A dichotomy theorem for turbulence

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1 Introduction

In this note we show:

Theorem 1.1 *Let G be a Polish group and X a Polish G-space with the induced orbit equivalence relation E_G Borel as a subset of $X \times X$. Then exactly one of the following:*
(I) There is a countable language $\mathcal{L}$ and a Borel function

$$\theta : X \to \mathrm{Mod}(\mathcal{L})$$

such that for all $x_1, x_2 \in X$

$$x_1 E_G x_2 \Leftrightarrow \theta(x_1) \cong \theta(x_2);$$

or
(II) there is a turbulent Polish G-space Y and a continuous G-embedding

$$\tau : Y \hookrightarrow X.$$

There are various bows and ribbons which can be woven into these statements. We can strengthen (I) by asking that θ also admit a Borel orbit inverse, that is to say some Borel function

$$\rho : B \to X,$$

for some Borel set $B \subset \mathrm{Mod}(\mathcal{L})$, such that for all $x \in X$

$$x E_G \rho(\theta(x));$$

and then after having passed to this strengthened version of (I) we still obtain the exact same dichotomy theorem, and hence the conclusion that the two competing versions of (I) are equivalent. Similarly (II) can be relaxed to just asking that τ be a Borel G-embedding, or even simply a Borel reduction of the relevant orbit equivalence relations. It is in fact a consequence of 1.1 that all the plausible weakenings and strengthenings of (I) and (II) are respectively equivalent to one another.

I will not closely examine these possible variations here. The equivalences alluded to above follow from our main theorem and the results of [3]. That monograph had previously shown that (I) and (II) are incompatible, and proved a barbaric forerunner of 1.1, and gone on to conjecture the dichotomy result above.

[3] is the proper place to find further discussion of the notation used in the proofs below. $\mathrm{Mod}(\mathcal{L})$ is the space of $\mathcal{L}$-structure on $\mathbb{N}$ equipped with the topology generated by quantifier free formulas. E_G refers to the orbit equivalence relation arising from the indicated action of G on the indicated space.

2 The Gandy-Harrington topology and a derived Polish G-space

We recall the notions of *recursive* Polish space and *recursive* continuous function from [5]. We assume that all our spaces are recursive Polish spaces, our group is a recursive Polish space, and multiplication viewed as a function

$$G \times G \to G$$

is a recursive continuous function, as is the evaluation of the action of G on X viewed as a function

$$G \times X \to X.$$

Indeed in the case of X we want to do more than simply assume that X has a recursive basis; we fix from now on a sequence $(U_\ell)_{\ell\in\omega}$ of open subsets of X which forms a basis and such that the assignment

$$\ell \mapsto U_\ell$$

is suitably effective; for instance, and this is only one method of many to make this precise, we could assume that

$$\bigcup_{\ell\in\omega} \{\ell\} \times X$$

is a recursive open subset of $\omega \times X$. We also fix a complete compatible metric $d_G(\cdot,\cdot)$ on G and assume it provides a recursive continuous function from $G \times G$ to $\mathbb{R}$.

We assume that there is a countable dense subgroup $G_0 = \{g_n : n \in \mathbb{N}\}$ of G such that the enumeration $n \mapsto g_n$ is recursive continuous as a function from $\mathbb{N}$ to G and such that $g \cdot U_\ell \in \{U_k : k \in \mathbb{N}\}$ all $g \in G_0$ and $\ell \in \mathbb{N}$. We then fix a basis for G closed under left and right translation by G_0 consisting purely of recursive open sets. We assume it includes a neighborhood basis at the identity of symmetric sets. We further assume that E_G is Δ^1_1 as a subset of $X \times X$.

Of course in general these assumptions might not be true, and in the arguments below we would need to laboriously replace each instance of "recursive" with "recursive in z" for some parameter z.

Notation Let $X_{\text{low}} = \{x \in X : \forall A \in \Sigma^1_1(x \in A \vee \exists B \in \Sigma^1_1(x \in B \wedge B \cap A = 0))\}$. Let τ_∞ be the topology generated by taking as a basis all Σ^1_1 sets.

In the case that $X = 2^\omega$ one actually has $X_{\text{low}} = \{x \in X : \omega_1^{\text{ck}} = \omega_1^{\text{ck}(x)}\}$.

Note that X_{low} is a Borel subset of X; on the other hand, while it is lightface Σ^1_1 it is *not* lightface Δ^1_1. Recall that Gandy's low basis theorem states that any non-empty Σ^1_1 contains an element of X_{low}.

The formulation of the next lemma was influenced by some parallel remarks made by Ramez Sami during the course of a seminar. It has been previously proved in [4] using the notion of strong Choquet space; the proof below is longer but more direct.

Lemma 2.1 *τ_∞ is a Polish topology on X_{low}.*

Proof We show $(X_{\text{low}}, \tau_\infty)$ to be homeomorphic to a G_δ subset of some well known Polish space. For ease of exposition we assume $X = 2^\omega$; it should be clear how to extend the argument to general recursive Polish spaces.

Let S be the set of Σ^1_1 subsets of X. To each $x \in X_{\text{low}}$ associate

$$f_x : S \to 2$$

by the formulation

$$f_x(A) = 1$$

if and only if

$$x \in A.$$

This map $x \mapsto f_x$ is clearly one-to-one. The preimage of a subbasic open set in 2^S of the form

$$\{x : f_x(A) = 1\}$$

is obviously open in $(X_{\rm low}, \tau_\infty)$; the preimage of a subbasic open set of the form

$$\{x : f_x(A) = 0\}$$

equals

$$\bigcup\{D : D \in \Delta^1_1, D \cap A = 0\} \cap X_{\rm low},$$

and is thus again open with respect to τ_∞. Therefore the map

$$x \mapsto f_x$$

is a homeomorphism onto its image, and we need only show that $\{f_x : x \in X_{\rm low}\}$ is a G_δ subset of 2^S in the product topology to complete the proof.

For this purpose let us define at each $n \in \omega$ the set A^*_n to be the set of $x \in X_{\rm low}$ for which the $n^{\rm th}$ x-recursive[1] linear order of ω is not a wellorder of rank less than $\omega_1^{\rm ck}$; and then at each m we let $C^*_{n,m}$ be the set of $x \in X_{\rm low}$ for which m is in the illfounded part of the $n^{\rm th}$ x-recursive linear order of ω. These sets are of course all Σ^1_1.

I claim that

$$f \in \{f_x : x \in X_{\rm low}\}$$

if and only if the following four things happen:

1. for all $n \in \omega$ and basic open W with
$$f(W) = 1,$$
there is some $U \subset X$ whose diameter is less than $\frac{1}{n}$ with
$$\overline{U} \subset W$$
and
$$f(U) = 1;$$
2. for all $k \in \omega$, $A_0, A_1, ..., A_k, C_0, C_1, ..., C_k \in \Sigma^1_1$, with
$$\forall i \leq k(f(A_i) = 1, f(C_i) = 0),$$
there is some $x \in X_{\rm low}$ with
$$\forall i \leq k(x \in A_i, x \notin C_i);$$
(in particular this rules out grievously absurd choices for $f : S \to \{0,1\}$, such as the two constant functions);
3. for all n with
$$f(A^*_n) = 1$$
there exists m with
$$f(C^*_{n,m}) = 1;$$

[1] Here for the only time we use $X = 2^\omega$; with suitable care the notion of recursive in x can be translated across to arbitrary recursive spaces.

4. for each Σ^1_1 set A we fix a Lusin scheme
$$(C^A_s)_{s\in\omega^{<\omega}}$$
of Π^1_0 sets such that
$$A=\{x\in X|\exists h\in\omega^\omega\forall n\in\omega(x\in C^A_{h|_n})\};$$
we then demand that
$$f(A)=1\Rightarrow f(C^A_0)=1,$$
and that for all $s\in\omega^\omega$
$$f(C^A_s)=1\Rightarrow\exists k\in\omega(f(C^A_{s^\frown k})=1).$$

The first, third, and fourth of these conditions correspond to G_δ sets, while the second is closed.

