A new zero-dimensional Polish group

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August 6, 1998

0. Introduction

Here we present an example of an abelian zero-dimensional Polish group G that is generated by every open neighborhood of the identity – that is to say, if $\mathcal{O} \subset G$ is open and $x \in G$ then there is some finite sequence $\epsilon_1, \epsilon_2, ..., \epsilon_n \in \mathcal{O}$ such that

$$x = \epsilon_1 + \epsilon_2 + ...\epsilon_n.$$

The main point here is that any zero-dimensional *locally compact* Polish group has a neigborhood basis at the identity consisting of (necessarily clopen) subgroups.

1. The group

1.1 Definition For $x \in \mathbb{R}$ we say that a sequence of integers $(a_i)_{i \in \mathbb{N}^+}$ is a *representation* of x if:

(i) at each n we have $-10^{-n} < a_n < 10^n$;

(ii) there is some integer k with

$$k + \sum_{n \in \mathbb{N}^+} \frac{a_n}{10^1 \cdot 10^2 \cdot ... \cdot 10^n} = x.$$

For $\vec{a} = (a_i)_{i \in \mathbb{N}^+}$ a representation of some real we let

$$||\vec{a}|| = \max_{n \in \mathbb{N}^+} |\frac{a_n}{10^n}|.$$

We let $||x||$ be the infinum of the set $\{||\vec{a}|| : a \text{ is a representation of } x\}$. We then let G be the set of $x \in \mathbb{R}$ for which there is some representation $\vec{a}$ of x with

$$\frac{a_n}{10^n} \to 0 \quad \text{as} \quad n \to \infty.$$

A representation $\vec{a}$ of a real x is *legitimate* if we do indeed have that

$$\frac{a_n}{10^n} \to 0$$

as $n \to \infty$.

1.2 Lemma G is a subgroup of $(\mathbb{R}, +)$.

Proof. It is immediately clear from the structure of the definitions that $-x \in G$ if $x \in G$. For additive closure, suppose $x, y \in G$ with legitimate representations $\vec{a}, \vec{b}$. Then we may find a representation $\vec{c}$ of $x + y$ with

$$|c_n| \leq |a_n| + |b_n| + 1$$

for each $n \in \mathbb{N}^+$. $\square$

1.3 Lemma For all $x \in G$ we have

$$||x|| \leq \frac{6}{10}.$$

Proof. Suppose $\vec{a}$ is a legitimate representation of x and

$$k + \sum_{n \in \mathbb{N}^+} \frac{a_n}{10^1 \cdot 10^2 \cdot \ldots \cdot 10^n} = x.$$

Then we may successively produce integers l, $\hat{a}_1, \hat{a}_2, \ldots$ and then $b_1, b_2, b_3, \ldots$ such that at each M

$$l + \sum_{n<M} \frac{b_n}{10^1 \cdot 10^2 \cdot \ldots \cdot 10^n} + \hat{a}_M + \sum_{n>M} \frac{a_n}{10^1 \cdot 10^2 \cdot \ldots \cdot 10^n} = x$$

such that

$$|\frac{\hat{a}_n}{10^n}| \leq \frac{1}{2},$$

$$|b_n| \leq |\hat{a}_n| + 1.$$

The first step of this process is the obvious one: If $|a_1| > 5$ then we simply choose an integer $l \in \{k-1, k+1\}$ and $\hat{a}_1$ such that

$$l + \frac{\hat{a}_1}{10^1} = k + \frac{a_1}{10^1},$$

$$|\hat{a}_1| < 5.$$

If on the other hand $|a_1| \leq 5$ we have no work to do, and simply set $l = k$ and $\hat{a}_1 = a_1$.

At the next stage it is similar: If $|a_2| > 50$ then we choose $b_1 \in \{\hat{a}_1 - 1, \hat{a}_1 + 1\}$ and $\hat{a}_2$ such that

$$b_1 + \frac{\hat{a}_2}{10^2} = \hat{a}_1 + \frac{a_2}{10^2},$$

$$|\hat{a}_2| < 50.$$

Again if $|a_2| \leq 50$ then simply set $b_1 = \hat{a}_1$ and $\hat{a}_2 = a_2$.

