g)$ commutes with every element of B. Following 3.5, 3.6, 3.11VIII of [2], if we let M_0 be the orthogonal complement of the kernel of T and let M_1 be the range of T, we may find an isometry

$$U : M_0 \to M_1$$

and positive $S \in B$ with

$$U \circ S = T$$

$$S^2 = T * T.$$

Since *every* element of B commutes with $\tau_1(g)$ we have that

$$U \circ \tau_1(g) \circ S = U \circ S \circ \tau_1(g)$$

$$= T \circ \tau_1(g) = \tau_2(g) \circ T = \tau_2(g) \circ U \circ S$$

and thus $U \circ \tau_1(g)$ equals $\tau_2(g) \circ U$ on range of S. Since Ran$S$ is $\tau_1[G]$-invariant we have Ran$S = \emptyset, H$ by assumption of irreducibility. So we will be done once we see that S is non-degenerate. However, $T * T \neq 0$ by assumption

$$\langle T * \circ T(e), e \rangle = \langle T(e), Te \rangle$$

$$= \langle \delta e, \delta e \rangle = \neq 0.$$

□

1.3 Corollary Every $\approx$-equivalence calss is F_σ.

§2 A lemma

From [5]:

2.1 Theorem Let G_0 be a Polish group. Let S_∞ be the infinite symmetric group, let G_1 be a closed subgroup of S_∞. Suppose X, Y are Polish spaces on which G_0 and G_1 respectively act continuously. Suppose that

(i) every G_0-orbit in X is dense;
(ii) every G_0-orbit in X is meager;
(iii) for all $x \in X$, *the local orbits of* x are somewhere dense; that is to say, if V is an open neighbourhood of 1_{G_0}, U is an open set containing x, and if $O(x, U, V)$ is the set of all $\hat{x} \in [x]_{G_0}$ such that there is a finite sequence $(x_i)_{i \leq k} \subset U$ such that $x_0 = x$, $x_k = \hat{x}$, and each $x_{i+1} \in V \cdot x_i$, then the closure of $O(x, U, V)$ contains an open set.

Then there is no Borel function

$$\theta : X \to Y$$

such that for all $x_1, x_2 \in X$

$$\exists g \in G_0(g \cdot x_1 = x_2) \Leftrightarrow \exists g' \in G_1(g' \cdot \theta(x_1) = \theta(x_2)).$$

2.2 Lemma Let G be a countable group that is not abelian-by-finite – that is to say, there is no abelian subgroup $G_0 < G$ with finite index ($|\{gG_0 : g \in G\}| < \aleph_0$). Then there is a U_∞-invariant G_δ set X on which (i), (ii), and (iii) of 2.1 hold – that is to say, every U_∞-orbit in X is dense, meager, and has all its local orbits somewhere dense.
Proof. By [6], [4] there is no Borel function θ_0 :Irr$(G, H) \to \mathbb{R}$ such that for all $\tau_1, \tau_2 \in \mathrm{Irr}(G, H)$

$$\tau_1 \approx \tau_2 \Leftrightarrow \theta_0(\tau_1) = \theta_0(\tau_2).$$

Therefore in particular, the closure of an orbit does not provide a complete invariant, and so we may find $\sigma_1, \sigma_2 \in \mathrm{Irr}(G, H)$ such that

$$\sigma_1 \not\approx \sigma_2$$

but

$$\overline{U_\infty \cdot \sigma_1} = \overline{U_\infty \cdot \sigma_2}.$$

Let $C = \{\tau \in \mathrm{Irr}(G, H) : \overline{U_\infty \cdot \tau} = \overline{U_\infty \cdot \sigma_1}\}$. C is a G_δ subset of $\mathrm{Irr}(G, H)$ and hence Polish.

Now on we shall restrict our attention to the Polish space C. From 1.3 we have every orbit F_σ, and thus, by the Baire category theorem no orbit in C is G_δ. So by 2.2.3 [1], every orbit in C is meager. By definition every orbit is dense, so we are just left to show every *local* orbit somewhere dense in the sense of 2.1(iii) above.

