The Smooth/Non-Smooth Distinction

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§ 0 Definitions

0.1 Definition A topological space X is said to be *Polish* if it is separable and allows a complete metric. The *Borel sets* are those subsets of X appearing in the σ-algebra generated by the open sets.

Frequently we are interested in question relating to the Borel structure alone. Thus a *standard Borel space* is a Polish space stripped down to its Borel structure.

In particular, $(X, \mathcal{B})$ is a standard Borel space if there is some complete separable metric on X which gives rise to $\mathcal{B}$ as its σ-algebra of Borel sets.

0.2 Examples: (i) $\mathbb{R}$; $\mathbb{C}$; $C([0,1])$ in the sup norm topology are Polish spaces; $C(\mathbb{R})$ is not.

(ii) $\mathbb{N}$ in the discrete topology is Polish; $\aleph_1$ is not.

(iii) $L^2([0,1], \text{Lebesgue})$, and indeed any separable Banach space is Polish.

0.3 Examples: (i) U_∞ the "unitary group" [i.e. collection of all transformations which respect the inner product] of Hilbert space is a Polish group, in the strong operator topology, with subbasic open sets of the form $\{T : \|T(\xi) - \zeta\| < \epsilon\}$. In fact, this is a *Polish group*: A topological group that is Polish as a space.

(ii) M_∞, the group of measure preserving transformations of the unit interval. This can be thought of as a closed subgroup of $U_\infty(L^2([0,1], \text{Lebesgue})$, the group of unitary transformations of $L^2([0,1], \text{Lebesgue})$; thus it is Polish by the last example.

(iii) Group structures on $\mathbb{N}$ in the topology of pointwise convergence for the group operations. (Subbasic open sets look like $\{G : n \cdot^G m = k\}$.) Can be shown to be Polish.

0.4 Examples: (i) Open subsets of the complex plane, $\mathcal{O}(\mathbb{C})$; with say the σ-algebra generated by sets of the form $\{U : U \cap V \neq \emptyset\}$ for some $V \subset \mathbb{C}$ open. This turns out to be a standard Borel space: we can give a Polish topology with this above σ-algebra as its Borel algebra.

Equally, $\mathcal{F}(\mathbb{C})$, the closed subsets of $\mathbb{C}$, or any Polish space in fact, is a standard Borel space in the natural Borel structure.

(ii) Any Borel subset of a Polish space is again a standard Borel space, in the algebra of all Borel subsets.

0.5 Theorem (Classical) Any standard Borel space is either finite, of size $\aleph_0$, or of size $2^{\aleph_0}$. Any two standard Borel spaces of the same cardinality are isomorphic – that is to say there is a Borel bijection between them which allows a Borel inverse.

0.6 Definition A function between standard Borel spaces is said to be *Borel* if it is Borel measurable – i.e. it pulls Borel sets back to Borel sets.

§1 Smooth/Non-smooth

1.1 Definition Let X be a Polish space, or even just a standard Borel space. An equivalence relation E on X is said to be *smooth* if

$$X/E = \{[x]_E : x \in X\},$$

the set of equivalence classes, in the quotient Borel structure generated by all E-invariant Borel sets, is a subset of some standard Borel space.

1.2 Lemma The following are equivalent for E on X:

(i) E is smooth;

(ii) there is some standard Borel space Y and Borel function $\rho : X \to Y$ such that

$$x_1 E x_2 \Leftrightarrow \rho(x_1) = \rho(x_2);$$

(iii) there is a countable sequence of Borel E-invariant subsets of X, $(B_n)_n$, such that

$$x_1 E x_2 \Leftrightarrow \forall n (x_1 \in B_n \Leftrightarrow x_2 \in B_n).$$

1.3 Remark Using that any two uncountable Borel spaces are isomorphic, in (ii) we can actually assume $Y = \mathbb{R}$; thus smoothness is equivalent to the existence of some Borel ρ which "assigns reals as complete invariants."

§2 Examples

2.1 Example $X = R^2$ and $(x_1, y_1)E(x_2, y_2)$ if $y_1 = y_2$. Then X/E can naturally be identified with $\mathbb{R}$. So it is certainly smooth.

2.2 Example $U(\mathbb{C}^n)$, unitary operators on $\mathbb{C}^n$. E=similarity. (I.e. they are the same operator "up to a change in basis").