The purpose of the first condition is to ensure that we can indeed find a point x_f in some intersection of basic open V with $f(V)=1$; and in this light the second condition ensures that this x_f meets all the open and closed sets promised by f, and from this we can obtain by the fourth condition and the usual kinds of arguments that x_f actually meets every Σ^1_1 set with $f(A)=1$. The third condition is the key one. It guarantees that no recursive in x_f wellorder has rank exactly ω_1^{ck}. □

Notation We let $X_{*\mathrm{low}}$ be the set of all $x\in X$ such that there exists some $g\in G$ with
$$g\cdot x\in X_{\mathrm{low}}.$$
We endow this with the topology τ^*_∞ generated by basic open sets of the form
$$(A\cap X_{\mathrm{low}})^{\Delta V}$$
where $A\subset X$ is Σ^1_1 and $V\subset G$ is open.

In fact if $x\in X_{*\mathrm{low}}$ one does not just obtain one $g\in G$ with $g\cdot x\in X_{\mathrm{low}}$ but a comeager set of g which so move x to a real with $\omega_1^{\mathrm{ck}(g\cdot x)}=\omega_1^{\mathrm{ck}}$: We first use that
$$\omega_1^{\mathrm{ck}(x)}=\omega_1^{\mathrm{ck}}$$
to conclude that there will be a comeager collection of g with $\omega_1^{\mathrm{ck}(g,x)}=\omega_1^{\mathrm{ck}}$, and any such g will do.

Thus $X_{*\mathrm{low}}$ equals
$$(X_{\mathrm{low}})^{*G}=(X_{\mathrm{low}})^{\Delta G},$$
and is consequently Borel. Note moreover that in this space, for any $A\subset X_{\mathrm{low}}$ which is Σ^1_1 and for any $V\subset G$ open we in fact have a sequence of Δ^1_1 sets
$$(D_\alpha)_{\alpha\in\omega_1^{\mathrm{ck}}}$$
such that
$$A^{*V}=X_{*\mathrm{low}}\setminus(\bigcup_{\alpha\in\omega_1^{\mathrm{ck}}}(D_\alpha)^{\Delta V}),$$
and thus A^{*V} is closed.

The next proof crumbles into a sequence of calculations. The first few of these resemble similar points made in the course of the proof of 5.1.5 from [1].

Lemma 2.2 *$(X_{*\mathrm{low}},\tau^*_\infty)$ is a Polish G-space.*

Proof The action of G on this space is obviously continuous.

Claim: $(X_{*\mathrm{low}}, \tau^*_\infty)$ is Hausdorff.

Proof of claim: τ^*_∞ includes the subspace topology on $X_{*\mathrm{low}}$ induced by the original topology on X. Since this original topology is Hausdorff, so is τ^*_∞. (□Claim)

Claim: $(X_{*\mathrm{low}}, \tau^*_\infty)$ is regular.

Proof of claim: Let $A^{\Delta V}$ be an open subset of $(X_{*\mathrm{low}}, \tau^*_\infty)$ and $x \in A^{\Delta V}$. Then we may find some $W_0 \subset V$ open and non-empty with $x \in A^{*W_0}$. Appealing to the continuity of the group operations we may find non-empty open sets

$$W_2 \subset W_1 \subset W_0$$

and an open neighborhood W_3 of the identity with

$$W_1 W_3^{-1} \subset W_0$$

and

$$W_2 W_3^{-1} \subset W_1.$$

x is then a member of the open set $(A^{*W_1})^{\Delta W_3}$ since for *any* $g \in W_3$ we have

$$W_1 g^{-1} \subset W_0$$

and

$$\forall^* h \in W_1 g^{-1} (h \cdot (g \cdot x) \in A).$$

Thus the claim will be proved if we show $(A^{*W_1})^{\Delta W_3}$ is included in the τ^*_∞-closed set A^{*W_2}. But here we have that for any $h \in W_3$ and z with $h \cdot z \in A^{*W_1}$

$$\forall^* g \in W_1 (gh \cdot z \in A)$$

$$\therefore \forall^* g \in W_1 h (g \cdot z \in A).$$

But

$$W_2 W_3^{-1} \subset W_1$$

$$\therefore W_2 \subset W_1 h$$

$$\therefore z \in A^{*W_2}.$$

(□Claim)

Since $(X_{*\mathrm{low}}, \tau^*_\infty)$ is clearly separable, we thus have that it is metrizable by Urysohn.

To see that the space is Polish we first define a continuous map

$$\pi : G \times X_{\mathrm{low}} \to X_{*\mathrm{low}}$$

by

$$(g, x) \mapsto g \cdot x.$$

Claim: π is continuous.

Proof of claim: We have $g \cdot x \in A^{\Delta V}$ if and only if $x \in A^{\Delta V g}$. (□Claim)

Claim: π is an open map.

Proof of claim: It suffices to show that whenever $A \subset X_{\mathrm{low}}$ is Σ^1_1 and $V \subset G$ is open,

$$V \cdot A = \bigcup_{W_0, W_1 \text{ open}, W_0 W_1 \subset V^{-1}} (\{y : \exists h \in W_0 (h \cdot y \in A)\})^{\Delta W_1}.$$

There are two parts to demonstrating this. We show these two inclusions separately.

$\subset$: Let

$$g \cdot x = z,$$

$g \in V$, $x \in A$. Then considering the continuity of the action we may choose open V_0 and V_1 containing g^{-1} and 1 which satisfy:

$$V_0(V_1)^{-1}V_1 \subset V^{-1}.$$

We can then let $W_0 = V_0 V_1^{-1}$ and $W_1 = V_1$.

With this arranged we have that *for all* $\epsilon \in W_1$

$$(g^{-1}\epsilon^{-1})\epsilon \cdot z = (g^{-1}\epsilon^{-1})\epsilon g \cdot x = x \in A.$$

Noting that any such $g^{-1}\epsilon^{-1}$ will be in W_0 we have in particular that

$$z \in (\{y : \exists h \in W_0(h \cdot y \in A)\})^{*W_1} \subset (\{y : \exists h \in W_0(h \cdot y \in A)\})^{\Delta W_1}.$$

$\supset$: Consider some

$$z \in (\{y : \exists h \in W_0(h \cdot y \in A)\})^{\Delta W_1},$$

where $W_0 W_1 \subset V^{-1}$. Then in particular, we can find $g_1 \in W_1$ with

$$g_1 \cdot z \in \{y : \exists h \in W_0(h \cdot y \in A)\}.$$

And then we may find some $g_2 \in W_0$ with

$$g_2 g_1 \cdot z \in A.$$

But then

$$g_2 g_1 \in V^{-1}$$

by assumption on W_0 and W_1, and hence

$$x = g_2 g_1 \cdot z$$

and

$$g = (g_2 g_1)^{-1}$$

witness that $z \in V \cdot A$. (Claim□)

In summary, the above claims have shown

$$\pi : G \times X_{\mathrm{low}} \to X_{*\mathrm{low}}$$

to be an open continuous map from a Polish space onto a metric space. Thus by Hausdorff's theorem, $(X_{*\mathrm{low}}, \tau_\infty^*)$ is Polish. □

3 The split in cases

As above, assume that G is a recursive Polish group and X is a recursive Polish G-space.

Definition A sequence

$$\vec{\mathcal{B}} = (\mathcal{B}_\alpha)_{\alpha \in \delta}$$

is *suitable* if

1. each $\mathcal{B}_\alpha$ is a subset of the Borel subsets of X;

2. $\delta < \omega_1^{\mathrm{ck}}$;
3. the function
$$\alpha \mapsto \mathcal{B}_\alpha$$
is Δ^1_1 in the codes for Borel sets;
4. in particular each $\mathcal{B}_\alpha$ is countable;
5. each $\mathcal{B}_\alpha$ includes the basic open sets in the original topology on X;
6. each $\mathcal{B}_\alpha$ is closed under G_0-translation.

