Classification Problems in Mathematics

Greg Hjorth

greg.hjorth@gmail.com

University of Melbourne

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1 Recall

Definition The *cardinality of X is less than or equal to Y*,

$$|X| \leq |Y|,$$

if there is an injection from X to Y.

This might suggest a notion of "cardinality" where we restrict our attention to some restricted class of injections.

This in turn could relate to the idea that an equivalence relation E on a set X is in some sense *classifiable* if there is a "reasonably nice" or "natural" or "explicit" function

$$f : X \to I$$

which induces (via $x_1 E x_2 \Leftrightarrow f(x_1) = f(x_2)$) an injection

$$\hat{f} : X/E \to I.$$

In the context of unitary group representation a definition exactly along these lines was proposed by G. W. Mackey.

Definition (Mackey) An equivalence relation E on a Polish space X is *smooth* if there is a Polish space Y and a Borel function $f : X \to Y$ such that

$$x_1 E x_2 \Leftrightarrow f(x_1) = f(x_2).$$

In the way of context and background

1. Borel functions are considered by many mathematicians to be basic and uncontroversial, and concrete in a way that a function summoned in to existence by appeal to the axiom of choice would not.

2. Many classification problems can be cast in the form of understanding an equivalence relation on a Polish space

2 The entry of descriptive set theorists

In the late 80's two pivotal papers suggested a variation and generalization of Mackey's definition.

A Borel Reducibility Theory for Classes of Countable Structures, H. Friedman and L. Stanley, **The Journal of Symbolic Logic**, Vol. 54, No. 3 (Sep., 1989), pp. 894-914

A Glimm-Effros Dichotomy for Borel Equivalence Relations, L. A. Harrington, A. S. Kechris and A. Louveau, **Journal of the American Mathematical Society**, Vol. 3, No. 4 (Oct., 1990), pp. 903-928

Neither paper referenced the other, and yet they used the exact same terminology and notation to introduce a new concept.

Definition Given equivalence relations E and F on X and Y we say that E is Borel reducible to F, written

$$E \leq_B F,$$

if there is a Borel function

$$f : X \to Y$$

such that

$$x_1 E x_2 \Leftrightarrow f(x_1) F f(x_2).$$

In other words, the Borel function f induces an injection

$$\hat{f} : X/E \to Y/F.$$

The perspective of Friedman and Stanley was to compare various classes of countable structures under the ordering $\leq_B$. The Harrington, Kechris, Louveau paper instead generalized earlier work of Glimm and Effros in foundational issues involving the theory of unitary group representations.

Definition Let E_0 be the equivalence relation of eventual agreement on $2^{\mathbb{N}}$. For X a Polish space let $\mathrm{id}(X)$ be the equivalence relation of equality on X.

Thus in the above notation we can recast Mackey's definition of *smooth*: An equivalence relation E is smooth if for some Polish X we have

$$E \leq_B \mathrm{id}(X).$$

It turns out that for any uncountable Polish space X we have

$$\mathrm{id}(\mathbb{R}) \leq_B \mathrm{id}(X)$$

and

$$\mathrm{id}(X) \leq_B \mathrm{id}(\mathbb{R}).$$

Definition An equivalence relation E on Polish X is *Borel* if it is Borel as a subset of $X \times X$.

Theorem 2.1 *(Harrington, Kechris, Louveau) Let E be a Borel equivalence relation. Then exactly one of the following two conditions holds:*

1. $E \leq_B \mathrm{id}(\mathbb{R})$*;*

2. $E_0 \leq_B E$*.*

Moreover 1. is equivalent to E being smooth.

This breakthrough result, this archetypal *dichotomy theorem*, suggested the possibility of understanding the structure of the Borel equivalence relations up to Borel reducibility, which in turn has become a major project in the last twenty years, which I will survey on Friday.

3 Examples of Borel reducibility in mathematical practice

It turns out that many classical, or near classical, theorems can be recast in the language of Borel reducibility.

Example Let (X, d) be a complete, separable, metric space. Let $K(X)$ be the compact subsets of X – equipped with the metric $D(K_1, K_2)$ equals

$$\sup_{x \in K_1} d(x, K_2) + \sup_{x \in K_2} d(x, K_1),$$

where $d(x, K) = \inf_{z \in K} d(x, z)$.

Let E be the equivalence relation of isometry on $K(X)$. Then Gromov showed that E is smooth. In other words,

$$E \leq_B \mathrm{id}(\mathbb{R}).$$

Example Let H be a separable Hilbert space and $U(H)$ the group of unitary operators of H.