Let $U \subset C$ be (relatively) open. Let $V \subset U_\infty$ be an open symmetric neighbourhood of 1_{U_∞}. Let $\tau_1 \in U$. Let $\{e_i : i \in \mathbb{N}\}$ be an orthonormal basis for H. Let $\delta > 0$, $A \subset G$ of (finite) size N, and $n \in \mathbb{N}$ be such that

$$\{\tau \in C : \langle \tau(g)(e_i), e_j\rangle - \langle \tau_1(g)(e_i), e_j\rangle| < \delta \text{ for all } i, j \leq n, g \in A\} \subset U$$

and let

$$U_0 = \{\tau \in C : \langle \tau(g)(e_i), e_j\rangle - \langle \tau_1(g)(e_i), e_j\rangle| < \delta/2 \text{ for all } i, j \leq n, g \in A\} \subset U$$

and

$$\{T \in U_\infty : \langle T(e_i), e_i\rangle > 1 - \delta \text{ all } i \leq n\} \subset V.$$

Choose $U_1 \subset U_0$ non-empty. We seek $O(\tau_1, U, V) \cap U_1 \neq \emptyset$. Choose $\tau_2 \in U_1$ with $\tau_1 \not\approx \tau_2$. Choose $T_i \in U_\infty$ with

$$T_i \cdot \tau_1 \to \tau_2.$$

Let $\{f_1, ..., f_k\}$ be an orthonormal basis for the subbasis generated by $\{(\tau_1^*(g))(e_i), (\tau_1(g))(e_i), e_i : g \in A, i \leq n\}$. By 1.2, for all $j_1, j_2 \in \mathbb{N}$ there is some $\epsilon > 0$ such that there are only finitely many $i \in \mathbb{N}$ with

$$|\langle e_{j_1}, T_i(f_{j_2})\rangle > |\epsilon.$$

Thus for $\delta_0 > 0$ arbitrarily small we may find i such that for all $j_1 \leq n, j_2 \leq k$ and $g \in A$

$$|\langle e_{j_1}, T_i(f_{j_2})\rangle| < \delta_0/k,$$

$$T_i \cdot \tau_1 \in U_1.$$

Claim: Let $f_1', ..., f_l'$ be a set of unit vectors in H, and let H_0 be a finite dimensional subspace of H with orthonormal basis $\hat{f}_1, ..., \hat{f}_{k'}$. If for each $i \leq k'$

$$\sum_{j \leq l} |\langle f_i', \hat{f}_j\rangle| < \epsilon,$$

then there is $T \in U_\infty$ such that for each $\langle f_i', T(f_i')\rangle$ is bigger than $1 - \epsilon$ and each $|\langle \hat{f}_i, T(f_i')\rangle| = 0$.

Proof of claim. Let $\{\hat{f}_i : i \in \mathbb{N}\}$ be some orthonormal basis of H extending the orthonormal basis of H_0 and such that $\{\hat{f}_i : i \leq k' + l\}$ provides a basis for the subspace generated by $H_0 \cup \{f_1', ..., f_{k'}'\}$. Then choose T so that it interchanges $\{\hat{f}_1, ..., \hat{f}_{k'}\}$ with $\{\hat{f}_{1+l+k'}, ..., \hat{f}_{k'+l+k'}\}$ and use that

$$\sum_{j \in \mathbb{N}} |\langle f_i', \hat{f}_j\rangle|^2 = 1.$$

(□Claim)

Applying the claim to the orthonormal set $\{T_i(f_j) : j \leq k\}$ we can find $T' \in U_\infty$ such that