Here we can completely classify by eigenvalues, listed with multiplicities in increasing order. It can all be calculated in a Borel way, and thus we obtain a Borel function

$$\mathcal{I} : U(\mathbb{C}^n) \to \mathbb{C}^n$$

such that T_1 similar T_2 if and only if

$$\mathcal{I}(T_1) = \mathcal{I}(T_2).$$

2.3 Example: $\mathbb{Z}$ acting on $\mathbb{R}/\mathbb{Z}$ by

$$n \cdot \bar{x} = n\sqrt{2} + \bar{x} \mod 2.$$

This is in fact not smooth.

Proof Let $\theta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be a Borel function with

$$\bar{x}_1 E_{\mathbb{Z}} \bar{x}_2 \Rightarrow \theta(\bar{x}_1) = \theta(\bar{x}_2).$$

Let $C \subset \mathbb{R}/\mathbb{Z}$ be a comeager set on which $\theta|_C$ is continuous. (Any Borel function is continuous on a comeager set; this follows from all Borel sets having property of Baire.)

Let

$$C_\infty = \bigcup_{n \in \mathbb{Z}} n \cdot C.$$

Again a comeager set, but now invariant under the $\mathbb{Z}$-action.

Choose some $\bar{x} \in C_\infty$. Then $[\bar{x}]_{\mathbb{Z}} = \mathbb{Z} \cdot \bar{x}$ is dense in C_∞ and $\theta|_{\mathbb{Z}\cdot\bar{x}}$ is constant.

Hence by continuity of $\theta|_{C_\infty}$ and density $\bar{x}$'s orbit, θ is constant on the comeager set C_∞.

$\square$

2.4 Fact Let G be a group acting by homeomorphisms on a Polish space X with orbit equivalence relation E_G. If every orbit dense and meager, then E_G is not smooth.

It turns out there is a funny kind of converse. (More on this later, if time allows.)

2.5 Fact Let G be a Polish group acting continuously on a Polish space X. Assume the orbit equivalence relation E_G is Borel as a subset of $X \times X$.

Then exactly one of the following two things happens:

(I) E_G is smooth;

(II) there is a Polish G-space Y (action of G on Y continuous), with every orbit dense and meager, and a continuous G-embedding from Y to X.

2.6 Definition For G a countable group and H a separable Hilbert space, let $\mathrm{Rep}(G, H)$ be the space of representations of G on H – that is to say the space of all homomorphisms from G to $U(H)$, the unitary group of H, with the topology of pointwise convergence.

$\mathrm{Irr}(G, H)$ the subspace of irreducible representations. (Can be shown to be Polish).

$\sim$ the equivalence relation of unitary equivalence – that is to say, two representations are equivalent if they are the same up to some unitary "change of coordinates".

2.7 Example Here one has $\sim$ on $\mathrm{Irr}(\mathbb{F}_2, H)$, the irreducible representations of the free group, non-smooth in general.

But for G abelian-by-finite, the "dual" is always smooth. In fact, one can show that the irreducible representations are all finite dimensional, and hence the unitary group responsible for any possible change in coordinates is compact, and such compact groups always give rise to smooth equivalence relations.

2.9 Fact Let G be a compact Polish group acting continuously on a Polish space X. Then E_G is smooth.

Idea Every orbit is compact, and hence closed. Then can show

$\rho : X \to \mathcal{F}(X)$ (= standard Borel space of closed subsets)

$$x \mapsto [x]_G$$

is Borel.

2.10 Aside Slawek Solecki has recently shown a converse: The above fact characterizes compactness for Polish groups: A Polish group is compact if and only if all its orbit equivalence relations are smooth.

§3 Smoothness as a measure of classifiability

3.1 Example George W. Mackey is sometimes associated with the view that we can only hope to classify the irreducible representations (up to equivalence, where as before we consider conjugation by the unitary group) when they are smooth.

His published writings seem to be somewhat more equivocal than this. For instance, in his survey paper *Infinite dimensional group representations,* **Bulletin of the AMS,** vol. 69(1963), he writes only

> *...there is no known case in which one has non-type I representations* [this corresponds to the non-smooth case] *and can describe (in any reasonable sense) the set of equivalence classes of the irreducibles...*[p. 633.]