This definition has glossed over the details of being Δ^1_1 *in the codes for Borel sets*; a protracted discussion is unlikely to be helpful. The reader will be able to provide their own coding for Borel sets, and any reasonable understanding of "in the codes" should suffice; for instance we can ask that there be some recursive wellorder $\prec$ of length δ and Δ^1_1 function
$$F : \omega \times \omega \to \omega^\omega$$
such that for any $n \in \omega$ and $\alpha = \|n\|_\prec$ we have that
$$\{F(n,m) : m \in \omega\}$$
gives a set of codes for the Borel sets appearing in the $\mathcal{B}_\alpha$.

Given such an admissible sequence, we define a sequence of objects
$$\hat{\varphi}^{\vec{\mathcal{B}}}_\alpha(x, B^{\Delta V_0}, V_1)$$
around each point $x \in X$ and at each $\alpha < \delta$.

Definition Given an admissible sequence $\vec{\mathcal{B}} = (\mathcal{B}_\beta)_{\beta\in\delta}$ as above, $x \in X$, $B \in \mathcal{B}_\alpha$, $V_0, V_1 \subset G$ basic open, V_1 containing 1_G, $V_1^{-1} = V_1$, we define $\hat{\varphi}^{\vec{\mathcal{B}}}_\beta(x, B^{\Delta V_0}, V_1)$ by induction on β:-

BASE CASE: for $x \in B^{\Delta V_0}$
$$\hat{\varphi}^{\vec{\mathcal{B}}}_0(x, B^{\Delta V_0}, V_1)$$
equals the set of $\ell \in \omega$ such that there exists $k \in \omega$, $x_0, x_1, ..., x_k \in X$ such that

$x_0 = x$;
$x' = x_k \in U_\ell$;
each $x_{i+1} \in V_1 \cdot x_i \cap B^{\Delta V_0}$.

INDUCTIVE STEP: If β is an ordinal lying strictly between 0 and δ, then
$$\hat{\varphi}^{\vec{\mathcal{B}}}_\beta(x, B^{\Delta V_0}, V_1)$$
equals the set of all
$$\langle \hat{\varphi}^{\vec{\mathcal{B}}}_\alpha(x', C^{\Delta W_0}, W_1), C, W_0, W_1\rangle,$$
where

$\alpha < \beta$;
$C \in \mathcal{B}_\alpha$;
$W_0, W_1 \subset G$ basic open;
W_1 contains 1_G, $W_1^{-1} = W_1$,
$x_0 = x$;
$x' = x_k \in C^{\Delta W_0}$;
each $x_{i+1} \in V_1 \cdot x_i \cap B^{\Delta V_0}$.

Notation For $\epsilon > 0$ we let W_ϵ be the set

$$\{g \in G | d_G(g, 1_G) < \epsilon\}.$$

Definition For $A \subset X$, $A \in \Sigma^1_1$, $V_0, V_1 \subset G$ basic open, $V_1^{-1} = V_1, 1_G \in V_1$, and $x \in A^{\Delta V_0}$, we let

$$\mathcal{O}(x, A^{\Delta V_0}, V_1)$$

be the set of x' for which there exists $k, x_0, x_1, ..., x_k$ such that
$x_0 = x$;
$x' = x_k$;
each $x_{i+1} \in V_1 \cdot x_i \cap A^{\Delta V_0}$.

Definition For $\vec{\mathcal{B}} = (\mathcal{B}_\alpha)_{\alpha<\delta}$ suitable, $\epsilon > 0$, $A \in \Sigma^1_1$, $V_0, V_1 \subset G$ basic open, $V_1^{-1} = V_1, 1_G \in V_1$, we say that

$$(A, V_0, V_1) \text{ is } \epsilon\text{-stable for } \vec{\mathcal{B}}$$

if for all $x, x', x'', \alpha < \delta, V_2, V_3, B$ with

1. $x \in A^{\Delta V_0}$,
2. $x', x'' \in \mathcal{O}(x, A^{\Delta V_0}, V_1)$,
3. $\alpha < \delta$,
4. $B \in \mathcal{B}_\alpha$,
5. $V_2, V_3 \subset G$ basic open,
6. $V_3^{-1} = V_3$,
7. $1_G \in V_3$,

$x' \in B^{\Delta V_2}$ implies

$$\exists^* g \in W_\epsilon(g \cdot x'' \in B^{\Delta V_2} \wedge \hat{\varphi}^{\vec{\mathcal{B}}}_\alpha(x', B^{\Delta V_2}, V_3) = \hat{\varphi}^{\vec{\mathcal{B}}}_\alpha(g \cdot x'', B^{\Delta V_2}, V_3)).$$

There are a few remarks about this definition. Firstly that if (A, V_0, V_1) is $\frac{1}{n}$-stable then it is ϵ-stable for any larger ϵ. The second observation is that we may replace

$$\exists^* g \in W_\epsilon(g \cdot x'' \in B^{\Delta V_2} \wedge \hat{\varphi}^{\vec{\mathcal{B}}}_\alpha(x', B^{\Delta V_2}, V_3) = \hat{\varphi}^{\vec{\mathcal{B}}}_\alpha(g \cdot x'', B^{\Delta V_2}, V_3))$$

by

$$\exists g \in W_\epsilon(g \cdot x'' \in B^{\Delta V_2} \wedge \hat{\varphi}^{\vec{\mathcal{B}}}_\alpha(x', B^{\Delta V_2}, V_3) = \hat{\varphi}^{\vec{\mathcal{B}}}_\alpha(g \cdot x'', B^{\Delta V_2}, V_3));$$

if one $g \in W_\epsilon$ brings about the desired result then it follows from the structure of the definitions that an open set of them will, and hence certainly a non-meager set will as well.

Lemma 3.1 *If $\vec{\mathcal{B}}$ is a suitable sequence, $n \in \omega$, $V_0, V_1 \subset G$ basic open, $V_1^{-1} = V_1$, $1_G \in V_1$, $A \in \Sigma^1_1$, and if*

$$(A, V_0, V_1)$$

is $\frac{1}{n}$-stable for $\vec{\mathcal{B}}$, then there is a Δ^1_1 set $D \supset A$ with

$$(D, V_0, V_1)$$

$\frac{1}{n}$-stable for $\vec{\mathcal{B}}$.

Proof For any $\Delta^1_1(x)$ set C we have that the set of x'' with

$$\exists g^* \in W_{\frac{1}{n}}(g \cdot x'' \in C)$$

is uniformly $\Delta^1_1(x)$; in particular, the set of x'' with

$$\exists^* g \in W_\epsilon(g \cdot x'' \in B^{\Delta V_2} \wedge \hat{\varphi}^{\vec{\mathcal{B}}}_\alpha(x', B^{\Delta V_2}, V_3) = \hat{\varphi}^{\vec{\mathcal{B}}}_\alpha(g \cdot x'', B^{\Delta V_2}, V_3))$$

is uniformly, in the indices for V_0, V_1, and B, $\Delta^1_1(x)$.

On the other hand, the set

$$\mathcal{O}(x, A^{\Delta V_0}, V_1)$$

is uniformly $\Sigma^1_1(x)$.

Thus the collection of Σ^1_1 sets A with (A, V_0, V_1) $\frac{1}{n}$-stable is uniformly Π^1_1 in the indices.