We continue in this manner. $\square$

In general if $\vec{a}$ and $\vec{b}$ are two legitimate representations of a real x then at each N

$$|\sum_{n>N}\frac{a_n}{10^1\cdot 10^2\cdot\ldots\cdot 10^n}-\sum_{n>N}\frac{b_n}{10^1\cdot 10^2\cdot\ldots\cdot 10^n}|\leq\frac{1}{10^1\cdot 10^2\cdot\ldots\cdot 10^N}.$$

Therefore for some $i\in\{0,1,-1\}$

$$|\sum_{n\leq N}\frac{a_n}{10^1\cdot 10^2\cdot\ldots\cdot 10^n}-\sum_{n\leq N}\frac{b_n}{10^1\cdot 10^2\cdot\ldots\cdot 10^n}+1|\leq\frac{1}{10^1\cdot 10^2\cdot\ldots\cdot 10^N},$$

and thus a_N must equal b_N-1, b_N, or b_N+1 modulo 10^N.

1.4 Lemma If $x\in G$ then there is a legitimate representation $\vec{a}$ with

$$||x||=||\vec{a}||.$$

Proof. Let $\vec{a}$ be a legitimate representation of x. By the previous lemma we may assume

$$||\vec{a}||<\frac{8}{10}.$$

Choose $N\in\mathbb{N}^+$ with

$$\frac{|a_n|+1}{10^n}<10^{-1}$$

all $n>N$. Let $\vec{b}$ be a competing legitimate representation of x with $||\vec{b}||\leq||\vec{a}||$.

Claim: For all $n>N$

$$a_n=b_n.$$

Proof of claim: Assume for a contradiction that $a_n\neq b_n$ and $n>N$. There are two possibilities.

Case(i) $a_n=b_n \bmod 10^n$. Then since $-10^n<a_n,b_n<10^n$ we must have $|a_n|+|b_n|=10^n$.

$$\therefore\frac{|b_n|}{10^n}=1-\frac{|a_n|}{10^n}>1-10^{-1},$$

by the assumption on $n>N$, with a contradiction to $||\vec{b}||\leq||\vec{a}||$.

Case(ii) $a_n=b_n-1 \bmod 10^n$ or $a_n=b_n-1 \bmod 10^n$.

There is a further split.

Case(iia) $a_n\neq b_n-1,b_n+1$. Then it is as case(i) with $|a_n|+|b_n|=10^n\pm 1$, and hence

$$\frac{|b_n|}{10^n}=1-\frac{|a_n|\pm 1}{10^n}>1-\frac{2}{10}=\frac{8}{10},$$

violating $||\vec{b}||\leq||\vec{a}||$.

Case(iib) $a_n=b_n-1$ or $a_n=b_n+1$. Thus for some η with $\eta\in\{-1,0,1\}$. Then

$$\frac{a_n}{10^n}+\frac{a_{n+1}}{10^n\cdot 10^{n+1}}=\frac{b_n}{10^n}+\frac{b_{n+1}}{10^n\cdot 10^{n+1}}+\frac{\eta}{10^n\cdot 10^{n+1}}.$$

Then we have a_{n+1} not equal to $b_{n+1}-1$, b_{n+1}, or $b_{n+1}+1$, and we are back into the contradiction of case(i). (Claim $\square$)

Thus in determining $||x||$ we are taking the infinum over a finite set. Thus this inf is attained. $\square$

1.5 Lemma For $x, y \in G$,

$$||x|| = ||-x||$$

and

$$||x+y|| \leq ||x|| + ||y||.$$

Proof. The equality $||x|| = ||-x||$ is obvious from the very form of the definitions.

For the inequality, we may as well assume $||x|| + ||y|| < 1$, or else we are done by lemma 1.3. Fixing representations $\vec{a}$, $\vec{b}$ with

$$||\vec{a}|| = ||x||$$

and

$$||\vec{b}|| = ||y||$$

we have in particular at each n that

$$\frac{|a_n|}{10^n} + \frac{|b_n|}{10^n} < 1.$$

Therefore if we just let $c_n = a_n + b_n$ we obtain a representation for $x+y$ with

$$|c_n| \leq |a_n| + |b_n|.$$

$\square$

1.6 Corollary If we set $\hat{d}(x,y) = ||x-y||$, then $\hat{d}$ defines a pseudo-metric on G.

1.7 Definition We let $d_{\mathbb{R}}$ be the usual euclidean metric on $\mathbb{R}$ and define d_G on G by

$$d_G(x,y) = d_{\mathbb{R}}(x,y) + \hat{d}(x,y).$$

Then d_G is the sum of a pseudo-metric and metric, and therefore is a metric.