3.2 Example A parallel problem is that of classifying measure preserving transformations, considered up to conjugacy – i.e. a measure respecting "relabellings" of the underlying space. (See Matt Foreman's talk.) In *Borel structure and invariants for measurable transformations*[1] Jacob Feldman writes:

> *One view is that a reasonable invariant for a set X of objects must be realizable as a real-valued Borel function for some naturally arising Borel structure on X...By a* complete *set of conjugacy invariants will be meant a countable family of conjugate invariants which, between them, can distinguish between any pair of nonconjugate transformations. (Obviously, one wants countability; else, why not use indicator functions for the conjugacy classes?)* [p. 390]

This notion amounts to smoothness. Feldman goes on to show conjugacy of m.p.t.s is non-smooth, and then therefore concludes they are unclassifiable.

[1] Proceedings of the AMS, vol. 46(1974)

§4. Harrington-Kechris-Louveau

4.1 Definition We consider the Polish space of infinite binary sequences:

$$\{0,1\}^{\mathbb{N}}$$

in the product topology. This is a compact metric space, and hence Polish. We define E_0 to be the equivalence of eventual agreement:

4.2 Fact E_0 is non-smooth.

One way to see this is to apply our earlier criterion for non-smoothness. If we take the group $G = \mathbb{Z}_2^{<\mathbb{N}}$, elements of the infinite product group

$$\prod_{\mathbb{N}} Z_2$$

with finite support, and we let this group act by addition mod 2 on each coordinate, then E_0 is the resulting orbit equivalence relation. Every equivalence class is dense and countable $\therefore$ meager.

4.3 Theorem (Harrington-Kechris-Louveau) Let E be a Borel equivalence relation on a standard Borel space X. Then exactly one of:

(I) E is smooth; or

(II) "E_0 embeds into E".

By (II) I mean there is a Borel $\rho : \{0,1\}^{\mathbb{N}} \to X$ such that for all $\vec{x}, \vec{y} \in \{0,1\}^{\mathbb{N}}$

$$\vec{x} E_0 \vec{y} \Leftrightarrow \rho(\vec{x}) E \rho(\vec{y}).$$

§5 Life without smoothness

Some examples of classification theorems without smoothness. These suggest that it might make sense to consider other *degrees of classifiability* other than the smooth/non-smooth distinction.

5.1 Example Consider $\mathrm{Hom}^+([0,1])$, the orientation preserving homeomorphisms of the unit interval. Here there is a kind of folklore theorem describing these homeomorphisms up to conjugacy.

For $\pi : [0,1] \to [0,1]$ a homeo with $\pi(0) = (0), \pi(1) = 1$, we take the maximal open intervals on which π is increasing ($\pi(x) > x$), decreasing ($\pi(x) < x$), or the identity ($\pi(x) = x$).

A complete invariant is given symbolically by describing the ordering between these intervals along with the behavior (increasing, decreasing, identity) for π.

Here the conjugacy equivalence relation is non-smooth. Yet we have in effect classified the homeomorphisms by countable linear orderings equipped with some unary predicates ($P_{\text{increasing}}, P_{\text{decreasing}}, P_{\text{identity}}$).

5.2 Example $U_\infty(L^2([0,1], \text{Lebesgue})$.

Infinite dimensional unitary operators considered up to conjugacy.

Unlike the finite dimensional case, we can not in general diagonalize. However it turns out that any such operator will be unitary equivalent to the rotation operator on the disjoint sum of infinitely sum of the unit circle in $\mathbb{C}$ equipped with some measure; and two operators will be conjugate if and only if the measures agree up to absolute continuity.

This "classification by measure up to absolute continuity" is generally considered to be some kind of solution to the classification problem for U_∞, despite the non-smoothness of the conjugacy equivalence relation.

5.3 Example A more recent example (from C^* algebras and topological dynamics). Consider minimal (no non-trivial closed invariant sets) homeomorphisms of $\{0,1\}^{\mathbb{N}}$ considered up to conjugacy of orbit equivalence relations: Thus $f_1 E f_2$ if there is some homeomorphism g conjugating their orbits:

$$\forall \vec{x}(g[\{f_1^{\ell}(\vec{x}) : \ell \in \mathbb{Z}\}] = \{f_2^{\ell}(g(\vec{x})) : \ell \in \mathbb{Z}\}).$$

T. Giordano, I.F. Putnam, and C.F. Skau [2] produce countable ordered abelian groups which, considered up to isomorphism, act as complete invariants.