And thus the lemma follows from Π^1_1 reflection.[2] □

Definition For $\vec{\mathcal{B}} = (\mathcal{B}_\alpha)_{\alpha\in\delta}$ suitable and $n \in \omega$, let $A_{\vec{\mathcal{B}},n}$ be the set of all $x \in X$ for which there is a non-meager collection of $g \in G$ with

$$\forall D \in \Delta^1_1, V_0, V_1 \subset G \text{ basic open}$$

$$(g \cdot x \in D^{\Delta V_0} \Rightarrow ((D, V_0, V_1) \text{ not } \frac{1}{n} - \text{stable for } \vec{\mathcal{B}}))).$$

For a given n and $\vec{\mathcal{B}}$, the set $A_{\vec{\mathcal{B}},n}$ is Σ^1_1.

Notation From now on let us endow $X_{*\mathrm{low}} \cap A_{\vec{\mathcal{B}},n}$ with the topology $\tau^*_{n,\infty}$ generated by all sets of the form

$$A^{\Delta V}$$

where $A \in \Sigma^1_1$, $A \subset X_{*\mathrm{low}} \cap A_{\vec{\mathcal{B}},n}$, and $V \subset G$ basic open. Let

$$(\hat{U}_\ell)_{\ell\in\omega}$$

enumerate a basis for $\tau^*_{n,\infty}$.

Lemma 3.2 *$(X_{*\mathrm{low}} \cap A_{\vec{b},n}, \tau^*_{n,\infty})$ is a Polish G-space.*

Proof Since it is an open subset of $(X_{*\mathrm{low}}, \tau^*_\infty)$. □

At this point the proof splits into two cases. Case(1) below leads to a turbulent Polish G-space Y as in (II) above. Case(2), on the other hand, will finish with a classification of the G-orbits by countable models considered up to isomorphism.

[2] See [2] for the statement and proof of this theorem.

4 Case(1)

For some suitable $\vec{\mathcal{B}} = (\mathcal{B}_\alpha)_{\alpha\in\delta}$ and $n \in \omega$, $A_{\vec{\mathcal{B}},n}$ is non-empty.

Let us fix such a $\vec{\mathcal{B}}$ and n. We will press forward, and eventually show that option (II) in 1.1 holds. Following §6.2 of [3] we may define "local invariants" for the Polish G-space $(X_{*\mathrm{low}} \cap A_{\vec{\mathcal{B}},n}, \tau^*_{n,\infty})$.

Notation For $x \in X_{*\mathrm{low}} \cap A_{\vec{\mathcal{B}},n}$ and U $\tau^*_{n,\infty}$-open, $V \subset G$ a symmetric open neighborhood of the identity, we let

$$\mathcal{O}(x, U, V)$$

be the set of all x' such that there exists $k, x_0, x_1, ..., x_k$ with

$$x_0 = x,$$

$$x_k = x',$$

and at each $i < k$

$$x_{i+1} \in U \cap V \cdot x_i.$$

Fix some enumeration $(\hat{U}_\ell)_{\ell\in\omega}$ of a basis for τ^*_∞. We then set off defining

$$\varphi_0(x, U, V)$$

be the set of ℓ such that

$$\mathcal{O}(x, U, V) \cap \hat{U}_\ell \neq 0.$$

And we continue in this direction by inductively defining

$$\varphi_\alpha(x, U, V)$$

to be the set of

$$\langle \varphi_\beta(x', U', V'), U', V' \rangle,$$

where β ranges over ordinals less than α, U' ranges over basic open subsets of U, V' ranges over symmetric basic open neighborhoods of 1_G included in V, and x' ranges over elements of $U' \cap \mathcal{O}(x, U, V)$.

In other words, $\varphi_0(x, U, V)$ encodes the closure with respect to $\tau^*_{n,\infty}$ of the local orbit $\mathcal{O}(x, U, V)$; and for $\alpha > 0$ the local invariant $\varphi_\alpha(x, U, V)$ encodes the set of all possible $\varphi_\beta(x', U', V')$ we can obtain by reducing α to β, U to U'. V to V', and passing to some other $x' \in \mathcal{O}(x, U, V)$.

Let us fix some particular $x \in X_{*\mathrm{low}} \cap A_{\vec{\mathcal{B}},n}$; it in fact does not matter which. Still following the relevant §6.2 of [3] we come, for any given x in the space, to some $\gamma(x) < \omega_1$ such that for all U that is $\tau^*_{n,\infty}$-open, for all $V \subset G$ that is a symmetric basic open neighborhood of the identity, and for all $x'.x'' \in [x]_G$

$$\varphi_{\omega_1}(x', U, V) = \varphi_{\omega_1}(x'', U, V)$$

iff

$$\varphi_{\gamma(x)}(x', U, V) = \varphi_{\gamma(x)}(x'', U, V).$$

We then obtain a Polish topology on the set

$$X^x =_{\mathrm{df}} \{y \in X_{*\mathrm{low}} \cap A_{\vec{\mathcal{B}},n} | \varphi_{\gamma(x)+2}(x, X_{*\mathrm{low}} \cap A_{\vec{\mathcal{B}},n}, G) = \varphi_{\gamma(x)+2}(y, X_{*\mathrm{low}} \cap A_{\vec{\mathcal{B}},n}, G)\}$$

by taking as a basis all sets of the form

$$\{y|\varphi_{\gamma(x)}(y,U,V)=\varphi_{\gamma(x)}(x',U,V)\},$$

as U ranges over $\tau^*_{n,\infty}$-open sets, V ranges over symmetric basic open neighborhoods of the identity, and x' ranges over $[x]_G$.

From this point on let us understand that X^x always refers to X^x in the indicated Polish topology.

Lemma 4.1 *For any $y\in X^x$,*

$$\overline{[x]_G}=\overline{[y]_G}.$$

Here we understand that the closure operation refers to the indicated topology on the space X^x.

Proof An immediate consequence of the definitions. □

Corollary 4.2 *For any $y\in X^x$, $A\subset X$ a Σ^1_1 set,*

$$[x]_G\cap A\neq 0 \text{ iff } [y]_G\cap A\neq 0.$$

Of course in this corollary it is important that we restrict ourselves to sets A which are lightface Σ^1_1 with respect to the original space X.

Lemma 4.3 *For any $y\in X^x$,*

$$\gamma(x)=\gamma(y).$$

Moreover for any U that is $\tau^*_{n,\infty}$ basic open and $V\subset G$ a symmetric basic open neighborhood of the identity,

$$\mathcal{O}(y,U,V)$$

is dense (with respect to the topology on X^x) in some basic open set of the form

$$\{y|\varphi_{\gamma(x)}(y,U,V)=\varphi_{\gamma(x)}(x',U,V)\}.$$

Proof These facts can be verified from the definitions in a reasonably routine manner; or see §6.2 [3] for the details. □

From now until the end of the section, let us just fix some $y\in X^x$. We will try to show that $[y]_G$ is meager in the space X^x.

If we succeed in this task, for some arbitrarily chosen $y\in X^x$, then we will have obtained that every orbit in the space must be meager. And then by 4.1 and 4.3 we have that the definition of turbulence is satisfied.

Now turning our attention back towards the fixed, but arbitrary, $y\in X^x$ we have from 4.2 that there will be some

$$\hat{y}\in[y]_G$$

such that for all[3] $D\in\Delta^1_1$, $V_0,V_1\subset G$ basic open, if

$$\hat{y}\in D^{\Delta V_0},$$

then (D,V_0,V_1) is not $\frac{1}{n}$-stable for $\vec{\mathcal{B}}$. Hence by 3.1, for all $A\in\Sigma^1_1$, $V_0,V_1\subset G$ basic open, if

$$\hat{y}\in A^{\Delta V_0},$$

then (A,V_0,V_1) is not $\frac{1}{n}$-stable for $\vec{\mathcal{B}}$.

