Cardinality and Equivalence Relations

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1 The broad outline of these talks

1. The basic idea of cardinality. (At the beginning at least, make very few mathematical assumptions of the audience.)

2. Slight modifications of this concept can lead to a spectrum of notions which resemble the notion of *size* or *cardinality*. ("Borel cardinality" or "effective cardinality", or even cardinality in $L(\mathbb{R})$).

3. Some of these notions are implicit in mathematical activity outside set theory. (For instance the work on *dual* of a group by George W. Mackey, or scattered references to *effective cardinality* in the writing of Alain Connes.)

4. In the 80's set theorists such as Harvey Friedman, Alexander Kechris, among others, began to suggest a way to explicate this idea around the concept of *Borel reducibility*

5. The outpouring of activity in this area over the last 15 -20 years. Su Gao, Alexander Kechris, Alain Louveau, Slawek Solecki, Simon Thomas, Boban Velickovic, among many others.

6. Dichotomy theorems for Borel equivalence relations (for instance, the Harrington, Kechris, and Louveau extension of Glimm-Effros).

7. Specific classification problems in mathematics (for instance the finite rank torsion free abelian groups)

8. Dynamical methods to analyze equivalence relations in the absence of reasonable dichotomy theorems (for instance turbulence)

9. Interactions between the theory of countable equivalence relations and the theory of orbit equivalence

Definition Let X be a set. $E \subset X \times X$ is said to be an *equivalence relation* if

1. it is reflexive (xEx all $x \in X$)
2. symmetric (xEy implies yEx)
3. transitive (xEy along with yEz implies xEz).

Example 1. Let X be the set of people. Let E be the equivalence relation of having the same height.

2. Let X again be the set of all people. Let E be the equivalence relation of having the same mother.

3. Let X be the set of all planets. Let E be the equivalence relation of being located in the same universe.

Definition For E an equivalence relation on a set X, a *complete invariant* or *classification* of E is a "reasonable" or "explicit" or "natural" function

$$f : X \to I$$

such that for all $x_1, x_2 \in X$ we have

$$x_1 E x_2$$

if and only if

$$f(x_1) = f(x_2).$$

This really a pseudo-definition, held completely hostage to how we best make sense of "reasonable" or "explicit".

Part of the story are the attempts to make this idea more precise, and this in turn connects in with variations on the concept of *cardinality*

Example 1. For E the equivalence relation of having the same height we clearly do have a natural classification. Namely: Height measured in feet and inches. Assign to each x in the set of people the height measured in feet and inches as $f(x)$.

2. For E the equivalence relation of having the same mother, it would likewise seem that there is a very explicit invariant: Assign to each person x their mother as $f(x)$.

3. For E being the equivalence relation of being in the same universe, the situation is not *so* clear.

   We could try to for instance assign to each planet the actual universe in which they live, but it is not clear that this is doing much more than assigning to each x the entire set of all y for which xEy.

Definition A function

$$f : X \to Y$$

is an *injection* if for all $x_1, x_2 \in X$,

$$x_1 \neq x_2$$

implies

$$f(x_1) \neq f(x_2).$$

In other words, an injection is a function which sends distinct points to distinct images.

Example 1. Let X be the set of people and let Y be the set of all human heads. Let

$$f : X \to Y$$

assign to each x its head. This is presumably (barring some very unusual case of siamese twins) an injection.

2. Let X be the set of all people who have ever lived and let

$$f : X \to X$$

assign to each X its mother. (Here we are ignoring minor chicken-egg questions about the first ever mother). This is clearly not an injection. There are cases of distinct people having the same mother.

Definition A function

$$f : X \to Y$$

is a *surjection* if for each $y \in Y$ there is some $x \in X$ with

$$f(x) = y.$$

A function which is both injective and surjective is called a *bijection*.

In rough terms, a bijection between X and Y is a way of marrying all the X's off with all the Y's with no unmarried Y's left over. (Here assuming no polygamy allowed).

Definition Given two sets X and Y, we say that the *cardinality of X is less than the cardinality of Y*, written

$$|X| \leq |Y|,$$

if there is an injection

$$f : X \to Y.$$

Theorem 3.1 *(Schroeder-Bernstein) If*

$$|X| \leq |Y|$$

and

$$|Y| \leq |X|,$$

then there is a bijection between the two sets.

Intuitively not so outrageous. If we say that a set has size 4, and count of the elements 1, 2, 3, 4, we are implicitly placing that set in a bijection with the set $\{1, 2, 3, 4\}$.

Definition A set α is an *ordinal* if:

1. it is transitive – $\beta \in \alpha$ along with $\gamma \in \beta$ implies $\gamma \in \alpha$; and
2. it is linearly ordered by $\in$ – if β, γ are both in α, then
$$\beta \in \gamma,$$
or
$$\gamma \in \beta,$$
or
$$\gamma = \beta.$$

$\emptyset$, the set having no members, which set theorists customarily identify with 0.