[2] *Topological orbit equivalence and C^*-crossed products* (**Journal Reine und Angewandt Mathematische,** vol. 469(1995), pp. 51-111)

§6 Other notions of classifiability

6.1 Definition For $\mathcal{L}$ a countable language we consider the standard Borel space of all $\mathcal{L}$-structures on $\mathbb{N}$, $\mathrm{Mod}(\mathcal{L})$.

Thus in the case that $\mathcal{L}$ consists of a single binary relation R we have $\mathrm{Mod}(\mathcal{L})$ the collection of all directed graphs on $\mathbb{N}$.

It turns out that all the plausible suggestions for a standard Borel structure on $\mathrm{Mod}(\mathcal{L})$ are the same.

6.2 Definition E on X is said to admit *classification by countable structures* if there is a Borel function

$$\theta : X \to \mathrm{Mod}(\mathcal{L})$$

(some countable language $\mathcal{L}$) such that

$$x_1 E x_2 \Leftrightarrow \theta(x_1) \cong \theta(x_2).$$

Equally we can ask when E can be classified by measures up to absolute continuity (Gelfand, the spectral theorem for infinite dimensional operators);

or countable sets of reals (Halmos and von Neumann, for "discrete spectrum" measure preserving transformations);

or reduction to an equivalence relation in which every equivalence class is countable (compare Kechris, and Feldman and Moore, for locally compact group actions);

or reduction to the continuous action of a Polish group (and for this last one we know just about nothing).

6.3 Theorem (Farah) There is nothing at all like Harrington-Kechris-Louveau for Borel equivalence relations.

(a) No canonical bad guy F such that E classifiable by countable structures if and only if no way to "embed" F into E;

(b) no small collection $\{F_1, ..., F_k\}$ of disjunctively canonical baddies such that E classifiable by countable structures if and only if no F_i embeds into E;

(c) in fact no countable or even less than continuum size set of canonical villains, even for E induced by the continuous action of a Polish group.

(d) And worse. (There are just not even enough "minimally" non-classifiable Borel equivalence relations.)

§7. A dynamical analysis of classification by countable structures

7.1 Theorem Let G be a Polish group and X a Polish space. Suppose that

(i) every orbit is dense;

(ii) every orbit is meager;

(iii) for $x \in X$, *the local orbits of* x are all somewhere dense; that is to say, if V is an open neighborhood of 1_G, U is an open set containing x, and if $O(x, U, V)$ is the set of all $\hat{x} \in [x]_G$ such that there is a finite sequence $(x_i)_{i \leq k} \subset U$ such that $x_0 = x$, $x_k = \hat{x}$, and each $x_{i+1} \in V \cdot x_i$, then the closure of $O(x, U, V)$ contains an open set.

Then there is no Borel (or even Baire measurable)

$$\theta : X \to \text{ Mod}(\mathcal{L})$$

such that for all $x, y \in X$

$$x E_G y \Leftrightarrow \theta(x) \cong \theta(y).$$

This seems to be the right way to analyze classification by countable structures (at least for equivalence relations induced by Polish group actions). (Note parallel to folklore theorem for non-smoothness.)

For instance:

7.2 Theorem Let G be a Polish group and X a Polish space with E_G Borel. Then exactly one of

(I) E_G is classifiable by countable structures (and in fact, by a reduction θ where the saturation of the image of θ is Borel, and [other desirable properties] BLAH BLAH BLAH etc etc etc);

(II) there is a Polish G-space Y satisfying (i), (ii), and (iii) from 7.1

Remark: Here, unlike most theorems of this kind, it is open whether we can prove the same result even for E_G non-Borel. If this *were* true then it would for instance reduce the "topological Vaught conjecture" to "Vaught's conjecture" for $\mathcal{L}_{\omega_1,\omega}$ [countably infinitary logic].

7.3 Aside Since then it has proved possible to provide a similar kind of analysis for when an orbit equivalence relation allows reduction to an equivalence relation where all the class are countable.

On the other hand, for various other natural dividing lines virtually nothing is known. For instance, reducibility of an arbitrary Borel equivalence relation to a Polish group action. Or reduction to measures up to absolute continuity.