Fix this $\hat{y}$. We need to "blur" this element of $[y]_G$ making it slightly more generic, without altering its important properties; for this purpose let us fix some $m>n$ such that

$$(W_{\frac{1}{m}})^{\pm}(W_{\frac{1}{m}})^{\pm}(W_{\frac{1}{m}})^{\pm}\subset W_{\frac{1}{n}}.$$

[3] Here, and everywhere else the issue might emerge, Σ^1_1, Δ^1_1, and Π^1_1 refer to $\Sigma^1_1(X)$, $\Delta^1_1(X)$, and $\Pi^1_1(X)$ respectively.

Claim 4.4 *There exists $y' \in [y]_G$ such that:*
(i) for all $A \in \Sigma^1_1$, if $y' \in A$, then

$$y' \in A^{*W}$$

for some basic open neighborhood W of 1_G;
(ii) $\omega_1^{\mathrm{ck}} = \omega_1^{\mathrm{ck}(y')}$; and hence, for all $A \in \Sigma^1_1$, either

$$y' \in A$$

or there is a Δ^1_1 set D with

$$y' \in D$$

and

$$D \cap A = 0;$$

(iii) for all $A \in \Sigma^1_1$, $V_0, V_1 \subset G$ basic open, if

$$y' \in A^{\Delta V_0},$$

then (A, V_0, V_1) is not $\frac{1}{m}$-stable for $\vec{\mathcal{B}}$.

Proof Note first of all that since $y \in X^x$ we may certainly find $g \in W_{\frac{1}{m}}$ such that

$$y' = g \cdot \hat{y}$$

satisfies (i) and (ii); any sufficiently generic $g \in W_{\frac{1}{m}}$ will do. For (iii) we need a further subclaim.

Subclaim: If (A, V_0, V_1) is $\frac{1}{m}$-stable and $g \in W_{\frac{1}{m}}$ with

$$y' = g \cdot \hat{y} \in A^{\Delta V_0},$$

then there is some $\frac{1}{n}$-stable (C, V_0', V_1') with

$$\hat{y} \in C^{\Delta V_0'}.$$

Proof of subclaim: Assume not, and then we may assume further that $g \in G_0$, since slightly nudging g into G_0 should not affect $\mathcal{O}(y', A^{\Delta V_0}, V_1)$.

Now consider some arbitrary $\hat{z} \in X$ with

$$z' = g \cdot \hat{z} \in A^{\Delta V_0},$$

and

$$\hat{z}^* \in \mathcal{O}(\hat{z}, A^{\Delta V_0 g}, g^{-1} V_1 g).$$

Let

$$z^* = g \cdot \hat{z}^* \in \mathcal{O}(z', A^{\Delta V_0}, V_1).$$

By assumption on (A, V_0, V_1), for any $\alpha < \delta$, $B \in \Sigma^1_1$, and V_2, V_3 basic open in G, with $B^{\Delta V_2}$ containing z, (and hence $B^{\Delta V_2 g^{-1}}$ containing $z^* = g \cdot \hat{z}^*$) we may find

$$h \in W_{\frac{1}{m}}$$

with

$$\hat{\varphi}^{\vec{\mathcal{B}}}_{\alpha}(hg \cdot \hat{z}, B^{\Delta V_2 g^{-1}}, g V_3 g^{-1})$$

(which by definition of z' equals

$$\hat{\varphi}^{\vec{\mathcal{B}}}_{\alpha}(h \cdot z', B^{\Delta V_2 g^{-1}}, gV_3 g^{-1}))$$

equal to

$$\hat{\varphi}^{\vec{\mathcal{B}}}_{\alpha}(z^*, B^{\Delta V_2 g^{-1}}, gV_3 g^{-1})$$

$$\therefore \hat{\varphi}^{\vec{\mathcal{B}}}_{\alpha}(g^{-1}hg \cdot \hat{z}, B^{\Delta V_2}, gV_3 g^{-1}) = \hat{\varphi}^{\vec{\mathcal{B}}}_{\alpha}(g^{-1}z^*, B^{\Delta V_2}, gV_3 g^{-1}),$$

which in turn by the definition of z^* equals

$$\hat{\varphi}^{\vec{\mathcal{B}}}_{\alpha}(\hat{z}^*, B^{\Delta V_2}, gV_3 g^{-1}).$$

Thus we have shown $(A, V_0 g, g^{-1}V_1 g)$ to be $\frac{1}{n}$-stable. (Claim□)

But this subclaim proves the claim, since $\hat{y}$ was assumed not to be so captured by any $\frac{1}{n}$-stable set. □

Let us then fix y' as above.

Claim 4.5 *For all $A \in \Sigma^1_1$, $V_0, V_1 \subset G$ basic open with $1_G \in V_1$, $V_1^{-1} = V_1$, if*

$$y' \in A^{\Delta V_0}$$

then

$$\exists y_1, y_2 \in \mathcal{O}(y', A^{\Delta V_0}, V_1)$$

$$\exists \alpha < \delta \exists B \in \mathcal{B}_\alpha$$

$$\exists V_2, V_3 \subset G \text{ basic open}, 1_G \in V_3, V_3^{-1} = V_3$$

with

$$y_1 \in B^{\Delta V_2}$$

but

$$\forall g \in W_{\frac{1}{m}}(g \cdot y_2 \in B^{\Delta V_2} \Rightarrow \hat{\varphi}^{\vec{\mathcal{B}}}_{\alpha}(y_1, B^{\Delta V_2}, V_3) \neq \hat{\varphi}^{\vec{\mathcal{B}}}_{\alpha}(g \cdot y_2, B^{\Delta V_2}, V_3)).$$

Proof Otherwise applying (ii) from our assumptions on y' we find $D \in \Delta^1_1$ containing y' such that

$$\forall z' \in A^{\Delta V_0} \cap D$$

$$\forall z_1, z_2 \in \mathcal{O}(z', A^{\Delta V_0}, V_1)$$

$$\forall \alpha < \delta \forall B \in \mathcal{B}_\alpha$$

$$\forall V_2, V_3 \subset G \text{ basic open}, 1_G \in V_3, V_3^{-1} = V_3$$

if

$$z_1 \in B^{\Delta V_2}$$

then

$$\exists g \in W_{\frac{1}{n}}(g \cdot y_1 \in B^{\Delta V_2} \wedge \hat{\varphi}^{\vec{\mathcal{B}}}_{\alpha}(z_1, B^{\Delta V_2}, V_3) = \hat{\varphi}^{\vec{\mathcal{B}}}_{\alpha}(g \cdot z_2, B^{\Delta V_2}, V_3)),$$

and hence as prior to 3.1

$$\exists^* g \in W_{\frac{1}{n}}(g \cdot y_1 \in B^{\Delta V_2} \wedge \hat{\varphi}^{\vec{\mathcal{B}}}_{\alpha}(z_1, B^{\Delta V_2}, V_3) = \hat{\varphi}^{\vec{\mathcal{B}}}_{\alpha}(g \cdot z_2, B^{\Delta V_2}, V_3)).$$