$1 = \{0\}$, the set whose only member is 0.

$2 = \{0, 1\}$, gives the usual set theoretical definition of 2. Then we keep going with $3 = \{0, 1, 2\}$, $4 = \{0, 1, 2, 3\}$, and so on.

We reach the first infinite ordinal with the set of natural numbers:

$$\omega = \{0, 1, 2, ...\}.$$

This again leads to a whole new ladder of ordinals:

$$\omega + 1 = \{0, 1, 2, ..., \omega\},$$
$$\omega + 2 = \{0, 1, 2, ..., \omega, \omega + 1\},$$
$$\omega + 3 = \{0, 1, 2, ..., \omega, \omega + 1, \omega + 2\},$$

and onwards:

$$\omega + \omega = \{0, 1, 2, ..., \omega, \omega + 1, \omega + 2, ...\},$$

$$\omega+\omega+1 = \{0, 1, 2, ..., \omega, \omega+1, \omega+2, ..., \omega+\omega\},$$

$$\omega\times\omega = \{0, 1, 2, ..., \omega, \omega+1, ...\omega+\omega, ...\omega+\omega+\omega, ...\},$$

ad infinitum.

Lemma 4.1 *The ordinals themselves are linearly ordered: If* β, γ *are both ordinals, then*

$$\beta \in \gamma,$$

or

$$\gamma \in \beta,$$

or

$$\gamma = \beta.$$

The Axiom of Choice: (In effect) Every set can be placed in a bijection with some ordinal.

Definition An ordinal is said to be a *cardinal* if it cannot be placed in a bijection with any smaller ordinal.

So for instance, ω (or $\aleph_0$ as it is sometimes called in this context) is indeed a cardinal: The smaller ordinals are finite.

But not $\omega + \omega$: Define

$$f : \omega + \omega \to \omega,$$

$$n \mapsto 2n,$$

$$\omega + n \mapsto 2n + 1.$$

The consequences of all this for the theory of cardinality:

> Every set can be placed in a bijection with an ordinal.
>
> The cardinals are a linearly ordered set.

This is parallel to a form of utilitarianism: There is only one good (human happiness) and that good can be compared in order and amount.

There is only one notion of "size" and the cardinals can be compared in order.

5 Variations on the notion of cardinality

Recall that we set $|X| \leq |Y|$ if there is an injection from X to Y.

However, it does make sense to look at parallel definitions for classes of injections which are more narrow than simply the class of *all* injections.

Fix Γ some class of functions (for instance, all Borel functions, all functions in $L(\mathbb{R})$).

We might want to say that the Γ-*cardinality of* X *is less than equal to that of* Y if there is some injection $f \in \Gamma$ with

$$f : X \to Y.$$

This should be compared to the problem of classifying an equivalence relation.

Definition For E and equivalence relation on X, and $x \in X$, let $[x]_E$ be the set of all $y \in X$ with

$$xEy.$$

Then let X/E be the set of all equivalence classes:

$$\{[x]_E : x \in X\}.$$

For instance, if X is the set of all people, and E is the equivalence relation of having the same height, then

$$[\text{Greg}]_E$$

would be the set of all people who are 5'6" tall.

X/E would consist of the set of all "groupings" of people where they are categorized strictly by height.

Then a classification of E would in some form be an "appropriate" function

$$f : X \to I$$

which induces an injection

$$\hat{f} : X/E \to I$$

via letting

$$\hat{f}(C)$$

take the value

$$f(y)$$

for any y in the equivalence class C.

The assumption $xEy \Rightarrow f(x) = f(y)$ ensures $\hat{f}$ is well defined.

The assumption $f(x) = f(y) \Rightarrow xEy$ ensures $\hat{f}$ is an injection.

However it remains to resolve the definition of what we should count as an "appropriate" or reasonable class of possible functions f.

One particular approach to theory of what should count as *reasonable* functions for the point of view of classification has been suggested by work of Mackey on group representations dating back to the middle of the last century.

The story now becomes considerably more mathematical. I will start first by describing the problem Mackey considered, and only then the explication of *reasonable* his work suggests.

Definition For H a Hilbert space, $U(H)$ denotes the group of *unitary operators* on H. That is to say, the set of all linear bijections

$$T : H \to H$$

such that for all $u, v \in H$

$$\langle T(v), T(u) \rangle = \langle u, v \rangle.$$

Definition For G a countable group, a *unitary representation of* G (on the Hilbert space H) is a homomorphism

$$\varphi : G \to U(H)$$

$$g \mapsto \varphi_g.$$

We then say that a representation is *irreducible* if the only closed subspaces of H which are invariant under

$$\{\varphi_g : g \in G\}$$

are the trivial ones: 0 and H.