1.8 Lemma d_G is a complete metric on G.

Proof. Let $(x_i)_{i \in \mathbb{N}}$ be a Cauchy sequence in (G, d_G). Fix at each i a representation $\vec{a}^i = (a^i_1, a^i_2, a^i_3, ...)$ of x_i with

$$||\vec{a}^i|| = ||x_i||.$$

After perhaps thinning out the original sequence of reals we may assume that for each n the sequence $(a^0_n, a^1_n, a^2_n, ...)$ is eventually constant. We may also assume that at each i

$$x_i = \sum_{n \in \mathbb{N}^+} \frac{a^i_n}{10^1 \cdot 10^2 \cdot ... \cdot 10^n};$$

in other words, the k is always zero. Letting a_n^∞ be the limit of the sequence $(a_n^0, a_n^1, a_n^2, ...)$, we will show that $\vec{a}^\infty$ is a legitimate representation of some $x_\infty \in G$.
Claim: For all $\epsilon > 0$ there is some N_ϵ and I_ϵ such that for all $m > N_\epsilon$ and $j > I_\epsilon$

$$\frac{a_m^j}{10^m} < \epsilon.$$

Proof of claim: Fix $\epsilon > 0$ and assume without loss of generality that $\epsilon < 10^{-1}$. Choose I_ϵ so that for all $i, j \geq I_\epsilon$ we have

$$d_G(x_i, x_j) < \frac{\epsilon}{3}$$

and

$$\frac{\epsilon}{3} > 10^{-I_\epsilon}.$$

Let N_ϵ be such that for all $m > N_\epsilon$

$$\frac{a_m^{I_\epsilon}}{10^m} < \frac{\epsilon}{3}.$$

Then for all $j > I_\epsilon$ we may obtain a legitimate representation $\vec{c}$ of x_j such that at each $m > N_\epsilon$

$$\frac{c_m}{10^m} < \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3}.$$

If

$$\frac{a_m^j}{10^m} \geq \epsilon,$$

then

$$|a_m^j - c_m| > 1,$$

and thus we must be in the situation that there is some $\eta \in \{-1, 0, 1\}$ such that

$$|a_m^j - c_m| = \eta \bmod 10^j$$

but

$$|a_m^j - c_m| \neq \eta$$

Thus

$$\frac{|a_m^j|}{10^m} > 1 - \frac{|c_m|}{10^m} - \frac{1}{10^m},$$

entailing the conclusion that

$$||\vec{a}^j|| > \frac{8}{10},$$

which was already prohibited by lemma 1.3 and the assumption $||x_j|| = ||\vec{a}^j||$. (Claim$\Box$)

Thus we may indeed obtain a real $x_\infty \in G$ with representation

$$x_\infty = \sum_{n\in\mathbb{N}^+} \frac{a_n^\infty}{10^1 \cdot ...10^n}.$$

To see that

$$x_i \to x_\infty$$

in (G, d_G) we appeal to the claim once more: Fix $\delta > 0$ and assume that $\delta < 1$. Then we may find $N_{\delta/2}$ and $I_{\delta/2}$ as in claim; in particular this gives

$$\frac{a_n^\infty}{10^n} < \frac{\delta}{2}$$

all $n > N_{\delta/2}$. By possibly enlarging $I_{\delta/2}$ we may assume that for all $i > I_{\delta/2}$ and $n \leq N_{\delta/2}$

$$a_n^i = a_n^\infty.$$

Then we obtain that for all $i > I_{\delta/2}$ and at *all* $n \in \mathbb{N}$

$$\frac{|a_n^i - a_n^\infty|}{10^n} < \delta,$$

and thus $||x_n - x_\infty||$ has a representation $\vec{c}$ with

$$||\vec{c}|| < \delta.$$

□

1.9 Lemma (G, d_G) is separable.

Proof. Let $A \subset G$ be the set of all reals in G with a representation that has finite support – that is to say, if $x \in A$ then there will be some $k \in \mathbb{Z}$ and $(a_i)_{i\in\mathbb{N}}$ such that at each n we have $-10^{-n} < a_n < 10^n$, $k + \sum_{n\in\mathbb{N}^+} \frac{a_n}{10^1 \cdot 10^2 \cdot ... \cdot 10^n} = x$, and $a_n = 0$ for sufficiently large n.