By (i) we obtain

$$y' \in D^{*W}$$

some basic open neighborhood W of 1_G. After possibly shrinking W and V_0 we can assume

$$W \subset V_1$$

and that there is some $g_0 \in G_0$ with

$$g_0 W = V_0.$$

Here it will be easier if we make a harmless simplifying assumption. I claim that we may assume $V_0 = W$ and that $g_0 = 1_G$. The point is that we can always replace A by the set

$$g_0^{-1} A,$$

and observe that

$$A^{\Delta g_0 W} = (g_0^{-1} A)^{\Delta W}.$$

So after making this minor notational reorganization let us assume that

$$1_G = g_0,$$

$$W = V_0,$$

$$1_G \in V_0,$$

$$V_0 \subset V_1,$$

$$y' \in A^{\Delta V_0} \cap D^{*V_0} \subset (A \cap D)^{\Delta V_0}.$$

Then for any

$$z \in (A \cap D)^{\Delta V_0},$$

$$z_1, z_2 \in \mathcal{O}(z, (A \cap D)^{\Delta V_0}, V_1),$$

and for any

$$\alpha < \delta$$

$$B \in \mathcal{B}_\alpha$$

$$V_2, V_3 \subset G \text{ basic open}$$

with

$$1_G \in V_3, V_3^{-1} = V_3$$

$$z_1 \in B^{\Delta V_2}$$

we may find $h_0 \in V_0 \subset V_1$ with

$$z' =_{\mathrm{df}} h_0 \cdot z \in D \cap (A \cap D)^{\Delta V_0},$$

$$\therefore \mathcal{O}(z', (A \cap D)^{\Delta V_0}, V_1) = \mathcal{O}(z, (A \cap D)^{\Delta V_0}, V_1),$$

$$\therefore z_1, z_2 \in \mathcal{O}(z', (A \cap D)^{\Delta V_0}, V_1) \subset \mathcal{O}(z', A^{\Delta V_0}, V_1),$$

and hence by the assumptions on D and the fact that $z' \in D$ we have

$$\exists^* g \in W_{\frac{1}{m}} (g \cdot y_1 \in B^{\Delta V_2} \wedge \hat{\varphi}_\alpha^{\vec{\mathcal{B}}}(z_1, B^{\Delta V_2}, V_3) = \hat{\varphi}_\alpha^{\vec{\mathcal{B}}}(g \cdot z_2, B^{\Delta V_2}, V_3)).$$

And thus we have seen that $(A \cap D, V_0, V_1)$ is $\frac{1}{m}$-stable. Which contradicts assumption (iii) on y'. $\square$

But the statement of this lemma is catastrophic with respect to y's hope of having a G_δ orbit. It implies in particular that for all $A \in \Sigma^1_1$, $V_0, V_1 \subset G$ basic open with $1_G \in V_1$, $V_1^{-1} = V_1$, if

$$y' \in A^{\Delta V_0}$$

then

$$\exists y_1, y_2 \in A^{\Delta V_0} \cap [y]_G$$

with

$$\mathcal{O}(y_1, A^{\Delta V_0}, V_1) = \mathcal{O}(y_2, A^{\Delta V_0}, V_1),$$
$$\therefore \varphi_{\gamma(x)}(y_1, A^{\Delta V_0}, V_1) = \varphi_{\gamma(x)}(y_2, A^{\Delta V_0}, V_1) = \varphi_{\gamma(x)}(y', A^{\Delta V_0}, V_1)$$

but

$$\forall g \in G(d_G(g, 1) < \frac{1}{m} \Rightarrow g \cdot y_1 \neq y_2).$$

But now since the sets of the form

$$\{z : \varphi_{\gamma(x)}(z, A^{\Delta V_0}, V_1) = \varphi_{\gamma(x)}(x', A^{\Delta V_0}, V_1)\}$$

(as x' ranges over $[x]_G$, A ranges over Σ^1_1, V_0, V_1 range over basic open sets with V_1 a symmetric neighborhood of the identity) form a basis of the topology on X_x, we have that the induced map

$$G \to [y]_G$$

$$g \mapsto g \cdot y$$

is not open. Thus by the Effros lemma, $[y]_G$ is not G_δ.

Thus we have shown that no orbit in X_x is G_δ. And hence all the orbits are meager.

And hence as in §6.2 of [3], all the orbits in X_x are dense, the local orbits of the form

$$\mathcal{O}(y, A^{\Delta V_0}, V_1)$$

are all dense inside the corresponding open set

$$\{z : \varphi_{\gamma(x)}(z, A^{\Delta V_0}, V_1) = \varphi_{\gamma(x)}(x', A^{\Delta V_0}, V_1)\}$$

for some $x' \in [x]_G$ with $\varphi_{\gamma(x)}(y, A^{\Delta V_0}, V_1) = \varphi_{\gamma(x)}(x', A^{\Delta V_0}, V_1)$, and by the calculation above every orbit is meager. Thus the action is turbulent, and we are done with this case.

5 Case (2)

Not case (1).

Claim 5.1 *There is a sequence*

$$\vec{\mathcal{B}} = (\mathcal{B}_\alpha)_{\alpha \in \omega_1^{\rm ck}}$$

such that

(a) $\alpha \mapsto \mathcal{B}_\alpha$ is Δ^1_1 in the codes;

(b) for each $\delta < \omega_1^{\rm ck}$ the sequence $\vec{\mathcal{B}}|_\delta = (\mathcal{B}_\alpha)_{\alpha \in \delta}$ is suitable;

(c) for each $\delta < \omega_1^{\rm ck}$, $\epsilon > 0$, $x \in X$ there is some $D \in \mathcal{B}_\delta$ along with basic open

$$V_0, V_1 \subset G,$$

V_1 a symmetric open neighborhood of the identity, (D, V_0, V_1) ϵ-stable for $\vec{\mathcal{B}}|_\delta$ and

$$x \in D^{\Delta V_0}.$$

Proof Recalling from Spector-Gandy that a subset of $L_{\omega_1^{\mathrm{ck}}}$ is Π^1_1 in the codes if and only if it is $\Sigma_1^{L_{\omega_1^{\mathrm{ck}}}}$, it suffices to show that there is a total $\Sigma_1^{L_{\omega_1^{\mathrm{ck}}}}$ function

$$\alpha \mapsto B_\alpha,$$

with each B_α included in the indices for Δ^1_1 sets, such that if we let $\mathcal{B}_\alpha$ be the corresponding collection of actual Δ^1_1 sets encoded by the B_α, then $(\mathcal{B}_\alpha)_{\alpha\in\delta}$ satisfies (c) above and 1, 2, 4, 5, 6 from the definition of suitability all $\delta < \omega_1^{\mathrm{ck}}$; note that these requirements are all Π^1_1 and hence $\Sigma_1^{L_{\omega_1^{\mathrm{ck}}}}$. Since the total $\Sigma_1^{L_{\omega_1^{\mathrm{ck}}}}$ functions are closed under transfinite $\Sigma_1^{L_{\omega_1^{\mathrm{ck}}}}$ recursion, it suffices to show that given any $(B_\alpha)_{\alpha\in\delta} \in L_{\omega_1^{\mathrm{ck}}}$ which gives rise to $(\mathcal{B}_\alpha)_{\alpha\in\delta}$ with the desired properties we can find some $B_\delta \in L_{\omega_1^{\mathrm{ck}}}$ such that the slightly extended corresponding $(\mathcal{B}_\alpha)_{\alpha\in\delta+1}$ still has (c) above and 1, 2, 4, 5, 6.

We know from the case assumption that given any $x \in X, \epsilon > 0$ there will be some $D \in \Delta^1_1$ and basic open V_0, V_1 with $x \in D^{\Delta V_0}$ and (D, V_0, V_1) having the $\Pi^1_1(\epsilon)$ property of ϵ-stability. The problem is to find a single set of such indices in $L_{\omega_1^{\mathrm{ck}}}$ which works for all such x and ϵ.