Let $\mathrm{Irr}(G, H)$ denote the collection of irreducible unitary representations of G on H.

Definition Two unitary representations

$$\varphi : G \to U(H_\varphi)$$

$$\psi : G \to U(H_\psi)$$

are *equivalent*, written

$$\varphi \cong \psi,$$

if they are *unitarily conjugate* in the sense that there is a unitary isomorphism

$$T : H_\varphi \to H_\psi$$

such that at every $g \in G$

$$\varphi_g = T \circ \psi_g \circ T^{-1}.$$

Theorem 6.1 *If $\varphi : \mathbb{Z} \to U(H)$ is an irreducible representation, then:*

1. *H is one dimensional; and*
2. *there is some $z \in \mathbb{C}$ such that at every $\ell \in \mathbb{Z}$ we have*
$$\varphi_\ell(v) = z^\ell \cdot v$$
all $v \in H$; and
3. *two distinct irreducible representations are equivalent if and only if they have the same $z \in \mathbb{C}$ associated to them.*

Thus we obtain a complete classification of irreducible representations of $\mathbb{Z}$ by their associated $z \in \mathbb{C}$. It is like we can view $\mathrm{Irr}(\mathbb{Z}, H)/\cong$ as a subset of $\mathbb{C}$.

It turns out that a similar, though somewhat more complicated, classification can be given for any abelian group.

In broad terms Mackey was led to ask: For which groups G can we *reasonably classify* the collection of equivalence classes

$$\mathrm{Irr(G,H)}/\cong$$

by points in some *concrete* space such as $\mathbb{C}$?

Before even groping towards an answer, one might first want to make the question precise.

Mackey did make the question precise, but this in turn requires the introduction of ideas lying at the foundations of descriptive set theory.

7 Polish spaces and Borel sets

Definition A topological space is said to be *Polish* if it is separable and it admits a complete compatible metric.

We then say that the *Borel sets* are those appearing in the smallest σ-algebra containing the open sets.

A set X equipped with a σ-algebra is said to be a *standard Borel space* if there is some choice of a Polish topology giving rise to that σ-algebra as its collection of Borel sets.

A function between two Polish spaces,

$$f : X \to Y,$$

is said to be *Borel* if for any Borel $B \subset Y$ the pullback $f^{-1}[B]$ is Borel.

Some examples

1. Any separable Hilbert space is Polish.
2. If H is a separable Hilbert space, then $U(H)$ is a closed subgroup of its isometry group and hence Polish.
3. If H is a separable Hilbert space and G is a countable group, then

$$\prod_G U(H)$$

is a countable product of Polish spaces and hence Polish.
4. Then the collection of unitary representations of G is a closed subspace of $\prod_G U(H)$, and hence Polish.
5. Finally it is a slightly non-trivial fact that the collection of irreducible representations is a G_δ subset of the collection of all representations, and hence Polish.

Definition (Mackey) An equivalence relation E on a Polish space X is *smooth* if there is a another Polish space Y and a Borel function

$$f : X \to Y$$

such that for all $x_1, x_2 \in X$ we have

$$f(x_1) = f(x_2)$$

if and only if

$$x_1 E x_2.$$

A countable group G has *smooth dual* if for any separable Hilbert space H, the equivalence relation $\cong$ on $\mathrm{Irr}(G, H)$ is smooth.

Question (Mackey, in effect) Which groups have smooth dual?

There are various answers in the literature to Mackey's question, including Glimm's solution of the *Mackey conjecture*, which applies not just to discrete groups but more generally lcsc topological groups.

In the case of discrete groups there is a completely algebraic characterization.

Theorem 7.1 *(Thoma) A countable group G has smooth dual if and only if it has an abelian subgroup with finite index.*

There are some very simple, from the point of view of Borel complexity, non-smooth equivalence relations.

Definition Equip

$$2^{\mathbb{N}} =_{\mathrm{df}} \prod_{\mathbb{N}} \{0, 1\}$$

with the product topology.

Let E_0 be the equivalence relation of eventual agreement on $2^{\mathbb{N}}$.

Lemma 7.2 E_0 *is not smooth.*

E_0 itself is F_σ as a subset of $2^{\mathbb{N}}$. The complexity of its classification problem has little do with any complexity it might have as a subset of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$.

It turns out that at the base of Glimm's proof of the Mackey conjecture is a theorem to the effect that under certain circumstances E_0 is the *canonical* obstruction to smoothness.

This was generalized by Ed Effros.

The final and ultimate generalization to the abstract theory of Borel equivalence relations was obtained by Leo Harrington, Alexander Kechris, and Alain Louveau in the late 1980's and in turn sparked a new direction of research in descriptive set theory.