Then given any $y \in G$ and $\epsilon > 0$ we need to show

$$\{x \in G : d_G(x, y) < \epsilon\} \cap A \neq 0.$$

Suppose that

$$l + \sum_{n\in\mathbb{N}^+} \frac{b_n}{10^1 \cdot 10^2 \cdot ... \cdot 10^n} = y$$

with $l \in Z$ and $\vec{b}$ a legitimate representation. Then we may find N such that for all $n > N$

$$\frac{|b_n|}{10^n} < \epsilon.$$

Then if we let

$$x = l + \sum_{n \leq N} \frac{b_n}{10^1 \cdot 10^2 \cdot \ldots \cdot 10^n}$$

we have our member of A. $\square$

1.10 Lemma The group operations in (G, d_G) are continuous.

Proof. Suppose that $x_0, y_0 \in G$. Then for any $\epsilon > 0$ and $x, y \in G$ with

$$||x_0 - x||, ||y_0 - y|| < \epsilon$$

then by 1.5 we have

$$||(-x_0) - (-x)|| = || - (x_0 - x)|| = ||x_0 - x|| < \epsilon$$

and

$$||(x_0 + y_0) - (x + y)|| = ||(x_0 - x) + (y_0 - y)|| \leq ||x_0 - x|| + ||y_0 - y|| < 2\epsilon.$$

$\square$

1.11 Lemma (G, d_G) is zero-dimensional.

Proof. It suffices by 1.10 to show that there is a neighborhood basis at the identity consisting of clopen sets. So fix $\epsilon > 0$, and we will show that there is some clopen set $\mathcal{O}$ such that

$$0 \in \mathcal{O} \subset \{x \in G : d_G(x, 0) < \epsilon\}.$$

Choose *irrational* $\delta > 0$ with

$$\delta < \frac{1}{10^2}, \epsilon.$$

Let $\mathcal{O} = \{x \in G : d_{\mathbb{R}}(x, 0) < 10\delta, ||x|| < \delta\}$.

Clearly $\mathcal{O}$ is open in (G, d_G), so we just need to show it is closed. For this purpose, fix $x \in G \setminus \mathcal{O}$.

If $||x|| \geq \delta$ then since $||x||$ rational by 1.4, and we have $||x|| > \delta$. Since $\{y \in G : ||y|| > \delta\}$ is an open set, we have shown that there is an open neighborhood around x disjoint from $\mathcal{O}$.

Alternatively if $||x|| < \delta$ then we must have $d_{\mathbb{R}}(x, 0) \geq 10\delta$. Then it follows that x has a representation $\vec{a}$ with $||\vec{a}|| < \delta$ and

$$k + \sum_{n \in \mathbb{N}^+} \frac{a_n}{10^1 \cdot 10^2 \cdot \ldots \cdot 10^n} = x,$$

and necessarily $k \neq 0$. Thus in fact

$$d_{\mathbb{R}}(x, 0) > \frac{1}{2}$$

and we have an open set containing x and disjoint from). $\square$

1.12 Lemma Every neighborhood of the identity in (G, d_G) generates the group.

Proof. Fix $\epsilon > 0$ and $x \in G$ with a legitimate representation $\vec{a}$ with

$$k + \sum_{n \in \mathbb{N}^+} \frac{a_n}{10^1 \cdot 10^2 \cdot \ldots \cdot 10^n} = x.$$

Choose $N \in \mathbb{N}$ with

$$\frac{|a_n|}{10^n}, \frac{1}{10^n} < \epsilon$$

all $n \geq N$.

Then we may certainly obtain

$$k + \sum_{n < N} \frac{a_n}{10^1 \cdot 10^2 \cdot \ldots \cdot 10^n}$$

in the subgroup of G generated by

$$\frac{1}{10^N}.$$

Since

$$d_G(0, \frac{1}{10^n}),\ d_G(0, \sum_{n \geq N} \frac{a_n}{10^1 \cdot 10^2 \cdot \ldots \cdot 10^n}) < \epsilon$$

we have shown that x is generated by $\{y \in G : d_G(0, y) < \epsilon\}$. □

1.13 Theorem In the topology generated by the metric d_G, G is a zero-dimensional Polish group.

Proof. This is exactly the content of the last five lemmas. □

References

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