Towards this purpose let $P \subset \omega \times X$ be Π^1_1 and universal for Π^1_1 subsets of X. We let

$$f : \omega \times X \to \mathrm{LO}$$

be recursive continuous such that $(n, x) \in P$ if and only if $f(n, x) \in \mathrm{WO}$. At each $\beta < \omega_1^{\mathrm{ck}}$ and $n \in \omega$ we let $P_{\beta,n} = \{x \in X : f(x) \in \mathrm{WO}, \|f(x)\| < \beta\}$[4]. Note that the Δ^1_1 subsets of X are exactly the $\{P_{\alpha,n} : \alpha < \omega_1^{\mathrm{ck}}, n \in \mathbb{N}\}$. By the case assumption and the number uniformization theorem[5], we may find a Δ^1_1 function

$$(\epsilon, x) \mapsto (\alpha_{\epsilon,x}, n_{\epsilon,x})$$

such that each $(P_{\alpha_{\epsilon,x},n_{\epsilon,x}}, V_0, V_1)$ is ϵ-stable (some choice of basic open V_0, V_1) for $(\mathcal{B}_\alpha)_{\alpha\in\delta}$, and

$$x \in P^{\Delta V_0}_{\alpha_{\epsilon,x},n_{\epsilon,x}}.$$

Boundedness gives

$$\sup\{\alpha_{\epsilon,x} : \epsilon > 0, x \in X\} < \omega_1^{\mathrm{ck}},$$

and so some such recursive ordinal $\gamma > \sup\{\alpha_{\epsilon,x} : \epsilon > 0, x \in X\}$ will give $\{P_{\beta,n} : n \in \mathbb{N}, \beta \leq \gamma\}$ as required. Under any reasonable system of indexing Δ^1_1 sets we can find $B_\delta \in L_{\omega_1^{\mathrm{ck}}}$ such that B_δ exactly gives the indices of the Δ^1_1 sets appearing in $\{P_{\beta,n} : n \in \mathbb{N}, \beta \leq \gamma\}$. □

Claim 5.2 *For all $x, y \in X$*

$$\hat{\varphi}^{\vec{\mathcal{B}}}_{\omega_1^{\mathrm{ck}}}(x, X, G) = \hat{\varphi}^{\vec{\mathcal{B}}}_{\omega_1^{\mathrm{ck}}}(y, X, G) \Rightarrow x E_G y.$$

Proof Suppose not. Then we may find $x, y \in X$ with

$$\hat{\varphi}^{\vec{\mathcal{B}}}_{\omega_1^{\mathrm{ck}}}(x, X, G) = \hat{\varphi}^{\vec{\mathcal{B}}}_{\omega_1^{\mathrm{ck}}}(y, X, G)$$

but

$$x \neg E_G y.$$

[4] Here $\|w\|$ denotes the ordinal isomorphic to w for $w \in \mathrm{WO}$.

[5] I.e. that if $A \subset \omega^\omega \times \omega$ is Π^1_1 with all vertical sections non-empty, then there is Δ^1_1 $\pi : \omega^\omega \to \omega$ such that for all z we have $(z, \pi(n)) \in A$.

But then using for the first time that E_G is Δ^1_1 we may apply the low basis theorem[6] for Σ^1_1 and find $x, y \in X$ with

$$\omega_1^{\mathrm{ck}(x,y)} = \omega_1^{\mathrm{ck}},$$

$$\hat{\varphi}^{\vec{\mathcal{B}}}_{\omega_1^{\mathrm{ck}}}(x, X, G) = \hat{\varphi}^{\vec{\mathcal{B}}}_{\omega_1^{\mathrm{ck}}}(y, X, G),$$

$$x \neg E_G y.$$

Now choose a recursive $\epsilon_0 > 0$ such that

$$\epsilon_0 < \frac{1}{2}$$

and

$$\forall g \in W_{\epsilon_0}(d_X(y, g \cdot y) < \frac{1}{2}).$$

Subclaim: For all $\beta < \omega_1^{\mathrm{ck}}$ there exists

$$x_0 \in [x]_G$$

such that for all $\alpha < \beta$, $B \in \mathcal{B}_\alpha$, $V_2, V_3 \subset G$ basic open with V_3 a symmetric neighborhood of 1_G,

$$\exists^* h \in W_{\epsilon_0}(\hat{\varphi}^{\vec{\mathcal{B}}}_\alpha(x_0, B^{\Delta V_2}, V_3) = \hat{\varphi}^{\vec{\mathcal{B}}}_\alpha(h \cdot y, B^{\Delta V_2}, V_3)).$$

Proof of subclaim: Choose $A \in \mathcal{B}_\beta$, $V_0, V_1 \subset G$ basic open sets with V_1 a symmetric neighborhood of the identity, (A, V_0, V_1) ϵ_0-stable for $\vec{\mathcal{B}}|_\beta$, and

$$y \in A^{\Delta V_0}.$$

Then choose $x_0 E_G x$ with

$$\hat{\varphi}^{\vec{\mathcal{B}}}_\beta(x_0, A^{\Delta V_0}, V_1) = \hat{\varphi}^{\vec{\mathcal{B}}}_\beta(y, A^{\Delta V_0}, V_1).$$

It follows from the iterative structure of the definition of the $\hat{\varphi}^{\vec{\mathcal{B}}}_\beta(\cdot, \cdot, \cdot)$ invariants that for each B, V_2, V_3, α as in statement of subclaim that

$$\exists \hat{y} \in \mathcal{O}(y, A^{\Delta V_0}, V_1)$$

with

$$\hat{\varphi}^{\vec{\mathcal{B}}}_\alpha(x_0, B^{\Delta V_2}, V_3) = \hat{\varphi}^{\vec{\mathcal{B}}}_\alpha(\hat{y}, B^{\Delta V_2}, V_3).$$

Then by definition of stability we have that

$$\exists^* h \in W_{\epsilon_0}(\hat{\varphi}^{\vec{\mathcal{B}}}_\alpha(h \cdot y, B^{\Delta V_2}, V_3) = \hat{\varphi}^{\vec{\mathcal{B}}}_\alpha(\hat{y}, B^{\Delta V_2}, V_3) = \hat{\varphi}^{\vec{\mathcal{B}}}_{\alpha^{\mathrm{ck}}}(x_0, B^{\Delta V_2}, V_3)).$$

(Subclaim□)

Now we may apply *overspill*[7] to find some

$$x_0 \in [x]_G$$

such that for all $\alpha < \omega_1^{\mathrm{ck}}$, $B \in \mathcal{B}_\alpha$, $V_2, V_3 \subset G$ basic open with V_3 a symmetric neighborhood of 1_G,

$$\exists^* h \in W_{\epsilon_0}(\hat{\varphi}^{\vec{\mathcal{B}}}_\alpha(x_0, B^{\Delta V_2}, V_3) = \hat{\varphi}^{\vec{\mathcal{B}}}_\alpha(h \cdot y, B^{\Delta V_2}, V_3)).$$

By the low basis theorem we may assume

$$\omega_1^{\mathrm{ck}(x_0,x,y)} = \omega_1^{\mathrm{ck}(x,y)} = \omega_1^{\mathrm{ck}}.$$

[6]I.e. any non-empty Σ^1_1 contains an x with $\omega_1^{\mathrm{ck}(x)} = \omega_1^{\mathrm{ck}}$.
[7]I.e. the converse of the usual boundedness theorem for Σ^1_1 subset of the recursive ordinals.

At this stage we can choose a recursive real δ_0 such that

$$\delta_0 < \frac{1}{2}$$

$$\forall g \in W_{\delta_0}(d_X(x_0, g \cdot x_0) < \frac{1}{2}).$$

Subclaim: For all $\beta < \omega_1^{\mathrm{ck}}$ there exists

$$y_0 \in W_{\epsilon_0} \cdot y$$

such that for all $\alpha < \beta$, $B \in \mathcal{B}_\alpha$, $V_2, V_3 \subset G$ basic open with V_3 a symmetric neighborhood of 1_G,

$$\exists^* h \in W_{\delta_0}(\hat{\varphi}_\alpha^{\vec{\mathcal{B}}}(y_0, B^{\Delta V_2}, V_3) = \hat{\varphi}_\alpha^{\vec{\mathcal{B}}}(h \cdot x_0, B^{\Delta V_2}, V_3)).$$

Proof of subclaim: Again we may choose some (A, V_0, V_1) which is δ_0-stable for $\vec{\mathcal{B}}|_\beta$ with $x_0 \in A^{\Delta V_0}$. And then we may choose

$$y_0 \in W_{\epsilon_0} \cdot y$$

with

$$\hat{\varphi}_\beta^{\vec{\mathcal{B}}}(x_0, A^{\Delta V_0}, V_1) = \hat{\varphi}_\beta^{\vec{\mathcal{B}}}(y, A^{\Delta V_0}, V_1),$$

and we may complete the argument as in the proof of the last subclaim. (Subclaim$\square$)