Let $\cong$ be the equivalence relation of conjugacy on $U(H)$, which is in effect the isomorphism relation considered in the last talk: $T_1 \cong T_2$ if

$$\exists S \in U(H)(S \circ T_1 \circ S^{-1} = T_2).$$

1. In the case that H is finite dimensional, every $T \in U(H)$ can be diagonalized. This gives a reduction of $\cong$ to the equality of finite subsets of $\mathbb{C}$, and hence a proof that $\cong$ is smooth.

2. In the case that H is infinite dimensional, the situation is considerably more subtle, but the spectral theorem allows us to write each element of $U(H)$ as a kind of direct integral of rotations.

Definition Let S^1 be the circle:

$$\{z \in \mathbb{C} : |z| = 1\}$$

in the obvious, and compact, topology. Let $P(S^1)$ be the collection of probability measures on S^1 – this forms a Polish space in the topology it inherits from being a closed subset $C(S^1)^*$ in the weak star topology (via the Riesz representation theorem). For $\mu, \nu \in P(S^1)$, set $\mu \sim \nu$ if they have the same null sets.

It then follows from the spectral theorem that

$$\cong \leq_B \sim .$$

The spectral theorem is often considered, though without the use of the language of Borel reducibility, to provide a *classification* of the infinite dimensional unitary operators up to conjugacy.

Example For S a countable set, may identify $\mathcal{P}(S)$ with

$$2^S = \prod_S \{0, 1\}$$

and thus view it as a compact Polish space in the product topology.

A torsion free abelian (TFA) group A is said to be of rank $\leq n$ if there are $a_1, a_2, ...a_n \in A$ such that every $b \in A$ has some $m \in \mathbb{N}$ with $m \cdot b \in \langle a_1, ..., a_n \rangle$.

Up to isomorphism, the rank $\leq n$ TFA groups are exactly the subgroups of $(\mathbb{Q}^n, +)$, and thus form a Polish space as a subset of $\mathcal{P}(\mathbb{Q}^n)$.

Let $\cong_n$ be the isomorphism relation on subgroups of $(\mathbb{Q}^n, +)$. In the language of Borel reducibility a celebrated classification theorem can be rephrased as:

Theorem 3.1 *(Baer)* $\cong_1 \leq_B E_0$.

Example Let $\mathrm{Hom}^+([0,1])$ be the orientation preserving homeomorphisms of the closed unit interval. In the sup norm metric, this forms a Polish space.

Let $\cong_{\mathrm{Hom}^+([0,1])}$ be the equivalence relation of conjugacy.

There is a kind of folklore observation to the effect that every element of $\mathrm{Hom}^+([0,1])$ can be classified *symbolically*, by recording the maximal open intervals on which it is increasing, decreasing, or the identity.

This translates into classifying

$$\mathrm{Hom}^+([0,1])/\cong_{\mathrm{Hom}^+([0,1])}$$

by countable linear orderings with equipped with unary predicates P_{inc} and P_{dec} up to isomorphism. Those in turn can be viewed as forming a closed subset of $2^{\mathbb{N}\times\mathbb{N}} \times 2^{\mathbb{N}} \times 2^{\mathbb{N}}$, and we obtain

$$\cong_{\mathrm{Hom}^+([0,1])} \leq_B \cong_{2^{\mathbb{N}\times\mathbb{N}}\times 2^{\mathbb{N}}\times 2^{\mathbb{N}}} .$$

Definition Let $\mathcal{L}$ be a countable language. Then $\mathrm{Mod}(\mathcal{L})$ is the set of all $\mathcal{L}$ structures with underlying set $\mathbb{N}$.

Definition Let τ_{qf} be the topology with basic open sets of the form

$$\{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}) : \mathcal{M} \models \varphi(\vec{a})\}$$

where $\varphi(\vec{x})$ is quantifier free and $\vec{a} \in \mathbb{N}^{<\infty}$.

τ_{fo} is defined in a parallel fashion, except with $\varphi(\vec{x})$ ranging over first order formulas, and more generally for $F \subset \mathcal{L}_{\omega_1,\omega}$ a countable *fragment* we define τ_F similarly with $\varphi(\vec{x}) \in F$.

It is not much more than processing the definitions to show τ_{qf} is Polish. For instance for $\mathcal{L}$ consisting of a single binary relation, we obtain a natural isomorphism with $2^{\mathbb{N}\times\mathbb{N}}$. It can be shown, however, that the others are Polish, and all these examples have the same Borel structure.

Definition For a sentence $\sigma \in \mathcal{L}_{\omega_1,\omega}$ we let $\cong_\sigma$ be isomorphism on $\mathrm{Mod}(\sigma)$, the set of $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ with $\mathcal{M} \models \sigma$.