(i) for all $j_1, j_2 \leq n$ and $g \in A$

$$\langle T'(e_{j_1}), T_i((e_{j_2}))\rangle = 0,$$
$$\therefore \langle T_i^{-1}(T'(e_{j_1})), e_{j_2}\rangle = 0,$$
$$\therefore \langle \tau_1(g)(T_i^{-1}(T'(e_{j_1}))), \tau_1(g)(e_{j_2})\rangle = 0,$$
$$\langle T'(e_{j_1}), T_i(\tau_1(g)(e_{j_2}))\rangle = 0,$$
$$\therefore \langle T_i^{-1}(T'(e_{j_1})), \tau_1(g)(e_{j_2})\rangle = 0,$$
$$\langle T'(e_{j_1}), T_i(\tau_1^*(g)(e_{j_2}))\rangle = 0$$
$$\therefore \langle \tau_1(g) \circ T_i^{-1} \circ T'(e_{j_1}), e_{j_2}\rangle = 0;$$

(ii) for all $j \leq n$

$$|\langle T'(e_j), e_j\rangle| > 1 - \delta_0.$$

From (ii) we obtain that

(iii) for all $j \leq n$

$$|\langle (T')^{-1}(e_j), e_j\rangle| > 1 - \delta_0.$$

and so $(T')^{-1} \in V$.

For future notation, let $\tau_3 = (T')^{-1} \circ T_i \cdot (\tau_1)$. Now note that for all $j_1, j_2 \leq n$

$$\langle \tau_3(g)(e_{j_1}), e_{j_2}\rangle$$
$$= \langle (((T')^{-1} \cdot (T_i \cdot (\tau_1)))(g)(e_{j_1})), e_{j_2}\rangle$$
$$= \langle (((T')^{-1}(T_i \cdot (\tau_1)))(g))(T'(e_{j_1}))), e_{j_2}\rangle$$
$$= \langle (T_i \cdot \tau_1(g))(T'(e_{j_1})), T'(e_{j_1})\rangle =$$
$$\langle (T_i \cdot \tau_1(g))(\delta_1 e_{j_1}), \delta_2 e_{j_2}\rangle + \langle (T_i \cdot \tau_1(g))(\delta_3 f), \delta_2 e_{j_2}\rangle$$
$$+\langle (T_i \cdot \tau_1(g))(\delta_1 e_{j_1}), \delta_4 f'\rangle + \langle (T_i \cdot \tau_1(g))(\delta_3 f), \delta_4 f'\rangle$$

where

$$1 > \delta_1 = |\langle T'(e_{j_1}), e_{j_1})\rangle| > 1 - \delta_0,$$
$$1 > \delta_2 = |\langle T'(e_{j_2}), e_{j_2})\rangle| \sqrt{(1 - \delta_0)},$$

and

$$(\delta_1)^2 + (\delta_3)^2 = 1,$$

$$(\delta_2)^2+(\delta_4)^2=1$$
$$f\perp e_{j_1},$$
$$f'\perp e_{j_2}.$$

Thus, since each $T_i\cdot\tau_1(g)$ is an isometry, for suitably small δ_0 we can obtain

$$\langle(((T')^{-1}\cdot(T_i\cdot(\tau_1)))(g)(e_{j_1})),e_{j_2}\rangle-\langle T_i\cdot\tau_1(g))((e_{j_1})),e_{j_2}\rangle|<\delta/2,$$

and thus $\tau_3=(T')^{-1}\circ T_i\cdot(\tau_1)\in U$. Since $T_i\cdot(\tau_1)\in O((T')^{-1}\circ T_i\cdot(\tau_1),U,V)$, it suffices to show that $(T')^{-1}\circ T_i\cdot(\tau_1)\in O(\tau_1,U,V)$.