Once more overspill allows us to find $y_0 \in W_{\epsilon_0}$ such that for all $\alpha < \omega_1^{\mathrm{ck}}$, $B \in \mathcal{B}_\alpha$, $V_2, V_3 \subset G$ basic open with V_3 a symmetric neighborhood of 1_G,

$$\exists^* h \in W_{\delta_0}(\hat{\varphi}_\alpha^{\vec{\mathcal{B}}}(y_0, B^{\Delta V_2}, V_3) = \hat{\varphi}_\alpha^{\vec{\mathcal{B}}}(h \cdot x_0, B^{\Delta V_2}, V_3)).$$

Continuing in a like minded fashion we may produce sequences

$$g_0, g_1, \cdots,$$

$$x_0, x_1, \cdots,$$

$$g'_0, g'_1, \cdots,$$

$$y_0, y_1, \cdots,$$

$$\delta_0, \delta_1, \cdots,$$

$$\epsilon_0, \epsilon_1, \cdots,$$

such that:

(0) for all $j \in \omega$,

$$\omega_1^{\mathrm{ck}(x_j, y_j)} = \omega_1^{\mathrm{ck}(x_{j+1}, y_j)} = \omega_1^{\mathrm{ck}};$$

(i) for all $j \in \omega$, $\alpha < \omega_1^{\mathrm{ck}}$, $B \in \mathcal{B}_\alpha$, $V_2, V_3 \subset G$ basic open with V_3 a symmetric neighborhood of 1_G,

$$\exists^* h \in W_{\delta_j}(\hat{\varphi}_\alpha^{\vec{\mathcal{B}}}(y_j, B^{\Delta V_2}, V_3) = \hat{\varphi}_\alpha^{\vec{\mathcal{B}}}(h \cdot x_j, B^{\Delta V_2}, V_3));$$

(ii) for all $j \in \omega$, $\alpha < \omega_1^{\mathrm{ck}}$, $B \in \mathcal{B}_\alpha$, $V_2, V_3 \subset G$ basic open with V_3 a symmetric neighborhood of 1_G,

$$\exists^* h \in W_{\delta_{j+1}}(\hat{\varphi}_\alpha^{\vec{\mathcal{B}}}(x_{j+1}, B^{\Delta V_2}, V_3) = \hat{\varphi}_\alpha^{\vec{\mathcal{B}}}(h \cdot y_j, B^{\Delta V_2}, V_3));$$

(iii) each x_j equals $g_{j-1} g_{j-2} \cdots g_0 \cdot x_0$;
(iv) each y_j equals $g'_{j-1} g'_{j-2} \cdots g'_0 \cdot y_0$;

(v) each $g_j \in W_{\delta_j}$;
(vi) each $g'_j \in W_{\epsilon_{j+1}}$;
(vii) at each j we have $\forall g \in W_{\delta_j}$

$$d_G(gg_{j-1}g_{j-2}\cdots g_0, g_{j-1}\cdots g_0) < 2^{-j-1};$$

(viii) at each j we have $\forall g \in W_{\epsilon_j}$

$$d_G(gg'_{j-1}g'_{j-2}\cdots g'_0, g'_{j-1}\cdots g'_0) < 2^{-j-1};$$

(ix) at each j

$$\forall g \in W_{\delta_j}(d_X(x_j, g\cdot x_j) < 2^{-j-1});$$

(x) at each j

$$\forall g \in W_{\epsilon_{j+1}}(d_X(y_j, g\cdot y_j) < 2^{-j-1}).$$

All the relevant sequences are Cauchy, so we can find g_∞, g'_∞ with

$$g_j \to g_\infty,$$

$$g'_j \to g'_\infty,$$

and hence by the continuity of the action

$$x_j \to g_\infty x_0,$$

$$y_j \to g'_\infty y_0.$$

We will be finished with the proof of the claim if we show that

$$g_\infty x_0 = g'_\infty y_0.$$

But note that at any j we can apply (i) to find $h \in W_{\delta_j}$ with

$$d_X(h\cdot x_j, y_j) < 2^{-j-1};$$

which by (v) yields

$$d_X(x_j, y_j) \leq d_X(x_i, h\cdot x_i) + d_X(h\cdot x_i, y_i) < 2^{-j-1} + 2^{-j-1} = 2^{-j}.$$

And so

$$x_0 E_G y_0$$

$$\therefore xE_G y,$$

after all. □

Notation Let HC denote the collection of hereditarily countable sets. $\mathrm{TC}(a)$ denotes the transitive closure of a set a.

Definition Let us say that

$$\theta : X \to \mathrm{HC}$$

is Δ^1_1 *in the codes* if there is an actual Δ^1_1 function

$$\Theta : X \to 2^{\omega\times\omega}$$

such that for all $x \in X$ we have that $\Theta(x)$ is a well founded and extensional relation on ω whose transitive collapse is isomorphic to

$$\mathrm{TC}(\theta(x)) \cup \{\theta(x)\};$$

that is to say, $\Theta(x)$ encodes the smallest transitive set containing $\theta(x)$ as an element.

Claim 5.3 *For all $\alpha < \omega_1^{\mathrm{ck}}$, $B \in \mathcal{B}_\alpha$, $V_0, V_1 \subset G$ basic open with V_1 a symmetric neighborhood of the identity, the function*

$$x \mapsto \hat{\varphi}_\alpha^{\vec{\mathcal{B}}}(x, B^{\Delta V_0}, V_1)$$

is Δ_1^1 in the codes.

Proof This follows routinely by induction on α. □

Thus we may obtain a Borel function

$$\theta : X \to 2^{\omega \times \omega}$$

such that for all $x \in X$ the relation $\theta(x)$ on ω is wellfounded, extensional, and has transitive collapse equal to

$$\{\hat{\varphi}_\alpha^{\vec{\mathcal{B}}}(x, X, G) : \alpha < \omega_1^{\mathrm{ck}}\} \cup \{\mathrm{TC}(\hat{\varphi}_\alpha^{\vec{\mathcal{B}}}(x, X, G)) : \alpha < \omega_1^{\mathrm{ck}}\}.$$

Hence

$$x E_G y$$

if and only if $\theta(x)$ and $\theta(y)$ are isomorphic as relations on ω. Thus E_G is reducible to

$$\cong|_{\mathrm{Mod}(\mathcal{L})},$$

for $\mathcal{L}$ a language generated by a single binary relation.

Finally some words on how we can extend the conclusion of this case to obtain a reduction that allows an orbit inverse.

To start we may apply theorem 5.1.5 of [1] to produce a Polish topology $\hat{\tau}$ on X which is compatible with the original Borel structure, includes all sets of the form

$$B^{\Delta V}$$

where $B \in \mathcal{B}_\alpha$ some $\alpha < \omega_1^{\mathrm{ck}}$, and is such that the action of G is still continuous with respect to this space.

From here we can consider the Polish G-space $(X, \hat{\tau})$. For this space we may directly evoke the generalized Scott invariants

$$x \mapsto \varphi_x$$

from §6.2 of [3]. Traipsing through the definitions there one sees that

$$\varphi_x = \varphi_y \Rightarrow \forall \alpha < \omega_1 (\hat{\varphi}_\alpha^{\vec{\mathcal{B}}}(x, X, G) = \hat{\varphi}_\alpha^{\vec{\mathcal{B}}}(y, X, G)),$$

and so by 5.2

$$x \mapsto \varphi_x$$

gives a complete invariant. It was shown in §6.2[3] that whenever we have these φ_x's serving as complete invariants then we can indeed obtain a Borel reduction to isomorphism on a class of countable structures which admits an orbit inverse. And thus case(2) leads to a strengthened version of (I) as its outcome.

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