$\cong_\sigma$ is *universal for countable structures* if given any countable language $\mathcal{L}'$ we have

$$\cong_{\mathrm{Mod}(\mathcal{L}')} \leq_B \cong_\sigma .$$

Theorem 4.1 *(Friedman, Stanley) The following are universal for countable structures: Isomorphism of countable trees, countable fields, and countable linear orderings. Isomorphism of countable* torsion *abelian groups is not universal for countable structures.*

One tends to obtain universality[1] for such a class of countable structures , except when there is an "obvious" reason why this must fail.

For instance, if the isomorphism relation is "essentially countable".

[1]The major being torsion abelian groups. The case for torsion free abelian groups remains puzzlingly open, despite strong indicators it should be universal

Definition A Borel equivalence relation F on Polish Y is *countable* if every equivalence class is countable.

An equivalence relation E on a Polish space is *essentially countable* if it is Borel reducible to a countable equivalence relation.

An equivalence relation E is *universal for essentially countable* if it is essentially countable and for any other countable Borel equivalence F we have $F \leq_B E$.

Theorem 5.1 *(Jackson, Kechris, Louveau) Universal essentially countable equivalence relations exist. In fact, for $\mathbb{F}_2$ the free group on two generators, the orbit equivalence relation of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$ is essentially countable.*

Fact 5.2 *If an equivalence relation E is essentially countable, then for some*[2] *countable languages $\mathcal{L}$ we have $E \leq_B \cong_{\mathrm{Mod}(\mathcal{L})}$.*

Theorem 5.3 *(Kechris) If G is a locally compact Polish group acting in a Borel manner on a Polish space X, then the resulting equivalence relation is essentially countable.*

In the context of isomorphism types of classes of countable structures, one can characterize when an equivalence relation is essentially countable in model theoretic terms.

Roughly speaking a class of countable structures with an appropriately "finite character" will be essentially countable.

In particular, if $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ satisfying σ is finitely generated, then $\cong_\sigma$ is essentially countable.

[2] In fact, "most"

Theorem 5.4 *(Thomas, Velickovic) Isomorphism of finitely generated groups is universal for essentially countable.*

As in the case of general countable structures, the tendency is for classes of essentially countable structures to be universal unless there is some relatively obvious obstruction.

In a paper with Kechris, we made a rather arrogant, reckless, and totally unsubstantiated, conjecture that isomorphism for rank two torsion free abelian groups would be universal for essentially countable.

Since E_0 is *not* universal for essentially countable, this was hoped to explain the inability of abelian group theorists to find a satisfactory classification for the higher finite rank torsion free abelian groups.

6 The saga of finite rank torsion free abelian groups

Although many well known mathematical classification theorems have a direct consequence for the theory of Borel reducibility, a major motivation has been to use the theory of Borel reducibility to explicate basic obstructions to the classification of certain classes of isomorphism.

One of the most clear cut cases has been the situation with finite rank torsion free abelian groups.

Recall that $\cong_n$ is being used to describe the isomorphism relation on rank $\leq n$ torsion free abelian groups, where we provide a model of the full set of isomorphism types by considering the subgroups of $\mathbb{Q}^n$.

Theorem 6.1 *(Baer, implicitly, 1937)*

$$\cong_1 \leq_B E_0.$$

A kind of mathematically precise justification for the vague feeling that rank two torsion free abelian groups did not admit a similar classification was provided by:

Theorem 6.2 *(Hjorth, 1998)* $\cong_2$ *is* not *Borel reducible to* E_0*.*

In some sense this addressed the soft philosophical motivation behind the conjecture with Kechris, but not the hard mathematical formulation with which it faced the world. This was left to Simon Thomas, who in a technically brilliant sequence of papers showed:

Theorem 6.3 *(Thomas, 2002, 2004) At every* n

$$\cong_n <_B \cong_{n+1} .$$

In general, and this lies at the heart of the technical mountains Thomas had to overcome, almost all the results to show that one essentially countable equivalence relation is *not* Borel reducible to another rely on techniques coming entirely outside logic, such as geometric group theory, von Neumann algebras, and the rigidity theory one finds in the work of Margulis and Zimmer.

In recent years the work of logicians in this area has begun to communicate and interact with mathematicians in quite diverse fields.

However, it has gradually become clear that many of the problems we would most dearly like to solve will not be solvable by the measure theoretic based techniques being used in these other fields. For instance....

Question Let G be a countable nilpotent group acting in a Borel manner on a Polish space with induced orbit equivalence relation E_G. Must we have

$$E_G \leq_B E_0?$$

The problem here is that with respect to any measure we will have $E_G \leq_B E_0$ on some conull set, and thus measure will not be a suitable method for proving the existence of a counterexample. In an enormously challenging and strikingly original fifty page manuscript, Su Gao and Steve Jackson showed $E_G \leq_B E_0$ when G is abelian.