Let $\{f_j:j\in\mathbb{N}\}$ be an orthonormal basis extending $\{e_j,(T_j)^{-1}\circ T'(e_j):j\leq n\}$. For $\varphi\in[0,2\pi]$ define $S_\varphi\in U_\infty$ by the requirement that for all $j\leq n$

$$S_\varphi(e_j)=\cos(\varphi)(e_j)-\sin(\varphi)((T_i)^{-1}\circ T'(e_j)),$$

$$S_\varphi((T_i)^{-1}\circ T'(e_j))=\sin(\varphi)(e_j)+\cos(\varphi)((T_i)^{-1}\circ T'(e_j)),$$

while for f_j outside $\{e_j,(T_i)^{-1}\circ T'(e_j):j\leq n\}$ we just have

$$S_\varphi(f_j)=f_j.$$

Since $(S_{\pi/2})^{-1}(e_j)=(T_i)^{-1}\circ T'(e_j)$ for each $j\leq n$ it follows that $(T')^{-1}\circ T_i\circ(S_{\pi/2})^{-1}\in V$. Thus if we can only show $S_{\pi/2}\cdot\tau_1\in O(\tau_1,U,V)$ then we obtain $\tau_3(=_{df}(T')^{-1}\circ T_i\cdot(\tau_1))$ is in $O(S_{\pi/2}\cdot\tau_1,U,V)$, and hence $T_i\cdot\tau_1\in O(\tau_3,U,V)$ and the transitivity of this relation $O(\cdot,U,V)$ yields $T_i\cdot\tau_1\in O(\tau_1,U,V)$ as required.

Then for all $j_1,j_2\leq n,g\in A$ we have

$$|\langle\tau_1(g)(S_\varphi^{-1}(e_{j_1})),S_\varphi^{-1}(e_{j_2})\rangle|$$

$$=|\langle\tau_1(g)(\cos(-\varphi)(e_{j_1})-\sin(-\varphi)((T_i)^{-1}\circ T'(e_{j_1})),\cos(-\varphi)(e_{j_2})-\sin(-\varphi)((T_i)^{-1}\circ T'(e_{j_2})\rangle|$$
$$=|\langle\cos^2(-\varphi)\langle\tau_1(g)(e_{j_1}),e_{j_2}\rangle+\sin^2(-\varphi)\langle\tau_1(g)((T_i)^{-1}\circ T'(e_{j_1})),(T')^{-1}\circ T_i(e_{j_2})\rangle|$$

by orthogonality of the sets $\{e_j,(\tau_1(g))(e_j):j\leq n,g\in A\}$ and $\{(T_i)^{-1}\circ T'(e_j),(\tau_1(g))(T_i)^{-1}\circ T'(e_j)):j\leq n,g\in A\}$ as expressed at (i) above. Thus

$$|\langle(S_\varphi\cdot(\tau_1))(g)(e_j),e_i\rangle-\langle(\tau_1(g))(e_{j_1}),e_{j_2}\rangle|=$$

$$=|\langle\cos^2(-\varphi)\langle\tau_1(g)(e_{j_1}),e_{j_2}\rangle+\sin^2(-\varphi)\langle\tau_1(g)((T_i)^{-1}\circ T'(e_{j_1})),(T_i)^{-1}\circ T'(e_{j_2})\rangle$$
$$-\langle(\tau_1)(g)(e_{j_1}),e_{j_2}\rangle|$$

But this final inequality is bounded by an expression of the form

$$\lambda|\langle\tau_1(g)((T_i)^{-1}\circ T'(e_{j_1})),(T_i)^{-1}\circ T'(e_{j_2})\rangle-\langle(\tau_1)(g)(e_{j_1}),e_{j_2}\rangle|$$

$$+(1-\lambda)|\langle\tau_1(g)(e_{j_1}),e_{j_2}\rangle-\langle(\tau_1)(g)(e_{j_1}),e_{j_2}\rangle|,$$

where $\lambda\in[0,1]$, and hence must be less than or equal to the bound δ obtained at the extreme points. Thus each $S_\varphi\cdot(\tau_1)$ is in U, and so we have traced out a continuous path of group elements connecting τ_1 to $S_{\pi/2}\cdot\tau_1$ inside U, and are therefore done. $\square$

The further non-classifiability results alluded to in the introduction are now a consequence of 2.1 and standard facts from the theory of equivalence relations – as may be found in [1].

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