A note on counterexamples to the Vaught conjecture

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1 Introduction

In [2] it was shown that:

Theorem 1.1 *There is a countable structure $\mathcal{M}$ with S_∞ dividing Aut($\mathcal{M}$) and whose Scott sentence has models of size $\aleph_1$ but no higher.*

This note makes the following corollary of that theorem explicit.

Corollary 1.2 *If there is some $\sigma \in L_{\omega_1\omega}$ providing a counterexample to Vaught's conjecture for $L_{\omega_1\omega}$, then there is some $\sigma' \in L_{\omega_1\omega}$ which again provides a counterexample and has no model of cardinality bigger than $\aleph_1$.*

Leo Harrington has shown that any counterexample to Vaught's conjecture will have, necessarily uncountable, models with arbitrarily high Scott ranks below ω_2.

2 Proof

For convenience, we work under ¬CH. A similar argument works for the version of the Vaught conjecture which asks for uncountably many non-isomorphic countable models without a perfect set of non-isomorphic countable models.

Definition S_∞ is said to *divide* a Polish group G if there is a closed subgroup $H < G$ and a continuous, onto, homomorphism

$$\pi : H \twoheadrightarrow S_\infty.$$

Notation For L a countable language we let X_L be the space of all L-structures with underlying set $\mathbb{N}$ and with the topology having basic open sets

$$\{\mathcal{N} : \mathcal{N} \models \psi(\vec{a})\}$$

for ψ quantifier free and $\vec{a}$ a finite sequence from $\mathbb{N}$.

It is well known and easily checked that X_L is a Polish space in the indicated topology. Compare §2.7 [1].

Let $\mathcal{M}$ be as in the statement of the theorem and let $\sigma \in L_{\omega_1\omega}$ have exactly $\aleph_1$ many non-isomorphic models. For convenience I will assume that $\mathcal{M}$ has $\mathbb{N}$ as its underlying set. Fix H a closed subgroup of Aut($\mathcal{M}$) which maps continuously and homomorphically onto S_∞.

In particular we have a continuous action of S_∞ on a Polish space X with exactly $\aleph_1$ many orbits. This obviously can be lifted up to a continuous action of the Polish group H with the same orbit structure, and then following [1] 2.3.5 we may lift the action of H on X up to an action of $\mathrm{Aut}(\mathcal{M})$ on a Polish space $Y \supset X$ such that, among other things, Y has the same number of orbits under $\mathrm{Aut}(\mathcal{M})$ as X does under H.

That is to say, $Y/\mathrm{Aut}(\mathcal{M})$ has size $\aleph_1$.

Then by 2.7.4 [1] we can find a richer countable language $L' \supset L$ such that the expansions of $\mathcal{M}$ to L', $Y^{\mathcal{M}}_{L'}$, form a universal Polish $\mathrm{Aut}(\mathcal{M})$ space. In particular there will be a injective and Borel $\mathrm{Aut}(\mathcal{M})$-map

$$\rho : Y \to Y^{\mathcal{M}}_{L'}.$$

Since the injective images of Borel sets are Borel, there will in particular be an invariant Borel set

$$B \subset Y^{\mathcal{M}}_{L'}$$

which contains exactly $\aleph_1$ many orbits under the action of $\mathrm{Aut}(\mathcal{M})$.

Now let C be the set of all $\mathcal{N} \in X_{L'}$ such that there exists $\mathcal{N}' \cong \mathcal{N}$ with $\mathcal{N}' \in B$. Note that C is then $\utilde{\Delta}^1_1$, and hence Borel, since, by invariance of B, we can equivalently describe C as the set of $\mathcal{N} \in X_{L'}$ which satisfy $\varphi_{\mathcal{M}}$, the Scott sentence of $\mathcal{M}$, and for which any $\mathcal{N}' \in Y^{\mathcal{M}}_{L'}$ isomorphic to $\mathcal{N}$ is in B.

Thus by [3], there is some $\sigma' \in L'_{\omega_1\omega}$ whose models with underlying set $\mathbb{N}$ are exactly the elements of C. Since $\sigma' \Rightarrow \varphi_{\mathcal{M}}$, σ' has no model of size $\aleph_2$. By construction, σ' has exactly $\aleph_1$ many non-isomorphic models.

References

[1] H. Becker and A. S. Kechris, *The descriptive set theory of Polish group actions*, London Mathematical Society Lecture Note Series, 232, Cambridge University Press, Cambridge, 1996.

[2] G. Hjorth, Knight's model, its automorphism group, and characterizing the uncountable cardinals, *Journal of Mathematical Logic,* vol. 2 (2002), 113-144.

[3] E.K.G. Lopez-Escobar, An interpolation theorem for denumerably long formulas, *Fundamenta Mathematicae,* vol. 57 (1957), 253-272.

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