Many of these issues relate to open problems in the theory of Borel dichotomy theorems and the global structure of the Borel equivalence relations under $\leq_B$.[3]

[3] Next talk

Definition An equivalence relation E on a Polish space X is *classifiable by countable structures* if there is a countable language $\mathcal{L}$ and a Borel function

$$f : X \to \mathrm{Mod}(\mathcal{L})$$

such that for all $x_1, x_2 \in X$

$$x_1 E x_2 \Leftrightarrow f(x_1) \cong f(x_2).$$

Here one might compare algebraic topology, where algebraic objects considered up to isomorphism are assigned as invariants for classes of topological spaces considered up to homeomorphism.

Again it turns out that some well known classification theorems have the direct consequence of showing that some naturally occurring equivalence relation admits classification by countable structures.

Example Recall $\cong_{\mathrm{Hom}^+([0,1])}$ as the isomorphism relation on orientation measure preserving transformations of the closed unit interval. Then the folklore observation mentioned from before in particular shows that $\cong_{\mathrm{Hom}^+([0,1])}$ is classifiable by countable structures.

Example A *Stone space* is a compact zero dimensional Hausdorff space. There is a fixed topological space X (for instance, the Hilbert cube), such that every separable Stone space can be realized as a compact subspace of X. Then $S(X)$, the set of all such subspaces, forms a standard Borel space, and we can let $\cong_{S(X)}$ be the homeomorphism relation on elements of $S(X)$.

Stone duality, the classification of Stone spaces by their associated Boolean algebras, in particular shows that $\cong_{S(X)}$ is classifiable by countable structures.

Example For λ the Lebesgue measure on $[0, 1]$, let M_∞ be the group of measure preserving transformations of $([0, 1]\lambda)$ (considered up to equality a.e.).

A measure preserving transformation T is said to be *discrete spectrum* if $L^2([0, 1], \lambda)$ is spanned by eigenvalues for the induced unitary operator

$$U_T : f \mapsto f \circ T^{-1}.$$

It follows from the work of Halmos and von Neumann that such transformations considered up to conjugacy in M_∞ are classifiable by countable structures.

Example The search for complete algebraic invariants has recurrent theme in the study of C^*-algebras and topological dynamics.

Consider minimal (no non-trivial closed invariant sets) homeomorphisms of $2^{\mathbb{N}}$

Let $\sim_{C(2^{\mathbb{N}})}$ be conjugacy of orbit equivalence relations: Thus

$$f_1 \sim_{C(2^{\mathbb{N}})} f_2$$

if there is some homeomorphism g conjugating their orbits:

$$\forall \vec{x}(g[\{f_1^{\ell}(\vec{x}) : \ell \in \mathbb{Z}\}] = \{f_2^{\ell}(g(\vec{x})) : \ell \in \mathbb{Z}\}).$$

Giordano, Putnam, and Skau produce countable ordered abelian groups which, considered up to isomorphism, act as complete invariants.

Their theorem implicitly shows $\sim_{C(2^{\mathbb{N}})}$ to be classifiable by countable structures.

This a theory, or rather a body of techniques, explicitly fashioned to show when equivalence relations are *not* classifiable by countable structures.

Definition Let G be a Polish group acting continuously on a Polish space X. For V an open neighborhood of 1_G, U an open set containing x, we let

$$O(x, U, V),$$

the U-V-local orbit, be the set of all $\hat{x} \in [x]_G$ such that there is a finite sequence

$$(x_i)_{i \leq k} \subset U$$

such that

$$x_0 = x, \qquad x_k = \hat{x},$$

and each

$$x_{i+1} \in V \cdot x_i.$$

Definition Let G be a Polish group acting continuously on a Polish space X. The action is said to be *turbulent* if:

1. every orbit is dense; and
2. every orbit is meager; and
3. for $x \in X$, the local orbits of x are all somewhere dense; that is to say, if V is an open neighborhood of 1_G, U is an open set containing x, then closure of $O(x, U, V)$ contains an open set.

Theorem 8.1 *(Hjorth) Let G be a Polish group acting continuously on a Polish space X with induced orbit equivalence relation E_G.*

If G acts turbulently on X, then E_G is not *classifiable by countable structures.*

This has been the engine behind a number of anti-classification theorems.

Example (Kechris, Sofronidis) Infinite dimensional unitary operators considered up to unitary conjugacy do not admit classification by countable structures.

Example (Hjorth) The homeomorphism group of the unit square,

$$\mathrm{Hom}([0,1]^2),$$

considered up to homeomorphism does not admit classification by countable structures.

Example (Gao) Countable metric spaces up to homeomorphism does not admit classification by countable structures.

Example (Törnquist) Measure preserving actions of $\mathbb{F}_2$ up to orbit equivalence do not admit classification by countable structures.